The Solution of the Truncated Harmonic Oscillator Using Lie Groups

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Abstract

In this paper we solve the problem of the harmonic truncated oscillator by using the symmetry Lie group method. We get good agreement with previous analytical results.

Keywords: Truncated harmonic oscillator, Lie groups

1 Introduction

The symmetry concept is one of the most stimulating and profound ideas in Mathematics and Physics. This concept has been used from the solid state crystallography groups, till the relativistic invariance of quantum field theories. Among the mathematical tools that used symmetry, as the key point, we can find Lie group theory. Basically, the method looks for solutions of differential equations treating them as on a hypersurface. Basically, it is used
the infinitesimal coefficients and symmetry generators of the equation. Of all possible particular solutions, the symmetry criterion allows to find only those, who are infinitesimal invariant under symmetry transformations, [1]-[3].

2 Truncated Harmonic Oscillator

We start with the Fokker-Planck equation for an harmonic truncated oscillator,

\[ \frac{\partial P(x,t)}{\partial t} = -\frac{\partial(f(x)P(x,t))}{\partial x} + D\frac{\partial^2 P(x,t)}{\partial x^2} \]  \(1\)

The oscillator is truncated under the next force condition:

\[ f(x) = \begin{cases} -kx & \rightarrow -d \leq x \leq d \\ 0 & \rightarrow -d > x > d \end{cases} \]  \(2\)

For each region, eq. (1), the Fokker-Planck is:

\[ \frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 (P(x,t))}{\partial x^2} \rightarrow -d > x > d \]  \(3\)

\[ \frac{\partial P(x,t)}{\partial t} = -\frac{\partial(f(x)P(x,t))}{\partial x} + D\frac{\partial^2 P(x,t)}{\partial x^2} \rightarrow -d \leq x \leq d \]  \(4\)

Where \(D\) is known as the diffusion constant [4], [6], \(k\) is the Hooke constant, and \(d\) is the position where the oscillator jumps.

3 The solution to \(P_t + DP_{xx} = 0\), for \(-d > x > d\)

The space for this equation is six dimensional, each for every variable and derivative. We build up the infinitesimal coefficients according to each dependent or independent variable in the differential equation: \(\xi(x,t,P)\) related to \(x\), \(\eta(x,t,P)\) to \(t\) y \(\phi(x,t,P)\) associated to \(P\). Then, we construct the general expression for the symmetry generator, [1], [7]:

\[ V = \xi(x,t,P)\frac{\partial}{\partial x} + \eta(x,t,P)\frac{\partial}{\partial t} + \phi((x,t,P))\frac{\partial}{\partial P} \]  \(5\)

The second prolongation of the symmetry generator, given in eq. (5), is:
Solution of the truncated harmonic oscillator

\[ P_t^2V = \xi(x, t, P) \frac{\partial}{\partial x} + \eta(x, t, P) \frac{\partial}{\partial t} + \phi((x, t, P)) \frac{\partial}{\partial P} + \phi_x \frac{\partial}{\partial P_x} \]

\[ + \phi_t \frac{\partial}{\partial P_t} + \phi_{xx} \frac{\partial}{\partial P_{xx}} + \phi_{tt} \frac{\partial}{\partial P_{tt}} + \phi_{xt} \frac{\partial}{\partial P_{xt}} \]  

(6)

The quantities \( \phi^{\alpha, J} \) are given by

\[ \phi^{\alpha, J} = D_J Q^\alpha + \sum_{i=1}^{p} \xi_i(x, u) u_{J,i}^\alpha \]

(7)

\[ Q^\alpha(x, u^1) = \phi^\alpha(x, u) - \sum_{i=1}^{p} \xi_i(x, u) \frac{\partial u^\alpha}{\partial x_i} \]

(8)

Where \( \alpha \) represents each dependent variable, \( J \) the independent variables and as a subindex represents the derivative respect those variables. \( Q \) is known as the characteristic equation of the symmetry generator and \( D_J \) is the derivative respect to \( J \). Now, we apply the symmetry condition who establishes that the prolongation of the symmetry generator applied to the differential equation must be zero, [1], [7]:

\[ (\xi(x, t, P) \frac{\partial}{\partial x} + \eta(x, t, P) \frac{\partial}{\partial t} + \phi((x, t, P)) \frac{\partial}{\partial P} + \phi_x \frac{\partial}{\partial P_x} \]

\[ + \phi_t \frac{\partial}{\partial P_t} + \phi_{xx} \frac{\partial}{\partial P_{xx}} + \phi_{tt} \frac{\partial}{\partial P_{tt}} + \phi_{xt} \frac{\partial}{\partial P_{xt}})(D \frac{\partial^2(P(x, t))}{\partial x^2} - \frac{\partial P(x, t)}{\partial t}) = 0 \]

(9)

The result is:

\[ D\phi^{xx} - \phi^t = 0 \]

(10)

Equations (7) and (8) let us to know the expressions \( \phi^{xx} \) and \( \phi^t \), which are:

\[ \phi^t = D_t Q + \xi P_{tx} + \eta P_{tt} \]

(11)

\[ \phi^{xx} = D_{xx} Q + \xi P_{xxx} + \eta P_{xxt} \]

(12)

\[ Q = \phi - \xi P_x - \eta P_t \]

(13)
Replacing them in eq. (9), we get:

$$-2P_t\xi_x + DP^2_x(\phi_{PP} - 2\xi_x P) + DP_x(2\phi_{xP} - \xi_x) - 3P_x P_t \xi_P \tag{14}$$
$$-DP^3_x \eta_{PP} - 2DP_x P_x \eta_P - 2DP_x \eta_x - P_t^2 \eta_P - DP^2_x P_t \eta_{PP}$$
$$-2DP_x P_t \eta_{xP} - DP_t \eta_{xx} - P_t \phi_t + P_x P_t \xi_P + P_x \xi_{xt} + P^2_t \eta_P + \eta_t = 0$$

Organizing this equation as a polynomial of $P_t$, the coefficients become a set of coupled system of differential equations for the infinitesimal coefficients. Then, we get:

$$\eta = -2Da_1 t^2 + 2a_2 t + a_4 \tag{15}$$

$$\xi = -2Da_1 x t - 2Da_3 t + a_2 x + a_5 \tag{16}$$

$$\phi = (a_1 P)/2x^2 + a_3 P x + a_6 P + a_7 g(x) + a_8 \tag{17}$$

Where the constants $a_i$ are known as the Lie constants. Then, the symmetry generator, also called symmetry algebra, is:

$$V = a_1(-2xt \frac{\partial}{\partial x} - 2Dt \frac{\partial}{\partial t} + (P x^2)/2 \frac{\partial}{\partial P})$$
$$+a_2 (x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t}) + a_3(-2Dt \frac{\partial}{\partial x} + P x \frac{\partial}{\partial P})$$
$$+a_4 \frac{\partial}{\partial t} + a_5 \frac{\partial}{\partial x} + a_6 P \frac{\partial}{\partial P} + a_7 g(x) \frac{\partial}{\partial P} + a_8 \frac{\partial}{\partial P}$$

Table 1: Table of Infinitesimal coefficients, [7].
The lineal independence of the infinitesimal coefficients, let us to find particular solutions to the chosen $a_i$. We choose the $a_3$ constant, then we build the characteristic equation of the particular symmetry generator and we apply the symmetry criteria, [1]-[3], [7]. Then, we get:

$$a_3 P_x + 2 Da_3 t \frac{\partial P}{\partial x} = 0 \tag{19}$$

$$DP_{xx} - P_t = 0 \tag{20}$$

The solution corresponds to the diffusion equation:

$$P(x,t) = \frac{1}{\sqrt{4\pi D t}} e^{-x^2/4Dt} \tag{21}$$

4 The solution to $DP_{xx} + kx P_x + kP - P_t = 0$

Following the same procedure as in section (3), we find the infinitesimal coefficients corresponding to $x, t$ and $P$, are:

$$\xi = a_1 e^{-kt} \tag{22}$$

$$\eta = a_2 \tag{23}$$

$$\phi = a_3 g(x,t) \tag{24}$$

We choose the constants $a_1$ and $a_2$, then, the symmetry generator is:

$$V = a_1 e^{-kt} \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial t} \tag{25}$$

After constructing the characteristic equation for the symmetry generator, and applying the criterion of symmetry, the solution is:

$$P(x,t) = c_1 e^{-x(kx+2ae^{-kt}))}/2D + c_2 \sqrt{(\pi/2Dk)e^{(-ae(-kt))}}/2Dk \times e^{(-x(kx+2ae^{-kt}))}/2D Erf \left[ (kx + ae^{-kt})/\sqrt{2Dk} \right] i \tag{26}$$
Figure 1: Probability density function for the harmonic truncated; both solutions eqs. (27) and (30), are on the same curve, [7].

Where $a = a_1/a_2$. When we choose only the real part of this solution, we normalize it and express the full solution of the truncated oscillator, then, we get, [7]:

$$P(x, t) = \begin{cases} 
N_1/\sqrt{4\pi D}te^{(-x^2)/4Dt}, & \rightarrow x < -d \\
N_2\sqrt{(k/2\pi D)}e^{(-(kx+ae^{-kt}))^2/2Dk} & \rightarrow -d \leq x \geq d \\
N_3/\sqrt{4\pi D}te^{(-x^2)/4Dt} & \rightarrow x > d 
\end{cases}$$

(27)

Here we have the normalization functions corresponding to the position $N_1$ and $N_2$, both of them are time dependent. These two functions are obtained from two conditions of the problem: the first one is that both parts of the solution must match in $x = |d|$, and the second one is the normalization condition, for the complete solution. Then, the two functions are, [1], [7]:

$$N_1 = N_2\sqrt{2kD}e^{(-(kd+ae^{-kt})^2)/2Dk+d^2/4Dt}$$

(28)

$$N_2 = \sqrt{(k/2\pi D)}(2e^{-(kd+ae^{-kt})^2/2Dk+d^2/4Dt} \int_d^\infty e^{(-x^2)/4Dt}dx + \int_{-d}^d e^{-(kx+ae^{-kt})^2/2Dk}dx)^{-1}$$

(29)

The two integrals in $N_2$ have solutions that involve error functions, so their accuracy is as great as the number of summands of the error function can be. On the other hand, the solution given in [4], for the truncated harmonic oscillator, is:
Figure 2: The continuous line corresponds to the error generated using the Lie group method. The dashed line is the error given in [4]. The discontinuity of the curves in \( d = 1, 55 \) is due to the potential jump of the oscillator at this point, [7].

\[
P(x, t) = \begin{cases} 
N/I/\sqrt{4\pi Dt}e^{-(x^2)/4Dt}, & \rightarrow x < -d \\
N_I g(t) \sum_{m=0}^{j} (1/(2^m m!))^{\frac{1}{2}}(k/2\pi D)) \times e^{-(kx^2)/2D}H_m(\sqrt{(k/2\pi D)} x)H_m(0)e^{-t|m|} & \rightarrow -d \leq x \geq d \\
N/I/\sqrt{4\pi Dt}e^{-(x^2)/4Dt}, & \rightarrow x > d 
\end{cases}
\]  

Where

\[
g(t) = e^{(-d^2/4Dt)/\sqrt{4\pi Dt}}(\sum_{m=0}^{j} (1/(2^m m!))^{\frac{1}{2}}(k/2\pi D))e^{-(kd^2)/2D} \times H_m(\sqrt{(k/2\pi D)} d)H_m(0)e^{-t|m|}^{-1}
\]

\[
N_I = (2/\sqrt{4\pi Dt}) \int_d^{\infty} e^{-(x^2)/4Dt} dx + g(t) \int_{-d}^{d} \sum_{m=0}^{j} (1/(2^m m!)) \times \int_{-d}^{d} H_m(\sqrt{(k/2\pi D)} d)H_m(0)e^{-t|m|} dx
\]

The Hermite polynomials of m-order are given by \( H_m(y) \). This kind of series for the Fokker-Planck is already known in a very interesting work, given in [4]-[5]. In order to compare the Lies group solution with the one given in reference [4], we use the next values \( d = 1, 55 \), \( D = 1 \) and \( k = 1, 4 \). Also, we took till the tenth summatory term of the error function, plotting for \( t=1 \). In figure (1),
we show the probability density for $m=0$; we remark the total coincidence of both curves, showing the validity of the Lie group method to solve the Fokker-Planck. In figure (2), we show the error generated in both solutions when we replace the solutions, equations (27) and (30) in the oscillator differential equations (3) and (4), as appears in eq. (33), [7].

$$\text{Err} = \begin{cases} 
DP_{xx} - P_t, & \rightarrow -d > x, x > d \\
DP_{xx} + kxP_x + kP - P_t, & \rightarrow -d \leq x \leq d
\end{cases}$$  \hspace{1cm} (33)

\section{5 Conclusions}

We solve the Fokker-Planck equation using Lie groups. The founded solution was contrasted with the results in reference [4]. Both solutions match quite well, fig. (1). In addition, the error because of the occurrence of error functions, during the normalization process, is lower in the case of Lie group method, fig. (2).

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\textbf{References}


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