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Measurability in Quantum Theory, Gravity and Thermodynamics and General Remarks to Hawking's Problems

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Abstract

In thermodynamics the measurability notion, introduced previously in a quantum theory, is defined on the basis of a minimal inverse temperature. Based on this notion, some implications are obtained for thermodynamics of black holes at all the energy scales and for quantum corrections of the basic quantities in the general case. Then on the basis of these results the Hawking Problems for black holes are subjected to an initial analysis.

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1 Introduction

This paper is a continuation of the earlier works published by the author [1]–[11] and preprints [12],[13]. The main target of published papers is to construct a correct quantum theory and gravity in terms of the variations (increments) *dependent on the existent energies*, i.e., the theory should not involve infinitesimal space-time variations (increments)

$$dt, dx_i, i = 1, ..., 3 \tag{1}$$

This mathematical apparatus based on the use of these variations comes from mathematical analysis [14], [15] and is completely adequate for classical mechanics [16],[17], where continuous space-time forms the base.
But in this approach, due to the introduction of ultraviolet and infrared divergences into a Quantum Theory (QT) [18] and also due to the absence of correct passage to the high-energy (ultraviolet) region in Gravity (GR)[19], we are facing very serious problems.
By the authors opinion, these problems are solvable but beyond the paradigm of continuous space-time.
To solve this problems, in the above-mentioned works, using the minimal length l_{min} (minimal time t_{min}), the author investigates a discrete space-time model, for which at low energies (far from the Planck energies) the results are to a high a curacy identical to those obtained with a continuous space-time model. And at high (Plancks) energies the indicated model is fundamentally discrete, leading to principally new results. All variations in any physical system considered in such a discrete model should be dependent on the existent energies.
The primary instrument for such a discrete model is the **measurability** notion introduced in [1]–[11].
In preprint [12] is demonstrated that a similar *(in essence dual)* notion may be also introduced in thermodynamics on the basis of a minimal inverse temperature, leading to very interesting inferences for thermodynamics of black holes at all the energy scales. Quantum corrections of the basic quantities were found in preprint [13] in the framework of Generalized Uncertainty Principle.
In this paper the author attempts the next step: based on the obtained results, Hawking's Problems for black holes are subjected to the in initial analysis.
Actually, all the required preliminary information is included in this text to gain a better understanding by the reader.

2 Generalized Uncertainty Principles in Quantum Theory and Thermodynamics

In this Section the author presents some of the results from Section 2 of the paper [20],because they are important for this work.
It is well known that in thermodynamics an inequality for the pair interior energy - inverse temperature that is completely analogous to the standard uncertainty relation in quantum mechanics [21] can be written [24] – [29]. The only (but essential) difference of this inequality from the quantum mechanical one is that the main quadratic fluctuation is defined by means of the classical partition function rather than by the quantum mechanical expectation values. In the last years a lot of papers appeared in which the usual

momentum-coordinate uncertainty relation has been modified at very high energies of order Planck energy E_p [30]–[41]. In this note we propose simple reasons for modifying the thermodynamic uncertainty relation at Planck energies. This modification results in existence of the minimal possible main quadratic fluctuation of the inverse temperature. Of course we assume that all the thermodynamic quantities used are properly defined so that they have physical sense at such high energies.

We start with usual Heisenberg Uncertainty Principle (relation) [21] for momentum - coordinate:

$$\Delta x \geq \frac{\hbar}{\Delta p}. \tag{2}$$

It was shown that at the Planck scale the high-energy term should be as follows:

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha' l_p^2 \frac{\triangle p}{\hbar} \tag{3}$$

where l_p is the Planck length $l_p^2 = G\hbar/c^3 \simeq 1,6\ 10^{-35}m$ and α' is a constant. In [30] this term is derived from the string theory, in [33] it results from the simple estimates of the Newtonian gravity and quantum mechanics, in [37] it comes from the black hole physics, other methods can also be used [36],[38],[39]. Relation (3) is quadratic in Δp

$$\alpha' l_p^2 \, (\Delta p)^2 - \hbar \, \Delta x \Delta p + \hbar^2 \leq 0 \tag{4}$$

and therefore leads to the fundamental length

$$\Delta x_{min} = 2\sqrt{\alpha'} l_p \tag{5}$$

Inequality (3) is called the Generalized Uncertainty Principle (GUP) in Quantum Theory.
Using relations (3), we can easily obtain a similar relation for the energy - time pair. Indeed, (3) gives

$$\frac{\Delta x}{c} \geq \frac{\hbar}{\Delta pc} + \alpha' l_p^2 \, \frac{\Delta p}{c\hbar}, \tag{6}$$

then

$$\Delta t \geq \frac{\hbar}{\Delta E} + \alpha' \frac{l_p^2}{c^2} \, \frac{\Delta pc}{\hbar} = \frac{\hbar}{\Delta E} + \alpha' t_p^2 \, \frac{\Delta E}{\hbar}. \tag{7}$$

where the smallness of l_p is taken into account so that the difference between ΔE and $\Delta(pc)$ can be neglected and t_p is the Planck time $t_p = l_p/c = \sqrt{G\hbar/c^5} \simeq 0,54\ 10^{-43} sec$. Inequality (7) gives, similar to (3), the lower boundary for time $\Delta t \geq 2t_p$ determining the fundamental time

$$t_{min} = 2\sqrt{\alpha'} t_p \tag{8}$$

Thus, we can write the inequalities in the standard form

$$\left\{ \begin{array}{ll} \Delta x & \geq \frac{\hbar}{\Delta p} + \alpha' \left(\frac{\Delta p}{P_{pl}} \right) \frac{\hbar}{P_{pl}} \\ & \\ \Delta t & \geq \frac{\hbar}{\Delta E} + \alpha' \left(\frac{\Delta E}{E_p} \right) \frac{\hbar}{E_p} \end{array} \right. \tag{9}$$

where $P_{pl} = E_p/c = \sqrt{\hbar c^3/G}$. Now we consider the thermodynamic uncertainty relations between the inverse temperature and interior energy of a macroscopic ensemble

$$\Delta \frac{1}{T} \geq \frac{k_B}{\Delta U}. \tag{10}$$

where k_B is the Boltzmann constant.
N.Bohr [22] and W.Heisenberg [23] first pointed out that such a kind of uncertainty principle is applicable in thermodynamics. The thermodynamic uncertainty relations (10) were proven by many authors in various ways [24] – [29]. Therefore, their validity is unquestionable. Nevertheless, relation (10) was proven in view of the standard model of the infinite-capacity heat bath encompassing the ensemble. But it is obvious from the above inequalities that at very high energies the capacity of the heat bath can no longer be assumed infinite at the Planck scale. Indeed, the total energy of the pair heat bath - ensemble may be arbitrary large but finite merely as the universe is born at a finite energy. Hence the quantity that can be interpreted as a temperature of the ensemble must have the upper limit and so does its main quadratic deviation. In other words, the quantity $\Delta(1/T)$ must be bounded from below. But in this case an additional term should be introduced into (10)

$$\Delta \frac{1}{T} \geq \frac{k_B}{\Delta U} + \eta \, \Delta U \tag{11}$$

where η is a coefficient. Dimension and symmetry reasons give

$$\eta \sim \frac{k_B}{E_p^2} \;\; or \;\; \eta = \alpha' \frac{k_B}{E_p^2} \tag{12}$$

As in the previous cases, inequality (11) leads to the fundamental (inverse) temperature.

$$T_{max} = \frac{\hbar}{2\sqrt{\alpha'} t_p k_B} = \frac{E_p}{2\sqrt{\alpha'} k_B} = \frac{T_p}{2\sqrt{\alpha'}} = \frac{\hbar}{t_{min} k_B},$$
$$\beta_{min} = \frac{1}{k_B T_{max}} = \frac{t_{min}}{\hbar} \tag{13}$$

In [42] the black hole horizon temperature is measured with the use of the Gedanken experiment. In the process the Generalized Uncertainty Relations

in Thermodynamics (11) are derived too. Expression (11) is considered in the monograph [43] within the scope of the mathematical physics methods.
Thus, we obtain a system of the generalized uncertainty relations in a symmetric form

$$\left\{\begin{array}{lll}\Delta x & \geq & \frac{\hbar}{\Delta p}+\alpha'\left(\frac{\Delta p}{P_{pl}}\right)\frac{\hbar}{P_{pl}}+...\\ \Delta t & \geq & \frac{\hbar}{\Delta E}+\alpha'\left(\frac{\Delta E}{E_p}\right)\frac{\hbar}{E_p}+...\\ \Delta\frac{1}{T} & \geq & \frac{k_B}{\Delta U}+\alpha'\left(\frac{\Delta U}{E_p}\right)\frac{k_B}{E_p}+...\end{array}\right. \tag{14}$$

or in the equivalent form

$$\left\{\begin{array}{lll}\Delta x & \geq & \frac{\hbar}{\Delta p}+\alpha' l_p^2\frac{\Delta p}{\hbar}+...\\ \Delta t & \geq & \frac{\hbar}{\Delta E}+\alpha' t_p^2\frac{\Delta E}{\hbar}+...\\ \Delta\frac{1}{T} & \geq & \frac{k_B}{\Delta U}+\alpha'\frac{1}{T_p^2}\frac{\Delta U}{k_B}+...\end{array}\right. \tag{15}$$

where the dots mean the existence of higher-order corrections as in [44]. Here T_p is the Planck temperature: $T_p = E_p/k_B$. (4)
In literature the relation (10) is referred to as the Uncertainty Principle in Thermodynamics (UPT). Let us call relation (11) the Generalized Uncertainty Principle in Thermodynamics (GUPT).
In this case, without the loss of generality and for symmetry, it is assumed that a dimensionless constant in the right-hand side of GUP (formula (3)) and in the right-hand side of GUPT (formula (11)) is the same – α'.

3 Minimal Length, Minimal Inverse Temperature, and Measurability in Quantum Theory and Thermodynamics

First, we consider in this Section the principal definitions from [2],[11] which are required to derive the key formulae in the second part of the Section and to obtain further results [12],[13].

Definition I. Let us call as **primarily measurable variation** any small variation (increment) $\widetilde{\Delta} x_\mu$ of any spatial coordinate x_μ of the arbitrary point $x_\mu, \mu = 1,...,3$ in some space-time system R if it may be realized in the form of the uncertainty (standard deviation) Δx_μ when this coordinate is measured within the scope of Heisenberg's Uncertainty Principle (HUP) [21] (formula (2) in general case):

$$\widetilde{\Delta} x_\mu = \Delta x_\mu, \Delta x_\mu \simeq \frac{\hbar}{\Delta p_\mu}, \mu = 1, 2, 3 \tag{16}$$

for some $\Delta p_\mu \neq 0$.

Similarly, for $\mu = 0$ for pair "time-energy" (t, E), let's call any small variation (increment) by **primarily measurable variation** in the value of time $\widetilde{\Delta} x_0 = \widetilde{\Delta} t_0$ if it may be realized in the form of the uncertainty (standard deviation) $\Delta x_0 = \Delta t$ and then

$$\widetilde{\Delta} t = \Delta t, \Delta t \simeq \frac{\hbar}{\Delta E} \tag{17}$$

for some $\Delta E \neq 0$. Formula (17) is nothing else as formula (7) for $\Delta E \ll E_p$ Here HUP is given for the nonrelativistic case. In the relativistic case HUP has the distinctive features [45] which, however, are of no significance for the general formulation of **Definition I.**, being associated only with particular alterations in the right-hand side of the second relation Equation (17).
It is clear that at low energies $E \ll E_P$ (momenta $P \ll P_{pl}$) **Definition I.** sets a lower bound for the **primarily measurable variation** $\widetilde{\Delta} x_\mu$ of any space-time coordinate x_μ.
At high energies E (momenta P) this is not the case if E (P) have no upper limit. But, according to the modern knowledge, E (P) are bounded by some maximal quantities E_{max}, (P_{max})

$$E \leq E_{max}, P \leq P_{max}, \tag{18}$$

where in general E_{max}, P_{max} may be on the order of Planck quantities $E_{max} \propto E_P, P_{max} \propto P_{pl}$ and also may be the trans-Planck's quantities.

In any case the quantities P_{max} and E_{max} lead to the introduction of the minimal length l_{min} and of the minimal time t_{min}.
Supposition II. There is the minimal length l_{min} as *a minimal measurement unit* for all **primarily measurable variations** having the dimension of length, whereas the minimal time $t_{min} = l_{min}/c$ as *a minimal measurement unit* for all quantities or **primarily measurable variations (increments)** having the dimension of time, where c is the speed of light.

l_{min} and t_{min} are naturally introduced as $\Delta x_\mu, \mu = 1, 2, 3$ and Δt in Equations (16) and (17) for $\Delta p_\mu = P_{max}$ and $\Delta E = E_{max}$.

For definiteness, we consider that E_{max} and P_{max} are the quantities on the order of the Planck quantities, then l_{min} and t_{min} are also on the order of Planck quantities $l_{min} \propto l_P$, $t_{min} \propto t_P$.

Definition I. and **Supposition II.** are quite natural in the sense that there are no physical principles with which they are inconsistent.

The combination of **Definition I.** and **Supposition II.** will be called the **Principle of Bounded Primarily Measurable Space-Time Variations (Increments)** or for short **Principle of Bounded Space-Time Variations (Increments)** with abbreviation (PBSTV).
As the minimal unit of measurement l_{min} is available for all the **primarily measurable variations** ΔL having the dimensions of length, the "Integrality Condition" (IC) is the case

$$\Delta L = N_{\Delta L} l_{min}, \tag{19}$$

where $N_{\Delta L} > 0$ is an integer number.
In a like manner the same "Integrality Condition" (IC) is the case for all the **primarily measurable variations** Δt having the dimensions of time. And similar to Equation (19), we get the for any time Δt:

$$\Delta t \equiv \Delta t(N_t) = N_{\Delta t} t_{min}, \tag{20}$$

where similarly $N_{\Delta t} > 0$ is an integer number too.

Definition 1 (**Primary or Elementary Measurability**)
(1) *In accordance with the PBSTV let us define the quantity having the dimensions of length or time as* **primarily (or elementarily) measurable**, *when it satisfies the relation Equation (19) (and respectively Equation (20))*.
(2)*Let us define any physical quantity* **primarily (or elementarily) measurable**, *when its value is consistent with points (1) of this Definition.*

It is convenient to use the deformation parameter α_a. This parameter has been introduced earlier in the papers [46],[20],[47]–[50] as a *deformation parameter* (in terms of paper [51]) on going from the canonical quantum mechanics to the quantum mechanics at Planck's scales (Early Universe) that is considered to be the quantum mechanics with the minimal length (QMML):

$$\alpha_a = l_{min}^2/a^2, \tag{21}$$

where a is the measuring scale. It is easily seen that the parameter α_a from Equation (21) is discrete as it is nothing else but

$$\alpha_a = l_{min}^2/a^2 = \frac{l_{min}^2}{N_a^2 l_{min}^2} = \frac{1}{N_a^2}. \tag{22}$$

At the same time, from Equation (22) it is evident that α_a is irregularly discrete.
Similarly to the minimal length l_{min} and the minimal time t_{min}, within the

scope of UPT (formula (10) and of the existent T_{max} (formula (13)), we can introduce a minimal inverse temperature as follows:

$$(\frac{1}{T})_{min} = (\Delta\frac{1}{T})_{min} = \frac{1}{T_{max}} \doteq \widetilde{\tau}. \tag{23}$$

Trying to find from formula (23) a minimal unit of measurability for the inverse temperature and introducing the "Integrality Condition" (IC) in line with the conditions (19),(20)

$$\frac{1}{T} = N_{1/T}\widetilde{\tau}, \tag{24}$$

where $N_{1/T} > 0$ is an integer number, we can introduce an analog of the **primary measurability** notion into thermodynamics.

Definition 1* (**Primary Thermodynamic Measurability**)
(1) *Let us define a quantity having the dimensions of inverse temperature as* **primarily measurable** *when it satisfies the relation (24).*
(2)Let us define any physical quantity in thermodynamics as **primarily measurable** *when its value is consistent with point (1) of this Definition.*

Definition 1* in thermodynamics is analogous to the **Primary Measurability** in a quantum theory (**Definition 1**)

Let us now return to the quantum theory.
It should be noted that, physical quantities complying with **Definition 1** won't be enough for the research of physical systems.
Indeed, such a variable as

$$\alpha_{N_a l_{min}}(N_a l_{min}) = p(N_a)\frac{l_{min}^2}{\hbar} = l_{min}/N_a, \tag{25}$$

(where $\alpha_{N_a l_{min}} = \alpha_a$ is taken from formula (22) at $a = N_a l_{min}$, and $p(N_a) = \frac{\hbar}{N_a l_{min}}$ is the corresponding **primarily measurable** momentum), is fully expressed in terms *only* **Primarily Measurable Quantities** of **Definition 1** and that's why it may appear at any stage of calculations, but apparently doesn't comply with **Definition 1**. That's why it's necessary to introduce the following definition generalizing **Definition 1**:

Definition 2. Generalized Measurability
We shall call any physical quantity as **generalized-measurable** or for simplicity **measurable** if any of its values may be obtained in terms of **Primarily Measurable Quantities** of **Definition 1**.

In what follows, for simplicity, we will use the term **Measurability** instead of **Generalized Measurability**.
It is evident that any **primarily measurable quantity (PMQ)** is **measurable**. Generally speaking, the contrary is not correct, as indicated by formula (25).
The **generalized-measurable** quantities are appeared from the **Generalized Uncertainty Principle (GUP)** (formula (3)) that naturally leads to the minimal length l_{min} [30]–[41]:

$$\Delta x_{min} = 2\sqrt{\alpha'} l_p \doteq l_{min}, \tag{26}$$

For convenience, we denote the minimal length $l_{min} \neq 0$ by ℓ and $t_{min} \neq 0$ by $\tau = \ell/c$.
Solving inequality (3), in the case of equality we obtain the apparent formula

$$\Delta p_{\pm} = \frac{(\Delta x \pm \sqrt{(\Delta x)^2 - 4\alpha' l_p^2})\hbar}{2\alpha' l_p^2}. \tag{27}$$

Next, into this formula we substitute the right-hand part of formula (19) for $L = x$. Considering (26), we can derive the following:

$$\Delta p_{\pm} = \frac{(N_{\Delta x} \pm \sqrt{(N_{\Delta x})^2 - 1})\hbar \ell}{\frac{1}{2}\ell^2} = \\ = \frac{2(N_{\Delta x} \pm \sqrt{(N_{\Delta x})^2 - 1})\hbar}{\ell}. \tag{28}$$

But it is evident that at low energies $E \ll E_p; N_{\Delta x} \gg 1$ the plus sign in the nominator (28) leads to the contradiction as it results in very high (much greater than the Plancks) values of Δp. Because of this, it is necessary to select the minus sign in the numerator (28). Then, multiplying the left and right sides of (28) by the same number $N_{\Delta x} + \sqrt{N_{\Delta x}^2 - 1}$, we get

$$\Delta p = \frac{2\hbar}{(N_{\Delta x} + \sqrt{N_{\Delta x}^2 - 1})\ell}. \tag{29}$$

Δp from formula (29) is the **generalized-measurable** quantity in the sense of **Definition 2.** However, it is clear that at low energies $E \ll E_p$, i.e. for $N_{\Delta x} \gg 1$, we have $\sqrt{N_{\Delta x}^2 - 1} \approx N_{\Delta x}$. Moreover, we have

$$\lim_{N_{\Delta x} \to \infty} \sqrt{N_{\Delta x}^2 - 1} = N_{\Delta x}. \tag{30}$$

Therefore, in this case (29) may be written as follows:

$$\Delta p \doteq \Delta p(N_{\Delta x}, HUP) = \frac{\hbar}{1/2(N_{\Delta x}+\sqrt{N_{\Delta x}^2-1})\ell} \approx \approx \frac{\hbar}{N_{\Delta x}\ell} = \frac{\hbar}{\Delta x}; \quad N_{\Delta x} \gg 1, \tag{31}$$

in complete conformity with HUP. Besides, $\Delta p \doteq \Delta p(N_{\Delta x}, HUP)$, to a high accuracy, is a **primarily measurable** quantity in the sense of **Definition 1**. And vice versa it is obvious that at high energies $E \approx E_p$, i.e. for $N_{\Delta x} \approx 1$, there is no way to transform formula (29) and we can write

$$\Delta p \doteq \Delta p(N_{\Delta x}, GUP) = = \frac{\hbar}{1/2(N_{\Delta x}+\sqrt{N_{\Delta x}^2-1})\ell}; N_{\Delta x} \approx 1. \tag{32}$$

At the same time, $\Delta p \doteq \Delta p(N_{\Delta x}, GUP)$ is a **Generalized Measurable** quantity in the sense of **Definition 2**.
Thus, we have

$$GUP \rightarrow HUP \tag{33}$$

for

$$(N_{\Delta x} \approx 1) \rightarrow (N_{\Delta x} \gg 1). \tag{34}$$

Also, we have

$$\Delta p(N_{\Delta x}, GUP) \rightarrow \Delta p(N_{\Delta x}, HUP), \tag{35}$$

where $\Delta p(N_{\Delta x}, GUP)$ is taken from formula (32), whereas $\Delta p(N_{\Delta x}, HUP)$ from formula (31).

Comment 2.*
From the above formulae it follows that, within GUP, the **primarily measurable** *variations (quantities) are derived to a high accuracy from the* **generalized-measurable** *variations (quantities) only in the low-energy limit $E \ll E_P$*

Next, within the scope of GUP, we can correct a value of the parameter α_a from formula (22) substituting a for Δx in the expression $1/2(N_{\Delta x}+\sqrt{N_{\Delta x}^2-1})\ell$. Then at low energies $E \ll E_p$ we have the **primarily measurable** quantity $\alpha_a(HUP)$

$$\alpha_a \doteq \alpha_a(HUP) = \frac{1}{[1/2(N_a+\sqrt{N_a^2-1})]^2} \approx \approx \frac{1}{N_a^2}; N_a \gg 1, \tag{36}$$

that corresponds, to a high accuracy, to the value from formula (22).
Accordingly, at high energies we have $E \approx E_p$

$$\alpha_a \doteq \alpha_a(GUP) = \frac{1}{[1/2(N_a + \sqrt{N_a^2 - 1})]^2}; N_a \approx 1. \tag{37}$$

When going from high energies $E \approx E_p$ to low energies $E \ll E_p$, we can write

$$\alpha_a(GUP) \stackrel{(N_a \approx 1) \rightarrow (N_a \gg 1)}{\longrightarrow} \alpha_a(HUP) \tag{38}$$

in complete conformity to *Comment 2**.

It is readily seen that an analog of **Definition 2. Generalized Measurability** exists in Thermodynamics too.
Indeed, returning back to the thermodynamic relation (11) in the case of equality, we have

$$\Delta \frac{1}{T} = \frac{k_B}{\Delta U} + \eta \, \Delta U, \tag{39}$$

that is equivalent to the quadratic equation

$$\eta \, (\Delta U)^2 - \Delta \frac{1}{T} \Delta U + k_B = 0. \tag{40}$$

The discriminant of this equation, with due regard for formula (12), is equal to

$$D = (\Delta \frac{1}{T})^2 - 4\eta k_B = (\Delta \frac{1}{T})^2 - 4\alpha' \frac{k_B^2}{E_p^2} \geq 0, \tag{41}$$

leading directly to $(\Delta \frac{1}{T})_{min}$ from formula (23)

$$(\Delta \frac{1}{T})_{min} = 2\sqrt{\alpha'} \frac{k_B}{E_p} = \frac{1}{T_{max}} = \widetilde{\tau} \tag{42}$$

or, due to the fact that k_B is constant, we have

$$(\Delta \frac{1}{k_B T})_{min} = \frac{2\sqrt{\alpha'}}{E_p}. \tag{43}$$

Now we consider the quadratic equation (40) in terms of **primarily measurable quantities** in the sense of **Definition 1***. In accordance with this definition and with formula (24) $\Delta(1/T)$, we can write

$$\Delta \frac{1}{T} = N_{\Delta(1/T)} \widetilde{\tau}, \tag{44}$$

where $N_{\Delta(1/T)}>0$ is an integer number.
The quadratic equation (40) takes the following form:

$$\eta\,(\Delta U)^2 - N_{\Delta(1/T)}\widetilde{\tau}\Delta U + k_B = 0. \tag{45}$$

Then, due to formula (43), we can find the "**measurable**" roots of equation (45) for ΔU as follows:

$$\begin{aligned}(\Delta U)_{meas,\pm} &= \frac{[N_{\Delta(1/T)} \pm \sqrt{N^2_{\Delta(1/T)}-1}]\widetilde{\tau}}{2\eta} = \\ &= \frac{2k_B[N_{\Delta(1/T)} \pm \sqrt{N^2_{\Delta(1/T)}-1}]\widetilde{\tau}}{\widetilde{\tau}^2} = \\ &\quad \frac{2k_B[N_{\Delta(1/T)} \pm \sqrt{N^2_{\Delta(1/T)}-1}]}{\widetilde{\tau}}.\end{aligned} \tag{46}$$

The last line in (46) is associated with the obvious relation $2\eta = \frac{\widetilde{\tau}^2}{2k_B}$.
In this way we derive a complete analog of the corresponding relation (28) from a quantum theory by replacement

$$\Delta p_{\pm} \Rightarrow \Delta U_{meas,\pm}; N_{\Delta x} \Rightarrow N_{\Delta(1/T)}; \hbar \Rightarrow k_B. \tag{47}$$

As, for **low temperatures and energies**, $T \ll T_{max} \propto T_p$, we have $1/T \gg 1/T_p$ and hence $\Delta(1/T) \gg 1/T_p$ and $N_{\Delta(1/T)} \gg 1$.
Next, in analogy with formula (28), in formula (46) we can have only the minus-sign root, otherwise, at sufficiently high $N_{\Delta(1/T)} \gg 1$ for $(\Delta U)_{meas,+}$ we can get $(\Delta U)_{meas,+} \gg E_p$. But this is impossible for low temperatures (energies). On the contrary, the minus sign in (46) is consistent with high and low energies.
So, taking the root value in (46) corresponding to this sign and multiplying the nominator and denominator in (46) by $N_{\Delta(1/T)}+\sqrt{N^2_{\Delta(1/T)}-1}$, we obtain

$$(\Delta U)_{meas} = \frac{2k_B}{(N_{\Delta(1/T)} + \sqrt{N^2_{\Delta(1/T)}-1})\widetilde{\tau}} \tag{48}$$

to have a complete analog of the corresponding relation from (29) in a quantum theory by substitution according to formula (47).
Then it is clear that, in analogy with formula (31), for low energies and temperatures $N_{\Delta(1/T)} \gg 1$ (48) may be rewritten as

$$\begin{aligned}(\Delta U)_{meas} &\doteq (\Delta U)_{meas}(T \ll T_{max}) = \\ &= \frac{2k_B}{(N_{\Delta(1/T)} + \sqrt{N^2_{\Delta(1/T)}-1})\widetilde{\tau}} \approx \\ &\approx \frac{k_B}{N_{\Delta(1/T)}\widetilde{\tau}}, N_{\Delta(1/T)} \gg 1,\end{aligned} \tag{49}$$

i.e. the Uncertainty Principle in Thermodynamics (UPT, formula (10)) is involved. In this case, due to the last formula, ΔU_{meas} represents a **primarily measurable** thermodynamic quantity in the sense of **Definition 1*** to a high accuracy.
Of course, at high energies the last term in the formula (49) is lacking and, for $T \approx T_{max}; N_{\Delta(1/T)} \approx 1$, we have:

$$(\Delta U)_{meas} \doteq (\Delta U)_{meas}(T \approx T_{max}) = \\ = \frac{k_B}{1/2(N_{\Delta(1/T)} + \sqrt{N^2_{\Delta(1/T)} - 1})\widetilde{\tau}}, \\ N_{\Delta(1/T)} \approx 1. \tag{50}$$

From (50) it follows that at high temperatures (energies) $(\Delta U)_{meas}$ could hardly be a **primarily measurable** thermodynamic quantity. Because of this, it is expedient to use a counterpart of **Definition 2**.

Definition 2*. Generalized Measurability in Thermodynamics
Any physical quantity in thermodynamics may be referred to as **generalized-measurable** or, for simplicity, **measurable** if any of its values may be obtained in terms of the **Primary Thermodynamic Measurability** of **Definition 1***.

In this way $(\Delta U)_{meas}$ from the formula (50) is a **measurable** quantity.
Based on the preceding formulae, it is clear that we have the limiting transition

$$(\Delta U)_{meas}(T \approx T_{max}) \stackrel{(N_{\Delta(1/T)}\approx 1)\rightarrow(N_{\Delta(1/T)}\gg 1)}{\longrightarrow} \\ \stackrel{(N_{\Delta(1/T)}\approx 1)\rightarrow(N_{\Delta(1/T)}\gg 1)}{\longrightarrow} (\Delta U)_{meas}(T \ll T_{max} \propto T_p), \tag{51}$$

that is analogous to the corresponding formula (38) in a quantum theory.
Therefore, in this case the analog of *Comment 2*.* is valid:

Comment 2 Thermodynamics*
From the above formulae it follows that, within GUPT (11), the **primarily measurable** *variations (quantities) are derived, to a high accuracy, from the* **generalized-measurable** *variations (quantities) only in the low-temperature limit* $T \ll T_{max} \propto T_p$.
To conclude this Section, it seems logical to make several important **remarks**.

R3.1 It is obvious that all the calculations associated with **measurability** of inverse temperature $\frac{1}{T}$ are valid for $\beta = \frac{1}{k_B T}$ as well. Specifically, introducing $\beta_{min} \doteq \widetilde{\beta} = \widetilde{\tau}/k_B$, we can rewrite all the corresponding formulae in the "**measurable**" variant replacing $1/T$ ($\Delta(1/T)$) by β,$\widetilde{\tau}$ by $\widetilde{\beta}$ and retaining

$N_{1/T}$ $(N_{\Delta(1/T)})$.

R3.2. Naturally, the problem of compatibility between the **measurability** definitions in quantum theory and in thermodynamics arises: is there any contradiction between **Definition 1** and **Definitions 1***?

On the basis of formulae (13) and (23),(24,(44) we can state:
measurability *in quantum theory and* ***thermodynamic measurability*** *are completely compatible and consistent as the minimal unit of inverse temperature* $\widetilde{\tau}$ *is nothing else but the minimal time* $t_{min} = \tau$ *up to a constant factor. And hence* $N_{1/T}, (N_{\Delta(1/T)})$ *is nothing else but* $N_t, (N_{\Delta t})$ *in (20). Then it is clear that* $N_t = N_{a=tc}$.

R3.3 Finally, from the above formulae (49), (50) it follows that the **measurable** temperature T is varying as follows:

$$T = \frac{T_{max}}{N_{1/T}}, T \ll T_{max} \propto T_p, N_{1/T} \gg 1;$$
$$T = \frac{T_{max}}{1/2(N_{1/T} + \sqrt{N_{1/T}^2 - 1})}, T \approx T_{max} \propto T_p,$$
$$N_{1/T} \approx 1. \qquad (52)$$

In such a way **measurable** temperature is a **discrete quantity** but at low energies it is almost constantly varying so, the theoretical calculations are very similar to those of the well-known continuous theory. Actually, discreteness manifests itself in the case of high energies only.

R3.4 What is the main difference between **Primarily Measurable Quantities (PMQ)** and **Generalized Measurable Quantities (GMQ)** in Quantum Theory and Thermodynamics? **PMQ** defines variables which may be obtained as a result of an immediate experiment. **GMQ** defines the variables which may be *calculated* based on **PMQ**, i.e. based on the data obtained in previous clause.

R3.5 It should be noted that the formula derived in the "measurable form" in the present work for $\Delta p(N_{\Delta x}, GUP)$ is somewhat different from the analogous formula derived earlier, for example, in [2],[11]. It is understandable because in [2],[11] at all the energy scales we have used the same ("naive") value of α_a specified by formula (21). In this paper we use the GUP-corrected value $\alpha_a = \alpha_a(GUP)$ (formula (37)) that at low energies $E \ll E_p$, to a high accuracy, turns to α_a formula (21). From here it follows that at low energies $E \ll E_p$ the value $\Delta p(N_{\Delta x})$ obtained in the present work is not different from the corresponding values in [2],[11]. But even with the use of very simple computations we can demonstrate that at high energies $E \approx E_p$ the difference between these two quantities is minor.

4 Measurability in Gravity, Black Holes and Hawking's Problems

Now let us show the applicability of the results from previous Section to a quantum theory of black holes.
In this Section consideration is given to gravitational dynamics at low $E \ll E_p$ and at high $E \approx E_p$ energies in the case of the Schwarzschild black hole in terms of **measurable quantities** from the Section 3. The Items **4.1.** and **4.2.** of this Section are based on the results from [12].
First we consider an even more general case of the space with static spherically-symmetric horizon in space-time.
It should be noted that such spaces and even considerably more general cases have been thoroughly studied from the viewpoint of gravitational thermodynamics in remarkable works of professor T.Padmanbhan [52]–[63] (the list of references may be much longer).
First, the author has studied the above-mentioned case in [64] and from the suggested viewpoint in [1]. But, proceeding from Section 2 of the present paper, it is possible to extend the results from [1].
In what follows we use the symbols from [63] which have been also used in [1].
The case of a static spherically-symmetric horizon in space-time is considered, the horizon being described by the metric

$$ds^2 = -f(r)c^2dt^2 + f^{-1}(r)dr^2 + r^2d\Omega^2. \tag{53}$$

The horizon location will be given by a simple zero of the function $f(r)$, at the radius $r = a$.
Then at the horizon $r = a$ Einstein's field equations ([63], eq.(117))

$$\frac{c^4}{G}\left[\frac{1}{2}f'(a)a - \frac{1}{2}\right] = 4\pi Pa^2 \tag{54}$$

where $P = T^r_r$ is the trace of the momentum-energy tensor and radial pressure. Therewith, the condition $f(a) = 0$ and $f'(a) \neq 0$ must be fulfilled.
On the other hand it is known that for horizon spaces one can introduce the temperature that can be identified with an analytic continuation to imaginary time. In the case under consideration ([63], eq.(116))

$$k_BT = \frac{\hbar cf'(a)}{4\pi}. \tag{55}$$

In [63] it is shown that in the initial (continuous) theory the Einstein Equation for horizon spaces in the differential form may be written as a thermodynamic identity (the first principle of thermodynamics) ([63], formula (119)):

$$\underbrace{\frac{\hbar c f'(a)}{4\pi}}_{k_B T}\underbrace{\frac{c^3}{G\hbar}d\left(\frac{1}{4}4\pi a^2\right)}_{dS}\underbrace{-\frac{1}{2}\frac{c^4 da}{G}}_{-dE}=\underbrace{Pd\left(\frac{4\pi}{3}a^3\right)}_{P\,dV}. \tag{56}$$

where, as noted above, T – temperature of the horizon surface, S –corresponding entropy, E– internal energy, V – space volume.

It is impossible to use (56) in the formalism under consideration because, as follows from the results given in the previous section and in [1], da, dS, dE, dV are not **measurable quantities**.

First, we assume that a value of the radius r at the point a is a **primarily measurable quantity** in the sense of **Definition 1**, i.e. $a = a_{meas} = N_a \ell$, where $N_a > 0$ - integer, and the temperature T from the left-hand side of (55) is the **primarily measurable** temperature $T = T_{meas}$ in the sense of **Definition 1***.

Then, in terms of **measurable** quantities, first we can rewrite (54) as

$$\frac{c^4}{G}\left[\frac{2\pi k_B T}{\hbar c}a_{meas}-\frac{1}{2}\right]=4\pi P a_{meas}^2. \tag{57}$$

We express $a = a_{meas}$ in terms of the deformation parameter α_a (formula (21)) as

$$a=\ell\alpha_a^{-1/2}; \tag{58}$$

the temperature T is expressed in terms of $T_{max} \propto T_p$ from (52).

Then, considering that $T_p = E_p/k_B$, equation (57) may be given as

$$\frac{c^4}{G}[\frac{\pi E_p}{\sqrt{\alpha'}N_{1/T}\hbar c}\ell\alpha_a^{1/2}-\frac{1}{2}\alpha_a]=4\pi P\ell^2. \tag{59}$$

Because $\ell = 2\sqrt{\alpha'}l_p$ and $l_p = \frac{\hbar c}{E_p}$, we have

$$\frac{c^4}{G}[\frac{2\pi E_p}{N_{1/T}\hbar c}l_p\alpha_a^{1/2}-\frac{1}{2}\alpha_a]=\frac{c^4}{G}[\frac{2\pi}{N_{1/T}}\alpha_a^{1/2}-\frac{1}{2}\alpha_a]=4\pi P\ell^2. \tag{60}$$

Note that in its initial form [63] the equation (54) has been considered in a continuous theory, i.e. at low energies $E \ll E_p$. Consequently, in the present formalism it is implicitly meant that the "measurable counterpart" of equation (54) – (57) (or the same (59),(60)) is also initially considered at low energies, in particular, $N_a \gg 1, N_{1/T} \gg 1$.

Let us consider (60) at low and high energies in terms of **measurable** quantities as applied to Schwarzschild black holes.

4.1. *Measurable case for low energies: $E \ll E_p$.* Due to formula (31), $a = a_{meas} = N_a \ell$, where the integer number is $N_a \gg 1$ or similarly $N_{1/T} \gg 1$. In this case GUP, to a high accuracy, is extended to HUP (formula (33),(34)). As this takes place, $\alpha_a = \alpha_a(HUP)$ is a **primarily measurable** quantities (**Definition 1,Definition 1***), $\alpha_a \approx N_a^{-2}, N_{1/T}^{-1}$, though taking a discrete series of values but varying smoothly, in fact *continuously.*
In this case, the equation (60) may be rewritten as

$$\frac{c^4}{G}[\frac{2\pi}{N_{1/T}}\alpha_a^{1/2}(HUP) - \frac{1}{2}\alpha_a(HUP)] = 4\pi P\ell^2. \tag{61}$$

So, at low energies the equation (61) written for the discretely-varying $\alpha_a(HUP), N_{1/T}^{-1}$ may be considered in a continuous theory.
As a result, in the case under study we can use the basic formulae from a continuous theory considering them valid to a high accuracy.
As a result, in the case under study we can use the basic formulae from a continuous theory considering them valid to a high accuracy.
In particular, in the notation used for *Schwarzschild's black hole* [65], we have

$$r_s = N_a\ell = \frac{2GM}{c^2}; M = \frac{N_a \ell c^2}{2G}. \tag{62}$$

As its temperature is given by the formula

$$T_H = \frac{\hbar c^3}{8\pi G M k_B}, \tag{63}$$

at once we get

$$T_H = \frac{\hbar c}{4\pi k_B N_a \ell} = \frac{\hbar c \alpha_a^{1/2}}{4\pi k_B \ell}. \tag{64}$$

Comparing this expression to the expression with high $N_{1/T}$ ($N_{1/T} \gg 1$) for temperature from the equation (52) that is involved in (57), we can find that at low energies $E \ll E_p$, due to comment **R2.2.** from Subsection 2.3, the number $N_{1/T}$ is actually coincident with the number N_a:

$$N_{1/T} = N_a = \alpha_a^{-1/2}(HUP). \tag{65}$$

The substitution of the last expression from formula (64) into the quadratic equation (61) for $\alpha_a^{1/2}(HUP)$ makes it a linear equation for $\alpha_a(HUP)$ with a single parameter P:

$$(4\pi - 1)\frac{c^4}{G}\alpha_a(HUP) = 8\pi P\ell^2, \tag{66}$$

4.2. *Measurable case for high energies:* $E \approx E_p$. Then, due to (32), a is the **generalized measurable** quantity $a = a_{meas} = 1/2(N_a + \sqrt{N_a^2 - 1})\ell$, with the integer $N_a \approx 1$.
The quantity

$$\Delta a_{meas}(q) = 1/2(N_a + \sqrt{N_a^2 - 1})\ell - N_a\ell = 1/2(\sqrt{N_a^2 - 1} - N_a)\ell \qquad (67)$$

may be considered as a **quantum correction** for the **measurable** radius $r = a_{meas}$, that is infinitesimal at low energies $E \ll E_p$ and not infinitesimal for high energies $E \approx E_p$.
In this case there is no possibility to replace GUP by HUP. In equation (59) $\alpha_a = \alpha_a(GUP)$ is a **generalized measurable** quantity (**Definition 2**).
As noted in formula (52) of Comment **R3.3**, in this case the number $N_{1/T}$ in equation (61) is replaced by $1/2(N_{1/T} + \sqrt{N_{1/T}^2 - 1})$, i.e. the equation is of the form

$$\frac{c^4}{G}\left[\frac{2\pi}{1/2(N_{1/T} + \sqrt{N_{1/T}^2 - 1})}\alpha_a^{1/2}(GUP) - \frac{1}{2}\alpha_a(GUP)\right] = 4\pi P\ell^2. \qquad (68)$$

In so doing the theory becomes really discrete, and the solutions of (68) take a discrete series of values for every N_a or $(\alpha_a(GUP))$ sufficiently close to 1.
In this formalism for a "quantum" Schwarzschild black hole (i.e. at high energies $E \approx E_p$) formula (64) is replaced by

$$T_H(Q) = \frac{\hbar c}{4\pi k_B 1/2(N_a + \sqrt{N_a^2 - 1})\ell} = \frac{\hbar c \alpha_a^{1/2}(GUP)}{4\pi k_B \ell}, \qquad (69)$$

where $N_a = N_{1/T}$.
Then the gravitational equation on the event horizon of "quantum" Schwarzschild black hole are written as

$$(4\pi - 1)\frac{c^4}{G}\alpha_a(GUP) = 8\pi P\ell^2, \qquad (70)$$

Remark 4.1. It is readily seen that a minimal value of $N_a = 1$ is *unattainable* because in formula (32) we can obtain a value of the length l that is below the minimum $l < \ell$ for the momenta and energies above the maximal ones, and that is impossible. Thus, we always have $N_a \geq 2$. This fact was indicated in [46],[20], however, based on the other approach.

Remark 4.2. So, all the members of the gravitational equation (66) (and (70), respectively), apart from P, are expressed in terms of the measurable parameter $\alpha_a(HUP)$ ($\alpha_a(GUP)$, respectively). From this it follows that P should be also expressed in terms of the measurable parameter α_a, i.e. $P = P(\alpha_a)$: $E \ll E_p$, $P = P[\alpha_a(HUP)]$ at low energies and $E \approx E_p, P = P[\alpha_a(GUP)]$ at high energies. Then, due to the above formulae, we can have for a "quantum" Schwarzschild black hole the horizon gravitational equation in terms of **measurable** quantities

$$(4\pi - 1)\frac{c^4}{G}\alpha_a(GUP) = 8\pi P[\alpha_a(GUP)]\ell^2, \tag{71}$$

where $\alpha_a(GUP)$ takes a discrete series of the values $\alpha_a(GUP) = (1/2(N_a + \sqrt{N_a^2 - 1}))^{-2}$; $N_a \geq 2$ is a small integer.

4.3. *Measurability and general remarks to Hawking's problem.*

In the mid 70-ies of the last century Hawking in his work [66] demonstrated that in the semiclassical approximation black holes emit particles to be evaporated as a result. But it is known that the semiclassical approximation works well at low energies $E \ll E_p$ only.
Therefore, it seems logical to support the idea suggested in the Introduction to the recent overview presented by seven authors [67]: "Since for (asymptotically flat Schwarzschild) black holes the temperatures increase as their masses decrease, soon after Hawkings discovery, it became clear that a complete description of the evaporation process would ultimately require a consistent quantum theory of gravity. This is necessary as the semiclassical formulation of the emission process breaks down during the final stages of the evaporation as characterized by Planckian values of the temperature and spacetime curvature".
Naturally, it is important to study the transition from low to high energies and vice versa in the indicated case. However, just this problem was treated in terms of **measurable** quantities in points **4.1.**, **4.2.**.
The results of points **4.1.** and especially **4.2.** associated with the work [66] and with the above citation from [67] in terms of **measurable** quantities may be interpreted in the following way.

4.3.1. Total evaporation of the Schwarzschild black hole is impossible because in the process of evaporation it always has the Planck remnants with the **generalized measurable** radius $r = 1/2(N_r + \sqrt{N_r^2 - 1})\ell$, where, according to **Remark 4.1.**, $N_r \geq 2$ is an integer number.
This inference is completely compatible with the principal result obtained by [68]:"...that the generalized uncertainty principle may prevent their (black

holes) total evaporation in exactly the same way that the uncertainty principle prevents the hydrogen atom from total collapse..".
In the suggested formalism, a separate problem is to find the limiting value of the above-mentioned equality $\mathbf{N}_{bh} = MaxN_r$, for which a black hole becomes resistant to evaporation. Then, if
$\mathbf{N}_{bh} > 2$, black micro-holes, whose radii are accordingly equal to $N_r = 2, ..., N_r = \mathbf{N}_{bh} - 1$, can hardly represent the remnants resultant from evaporation of large black holes. Thus, the only variant for the realization of such black micro-holes is their origination in the Early Universe.

4.3.2. Similarly, from formula (62) we can calculate for the **"measurable"** mass M of a *large Schwarzschild black hole* its **"measurable"** quantity $M(Q)$ in the "quantum" case [13]:

$$M(Q) = \frac{1/2(N_{r_s} + \sqrt{N_{r_s}^2 - 1})\ell c^2}{2G} = \frac{\alpha_{r_s}^{-1/2}(GUP)\ell c^2}{2G}. \tag{72}$$

In the low-energy case ($E \ll E_p; N_{r_s} \gg 1$) the well-known semiclassical Bekenstein-Hawking formula [65]

$$S_{Schw} = \frac{4\pi r_s^2}{4l_p^2} = \frac{\pi r_s^2}{l_p^2} \tag{73}$$

for entropy of a *large Schwarzschild black hole*, considering formula (5), may be written in terms of **measurable** quantities as follows [13]:

$$S_{Schw,meas} = \frac{4\pi(N_{r_s}\ell)^2}{\ell^2/\alpha'} = 4\pi\alpha' N_{r_s}^2. \tag{74}$$

Thereafter, in the case of high energies ($E \approx E_p; N_{r_s} \approx 1$) the semiclassical **"measurable"** Bekenstein-Hawking entropy $S_{Schw,meas}$ (formula (74)) for a "quantum" Schwarzschild black hole is replaced by the quantum entropy $S_{Schw,meas-q}$ and we have [13]

$$S_{Schw,meas-q} = \frac{4\pi(\frac{1}{2}(N_{r_s} + \sqrt{N_{r_s}^2 - 1})\ell)^2}{\ell^2/\alpha'} = \\ = \pi\alpha'(N_{r_s} + \sqrt{N_{r_s}^2 - 1})^2. \tag{75}$$

Then we can derive a **"measurable quantum correction"** for for all quantities: the temperature T, the mass M and entropy S of a Schwarzschild black hole [13]:

$$\begin{cases} \Delta_Q T \doteq T_H(Q) - T_H = \frac{\hbar c}{4\pi k_B \ell}(\alpha_{r_s}^{1/2}(GUP) - \alpha_{r_s}^{1/2}(HUP)), \\ \Delta_Q M \doteq M(Q) - M = \frac{\ell c^2}{2G}(\alpha_{r_s}^{-1/2}(GUP) - \alpha_{r_s}^{-1/2}(HUP)) = \\ = \frac{\ell c^2}{4G}(\sqrt{N_{r_s}^2 - 1} - N_{r_s}) = \frac{c^2}{4G}(\sqrt{r_s^2 - \ell^2} - r_s), \\ \Delta_Q S \doteq S_{Schw,meas-q} - S_{Schw,meas} = \\ = \pi\alpha'(N_{r_s} + \sqrt{N_{r_s}^2 - 1})^2 - 4\pi\alpha' N_{r_s}^2 = \\ = \pi\alpha'(2N_{r_s}\sqrt{N_{r_s}^2 - 1} - 2N_{r_s}^2 - 1). \end{cases} \tag{76}$$

As indicated by the last formulae the **measurable quantum corrections** *are nothing else but the difference between the* **generalized measurable** *quantities and the* **primarily measurable** *quantities of Section 3.*

The last remark should be taken into consideration during the reformulation of the renowned Hawking's problem of Information Paradox [69]–[71] in terms of **measurable** quantities.

Obviously, by itself the notion *information* in the suggested discrete theory should be adequately modified as it is quite clear that at low energies this theory, though being very close to the initial continuous theory, is not absolutely identical and must have significant differences from the continuous theory.

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