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Exact Solutions of a Chaplygin Gas in an Anisotropic Space-Time of Petrov D

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Abstract

This document obtains two exact solutions to the anisotropic space-time of Petrov D by using the model of a perfect fluid. These solutions represent a scenario of a universe in which the pressure P and the energetic density μ of the fluid are inversely proportional (Chaplygin's type $P = -Q^2/\mu$), where Q is a constant of proportionality. It is established that the symmetry of those models, in the proximities when $t \to 0$, is equivalent to the analogues for the dust model, and might tend to behave as the solutions of the flat or vacuum LRS Kasner solution (Local Rotational Symmetry). Although the solutions are not flat or vacuum in any of the cases, in those proximities, the density tends to infinite and with no pressure. When $t \to \infty$, the models tend to behave as the isotropic flat model of the type FRWL. In the analysis of the Hubble and the deceleration parameters obtained that in those solutions, the Hubble's constant and the deceleration parameter, depend on time and manner in which their values, or tendency, significantly evolve. The deceleration parameter q changes its sign as times passes, so that it represents an initial deceleration process that, in continuity, constantly changes to a process of acceleration.

Keywords: cosmology, Chaplygin gas, Einstein, exact, solution

1 Introduction

Cosmology has become more dynamic considering the microwave radiation obtained from COBE, WMAP, PLANCK, and the discovery of the universe acceleration [1, 2]; these and other aspects have been discussed on [3].
Models from different equations and fluid states have been obtained using the perfect fluid, see example [4], [5], where several scenarios of the Universe are described, depending on the type of equation and the state. This has allowed exact analytical solutions for some limited cases of equations or with metrics that contain complex elements ($g_{\mu\nu} \in \mathbb{C}$). There have also been obtained solutions considering a binary mixture of dark energy, or quintessence, and an ideal fluid in conjoint movement [5], dark energy and dark mass with different velocities [6]. For example, the investigation on gravitational waves in a Bianchi-I space-time has been carried out fundamentally, but not restricted, to the Kasner solution, and important differences between a symmetry of the type Bianchi-I anisotropic and isotropic of the type of Robertson and Walker have been determined. An important influence has been established by the anisotropic expansion of space, on the gravitational waves and the influence of the energy of these waves in the anisotropic space in expansion [7][8]; in addition to the anisotropic role, in the coupling between gravitational waves and density modes, and the density and perturbations in density as possible causes of the formation of galaxies [9]. One of the fluid models that results equivalent, at a certain time, to a fluid given by the state equation $P = \lambda_1\mu$, where P and μ are the pressure and the energy density respectively, and λ_1 a constant and another time for $P = \lambda_2\mu$, so that the transition from one type of state equation to the other is proportional and continuous to the pattern P and μ is the gas model of Chaplygin [10], in the form $P = -Q^2/\mu$, where Q is a constant. In this model [11], it has been obtained that in a FRWL space-time of several scenarios of the universe are produced, and those are achieved when using the dust model, at the beginning, and the dark energy model, when a long time has passed. In addition, it is described a smooth transition from a decelerated universe in expansion to the current time of cosmic acceleration. The Chaplygin gas model is used for tachyon cosmological models with a constant value of the potential space. As consequence, it is interesting to study this model and its generalities [12] in several scenarios and cosmological symmetries.

2 The Metric Interval

The metric interval that will be used is the anisotropic and homogeneous Petrov D, close to LRS (Local Rotational Symmetry) used in Bianchi-I, and follows the pattern [3]

$$ds^2 = Fdt^2 - t^{2/3}K(dx^2 + dy^2) - \frac{t^{2/3}}{K^2}dz^2, \tag{1}$$

Where F and K are functions of t.

3 Solutions of the Einstein's Equations for the Chaplygin's Fluid Model

The model of the perfect fluid used in cosmology represents a fluid without viscosity, isotropic ($P = P(\mu)$) and with no shear stresses, which can be expressed as follows [13]

$$T_{\alpha\beta} = (\mu + P)\, u_\alpha u_\beta - g_{\alpha\beta} P, \tag{2}$$

Where $T_{\alpha\beta}$ is the stress energy tensor of the perfect fluid, u_α is the tetradimensional speed, $g_{\alpha\beta}$ is the metric tensor, μ and P are respectively the energy density and the fluid pressure.
The state equation for the analyzed fluid will be taken as $P = -Q^2/\mu$ (Chaplygin Fluid), where Q is a constant.
The components of the Einstein tensor $G_\alpha^\beta = R_\alpha^\beta - \frac{1}{2}\delta_\alpha^\beta R$ different from zero, are

$$G_0^0 = \frac{4\,K^2 - 9\,t^2\dot{K}^2}{12t^2K^2F}, \tag{3}$$

$$G_1^1 = -\frac{3Kt\dot{K}\left(2F - \dot{F}t\right) + 3Ft^2\left(2K\ddot{K} - 5\dot{K}^2\right) + 4K^2\left(\dot{F}t + F\right)}{12t^2K^2F^2}, \tag{4}$$

$$G_2^2 = G_1^1, \tag{5}$$

$$G_3^3 = -\frac{-6Kt\dot{K}\left(2F - \dot{F}t\right) - 3Ft^2\left(4K\ddot{K} - \dot{K}^2\right) + 4K^2\left(\dot{F}t + F\right)}{12t^2K^2F^2}, \tag{6}$$

It will be considered a fluid with tetradimensional speed $u_\alpha = (u_0, 0, 0, 0)$, therefore, the components of the stress energy tensor (2) different from zero are

$$T_0^0 = \mu, \tag{7}$$

$$T_1^1 = T_2^2 = T_3^3 = -P, \tag{8}$$

This implies that from the Einstein equations $G_\alpha^\beta = \kappa T_\alpha^\beta$, it must meet that $G_1^1 = G_3^3$, so that from (4) and (6), it is obtained

$$\dot{K}K\left(2F - \dot{F}t\right) - 2Ft\left(-K\ddot{K} + \dot{K}^2\right) = 0 \tag{9}$$

So

$$K = K_0 e^{C_1 \int \frac{F^{1/2}}{t} dt}, \tag{10}$$

Where K_0 and C_1 are constants. Without losing its generalities, the constant K_0 in (10) is considered equal to 1, and the constant $C_1 = \pm 2/3$; this represents two different models.
From the state equation $P = -Q^2/\mu$, it is obtained that $G_1^1 = -Q^2/G_0^0$, and from (4),(6) and (10) it is obtained the state equation, if it is satisfies that

$$-\dot{F}t\left(F-1\right) + F\left(\left(F-1\right)^2 + 9F^2Q^2t^4\right) = 0, \tag{11}$$

which solution is

$$F = \frac{1}{1 \pm 3t\sqrt{\alpha + t^2Q^2}}, \tag{12}$$

where α is an constant of integration.
The density and the pressure of (7) and (8), assuming that $\kappa = 1$ in the Einstein equations, follow the patterns

$$\mu = \pm\frac{\sqrt{\alpha + t^2Q^2}}{t} \tag{13}$$

And

$$P = \mp\frac{Q^2t}{\sqrt{\alpha + t^2Q^2}} \tag{14}$$

In (13), the density is positive if its sign is positive and the time is taken as positive, or if the sign before the equality is negative and the time is negative. In both cases, there is an interval of time when $g_{00} = F > 0$ but $\mu < 0$. If the positive time is considered, it means that

$$F = \left(1 + 3t\sqrt{\alpha + t^2Q^2}\right)^{-1},$$

and there is a time between

$$t \in]\left(3\alpha - \sqrt{9\alpha^2 + 4Q^2}\right) / \left(6Q^2\right), 0],$$

F is positive in that interval, although μ is negative or singular when $t = 0$, but the components g_{11}, g_{22}, g_{33} are complex, so the time t will be considered in the interval of $t \in [0, \infty[$, for which the density is positive or singular when $t = 0$.

For future analysis of the solutions, in the proximities with $t = 0$, it will be used the Kretschmann invariant. The condition $Krets < \infty$ is required and sufficient [14] for the finitude of all the invariants of an algebraic curvature, and this solution takes the pattern

$$Krets = R_{\alpha\beta\gamma\tau} R^{\alpha\beta\gamma\tau} = \frac{1}{27} \frac{\pm 32 F^{7/2} - 24 F \dot{F} t \left(F - 1\right) + 9 \dot{F}^2 t^2 + 4 F^2 \left(-6F + 9F^2 + 5\right)}{F^4 t^4}. \quad (15)$$

4 Analysis of the Solutions

The two possible solutions result in the same value for the density and the pressure, and diverge from each other, mainly when $t \to 0$. The prior is observed in (12), when considering $t \to 0$; in this case F can get closer to

$$F = \frac{1}{1 + 3t\sqrt{\alpha}} \quad (16)$$

Which agrees with the solutions of the dust model [3]. When considering (16) in $3\sqrt{\alpha} = \beta$, if $\xi = 2\,\frac{\sqrt{1+t\beta}}{\beta}$, then $t = \frac{-4+\xi^2\beta^2}{4\beta}$, the solution can be expressed as follows

$$ds^2 = d\xi^2 - \left(\frac{1}{2}\,\xi\,\beta \pm 1\right)^{4/3} \left(dx'^2 + dy'^2 + \left(\frac{\xi\,\beta - 2}{\xi\,\beta + 2}\right)^{\pm 2} dz'^2\right) \quad (17)$$

where $x' = 4^{\pm 1/3}\beta^{-1/3 \mp 1/3} x$, analogous y, $z' = 4^{\mp 2/3}\beta^{\pm 2/3 - 1/3} z$. From the solutions (17), it can be noticed that when $t \longrightarrow 0$, $\xi \longrightarrow 2/\beta$, therefore, it is assumed $\xi > 0$ and noticeable that $\xi \longrightarrow (\beta t + 2)/\beta$, then the metric can be expressed, returning to the time coordination t, with the pattern

$$ds^2 \longrightarrow dt^2 - t^{2/3 \pm 2/3}(dx^2 + dy^2) - t^{2/3 \mp 4/3} dz^2, \quad (18)$$

The solution in the proximities with $t = 0$, is singular in all the cases. Although in appearance the solution tends to a flat world when the negative sign is chosen ($C_1 = -2/3$) in (18), the Kretschmann invariant, when $t \longrightarrow 0$, tends to $Krets \longrightarrow 1/3\beta^2 t^{-2} \longrightarrow \infty$, therefore it is singular. This model represents a space-time that at the beginning when $t = 0$, extends one of its axis as an

infinite tube, but with a very small radio.
When the sign is taken as positive ($C_1 = 2/3$), the invariant tends to $Krets \longrightarrow 64/27t^{-4}$, which tends to the solution of the Kasner vacuum LRS (E_{D_1}) [3], therefore, it is also singular when $t = 0$. This solution represents a space-time that, at the beginning, had the shape of a very thin pancake with an infinite radio.

When $t \to \infty$, the solutions tend to behave as the solutions of the dark energy model [3], for which $P \to -\mu \to -Q$.
If it is considered the following change of temporal coordinate $t = \frac{\sinh\left(\eta\sqrt{3Q}\right)}{\sqrt{3Q}}$, the metric tends to become an isotropic type with the pattern

$$ds^2 \approx {ds_{isot}}^2 + {ds_{pert}}^2 \tag{19}$$

$${ds_{isot}}^2 = d\eta^2 - e^{2/3\sqrt{3Q}\eta}\left(dx'^2 + dy'^2 + dz'^2\right) \tag{20}$$

$${ds_{pert}}^2 = \pm\frac{4}{3}\frac{\left(dx'^2 + dy'^2 - 2\,dz'^2\right)}{e^{1/3\sqrt{3Q}\eta}}, \tag{21}$$

where $x' = 2^{-1/3\pm1/3}\,(3Q)^{-1/6\mp1/6}\,x$, analogous y, $z' = 2^{\mp2/3-1/3}\,(3Q)^{-1/6\pm1/3}\,z$. From the previous analysis, it is evident that ${ds_{pert}}^2$ decays faster, being this solution prompt to isotropic, and close to

$$ds^2 \approx d\eta^2 - e^{2/3\sqrt{3Q}\eta}\left(dx'^2 + dy'^2 + dz'^2\right) \tag{22}$$

and it presents the characteristics of a space-time in De Sitter inflation in the flat slicing coordinates.
The solution tends to be equivalent to the one of the dark energy model in an isotropic and flat space-time of the type FRWL.

5 Hubble parameters and deceleration

The Hubble parameter, for the given symmetry, can be defined as [15]

$$H = \frac{1}{3}\left(H_x + H_y + H_z\right) = \frac{1}{3}\left(\frac{\dot{\sqrt{g_{11}}}}{\sqrt{g_{00}}\sqrt{g_{11}}} + \frac{\dot{\sqrt{g_{22}}}}{\sqrt{g_{00}}\sqrt{g_{22}}} + \frac{\dot{\sqrt{g_{33}}}}{\sqrt{g_{00}}\sqrt{g_{33}}}\right), \tag{23}$$

where the point represents the derivative of t. For the solution (12) with positive sign, it meets that

$$H = \frac{\sqrt{1 + 3t\sqrt{\alpha + Q^2t^2}}}{3t}. \tag{24}$$

From (24), if $t \to 0$, the parameters tends to infinite, and if $t \to \infty$, $H \to \sqrt{Q/3}$; therefore, the value of Q can be determined through the parameter H

when $t \to \infty$.
The deceleration parameter q is defined as in [15], with the pattern

$$q = -\left(1 + \frac{\dot{H}}{H^2\sqrt{g_{00}}}\right) = -\left(1 + \frac{1}{3}(Q_x + Q_y + Q_z)\right), \tag{25}$$

where

$$Q_x = \frac{1}{\sqrt{g_{00}}H^2}\left(\frac{\dot{\sqrt{g_{11}}}}{\sqrt{g_{00}}\sqrt{g_{11}}}\right)^{\cdot}, Q_y = \frac{1}{\sqrt{g_{00}}H^2}\left(\frac{\dot{\sqrt{g_{22}}}}{\sqrt{g_{00}}\sqrt{g_{22}}}\right)^{\cdot},$$

$$Q_z = \frac{1}{\sqrt{g_{00}}H^2}\left(\frac{\dot{\sqrt{g_{33}}}}{\sqrt{g_{00}}\sqrt{g_{33}}}\right)^{\cdot}.$$

The deceleration parameter for the analyzed solutions takes the pattern

$$q = \frac{4\sqrt{\alpha + t^2Q^2} + 3\,t\alpha - 6\,t^3Q^2}{2\left(1 + 3\,t\sqrt{\alpha + t^2Q^2}\right)\sqrt{\alpha + t^2Q^2}}, \tag{26}$$

From (26), when $t \to 0$, the parameter $q \to 2$, and when $t \to \infty$, it is $q \to -1$, which implies a change in the sign given in $t_{change} = \frac{\sqrt{2}\sqrt{B\left((B+3\,\alpha)^2+48\,Q^2\right)}}{6QB}$, where

$$B = \sqrt[3]{1728\,\alpha\,Q^2 - 27\,\alpha^3 + 12\,\sqrt{-768\,Q^4 + 20304\,\alpha^2Q^2 - 729\,\alpha^4}Q}.$$

For this reason, the two models $C_1 = \pm 2/3$, present a common scenario in relation to the parameters q and H; the possible universes at their beginnings had a Hubble parameter that tends to infinite with the pattern $H \to 1/(3t)$ and decays to a constant value with the form $H \to \sqrt{Q/3}$ with time. In this case, the q parameter indicates an expansion of the Universe, initially decelerated, until the time $t = t_{change}$, and after accelerated; as time increases, it approaches -1, which is generally consistent with some of the previously mentioned observations Adam, Lima, Omer.

6 Conclusion

The two solutions for an homogeneous, anisotropic, and non static Petrov D space-time with the model of the perfect fluid and with an equation of state, where the pressure is inversely proportional to the energy density $P = -Q^2/\mu$, and results in two possible scenarios. The first one, when the axis z expands more intensely than the plane x, y, and when the opposite occurs.

When analyzing details of interest, for example, at the beginning when $t = 0$ or when $t \longrightarrow \infty$, it is obtained that for those models, the metric might tend to be equivalent to the Kasner vacuum LRS when $t \longrightarrow 0$, if $C_1 = 2/3$, and it is singular when $t = 0$; in the other solution, if the negative sign is taken as K, this tends to be equivalent to a flat world, but with a singularity in $t = 0$. Both solutions tend to the spatial isotropic regime equivalent to the dark energy model for a flat model of FRWL with time, and tend to be equivalent to those obtained for a model of dust, with small time.
Both the Hubble and the deceleration parameter depend on time, which marks a fundamental difference between the same parameters obtained for the flat universe of FRWL, or the solution of the Kasner vacuum [15], where the deceleration parameter is constant for each type of model or for all of them in Kasner's case. In the resulting solutions, the parameters are time dependent, so in certain limits they behave as the analogous obtained from the Kasner solution and from other limits, as the analogous to the FRWL solution of the flat model of dark energy, it exists a transitional path between them through time, so that the parameter q changes its value and sign, which means that a period of deceleration is presented in small values t, and then it changes to a process of acceleration as time passes. This agrees with some of the observations already stated in [16], [17] and [18].

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Annex

The integral M in (10) presents the following pattern

$$M = M_1 - M_\infty$$

where $M_\infty = const \in \mathbb{C}$,

$$M_1 = -R\left(t\right)\frac{N_{2,4,1}N_{2,1,4}N_{2,1,3}\left(\sqrt{\alpha+t^2Q^2}-\sqrt{\alpha}-e_2tQ\right)^2\left(e_1-e_4\right)}{\alpha\sqrt{1+3\sqrt{\alpha}t\sqrt{1+\frac{t^2Q^2}{\alpha}}}\left(\sqrt{1+\frac{t^2Q^2}{\alpha}}\right)\left(e_4-e_2\right)\left(e_1-e_2\right)e_2e_1}$$

$M_1 \in \mathbb{C}$ (but $M \in \mathbb{R}$) and

$$R\left(t\right) = \left({e_2}^2+1\right)e_1F\left(N_{2,4,1},h\right)+\Pi\left(N_{2,4,1},j,h\right)e_1e_2\left(e_1-e_2\right)-$$

$$\Pi\left(N_{2,4,1},\frac{e_2j}{e_1},h\right)\left(e_1-e_2\right)$$

$F\left(x,k\right) = \int_0^x \frac{d\xi}{(1-\xi^2)(1-k^2\xi^2)}$ is the incomplete elliptic integral of the first kind, and $\Pi\left(x,h,k\right) = \int_0^x \frac{(1-k^2\xi^2)^{-1/2}d\xi}{(1-h\xi^2)\sqrt{1-\xi^2}}$ is the incomplete elliptic integral of the third kind.

$$N_{i,n,k} = \sqrt{\frac{\sqrt{\alpha+t^2Q^2}\left(e_n-e_i\right)-\sqrt{\alpha}\left(e_n-e_i\right)-e_kQt\left(e_n-e_i\right)}{\sqrt{\alpha+t^2Q^2}\left(e_n-e_k\right)-\sqrt{\alpha}\left(e_n-e_k\right)-e_itQ\left(e_n-e_k\right)}},$$

$h = \sqrt{\frac{j(e_3-e_2)}{e_3-e_1}}$ and $j = \frac{e_4-e_1}{e_4-e_2}$ the constants e_k are the roots of the equation $x^4Q+6\,\alpha\,x^3-2\,x^2Q+6\,\alpha\,x+Q = 0$ so that the roots (usually complex) are taken in an anticlockwise sense, growing with the index $k = 1..4$, when graphed in a rectangular coordinate system x and y, where a root e_k has the pattern $e_k = x_k + y_ki$; when roots are multiple of others($e_k = Ae_l$), it is first taken the one with a lower absolute value.
The constant of integration M_∞ has the pattern

$$M_\infty = -\sqrt{\frac{Q}{3\alpha}}\frac{R_\infty L_{2,4,1}L_{2,1,4}L_{2,1,3}\left(1-e_2\right)^2\left(e_1-e_4\right)}{\left(e_4-e_2\right)\left(e_1-e_2\right)e_2e_1},$$

where

$$R_\infty = \left({e_2}^2 + 1\right) e_1 F\left(L_{2,4,1}, h\right) + \Pi\left(L_{2,4,1}, j, h\right) e_1 e_2 \left(e_1 - e_2\right) -$$

$$\Pi\left(L_{2,4,1}, \frac{e_2 j}{e_1}, h\right)\left(e_1 - e_2\right)$$

and

$$L_{i,n,k} = \sqrt{\frac{\left(-1 + e_k\right)\left(e_n - e_i\right)}{\left(-1 + e_i\right)\left(e_n - e_k\right)}}.$$

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