

Advanced Studies in Theoretical Physics  
Vol. 11, 2017, no. 4, 179 - 212  
HIKARI Ltd, [www.m-hikari.com](http://www.m-hikari.com)  
<https://doi.org/10.12988/astp.2017.61142>

## A *Physics-First* Approach to the Schwarzschild Metric

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### Abstract

As is well-known, the Schwarzschild metric cannot be derived based on pre-general-relativistic physics *alone*, which means using only special relativity, the Einstein equivalence principle and the Newtonian limit. The standard way to derive it is to employ Einstein's field equations. Yet, analogy with Newtonian gravity and electrodynamics suggests that a more constructive way towards the gravitational field of a point mass might exist. As it turns out, the additional physics needed is captured in two plausible postulates. These permit to deduce the exact Schwarzschild metric without invoking the field equations. Since they express requirements essentially designed for use with the spherically symmetric case, they are less general and powerful than the postulates from which Einstein constructed the field equations. It is shown that these imply the postulates given here but that the converse is not quite true. The approach provides a fairly fast method to calculate the Schwarzschild metric in arbitrary coordinates exhibiting stationarity and sheds new light on the behavior of waves in gravitational fields.

**PACS:** 04.20.-q, 04.20.Cv, 01.40.gb

**Keywords:** General relativity, spherical symmetry, Schwarzschild metric

## 1 Introduction

Newton's universal law of gravitation [20], describing the attractive force between two point masses, was found long before its generalization to arbitrary mass densities via the divergence theorem [12] and irrotationality of the gravitational field. Coulomb's law giving the force between two point charges [6, 7] was presented well before its field-theoretic foundation through the Poisson equation [21]. With general relativity (GR), the historical sequence was reverted. Einstein first gave the field equations [10], i.e., the general law, encompassing the most complicated cases, and only then the gravitational field of a point mass was found [31, 9]. Could it have been the other way round?

Clearly, the Schwarzschild metric is more difficult to infer than either Newton's or Coulomb's law. Moreover, knowing the law for point objects before the field equations would not have been as useful as in the predecessor theories. These have linear field equations and hence the general law may be constructed from the special one. A similar feat is difficult to imagine in Einstein's theory. The field equations of GR are nonlinear.

In addition, the question arises what it would take to find the Schwarzschild metric without the field equations. Are they not essential to its derivation? On the other hand, Coulomb's law can be derived from Maxwell's equations and that is what one would do with the full set of electromagnetic field equations at hand. Yet, it was obtained independently of, and before, these equations. Can this process of discovery of a particular physical law before the general framework of a theory be mimicked in the case of GR?

Let us first discuss a few things that do *not* work. Inspired by the simple form of the Schwarzschild metric in standard coordinates, a variety of attempts at obtaining the metric without reference to the field equations have been made. In particular, the observation that the diagonal time and radial metric components,  $g_{tt}$  and  $g_{rr}$ , are inversely proportional to each other seemed intriguing and has remained a source of continuing interest and confusion. Although it is well understood under which circumstances  $g_{tt}g_{rr} = -1$  arises [16], erroneous attributions of this feature to special relativistic effects may be found even in the recent literature [27, 3, 4]. Typically, the reciprocity of time dilation and length contraction is invoked, which is however *not* responsible for the property. This kind of argument goes back to Lenz and Schiff [32, 29]. The latter tried to show that not only gravitational redshift but also light deflection by the sun could be quantitatively accounted for by use of special relativity (SR), the Einstein equivalence principle (EP), the Newtonian limit (NL), and nothing more. The anomalous perihelion precession of Mercury would then be the only classical test of GR that really probes the field equations.

In the same year when Schiff's paper appeared, an article by Schild [30] clarified that in order to get the first-order coefficient in an expansion of  $g_{rr}$  in

powers of the (normally small) quantity  $GM/rc^2$ , more than the three ingredients SR, EP and NL are needed, which is sufficient to falsify Schiff's claim.<sup>1</sup>

As an aside, to obtain the general relativistic equations of motion in a *given* metric, even the first two of these three ingredients are fully sufficient. GR consists of two parts, one dealing with the way of spacetime telling energy how to move and the other with the way of energy telling spacetime how to curve. Half of the theory (the first part) is obtainable from SR and the EP. To expect the other half to arise from mere addition of the NL would be unreasonable.

In fact, as I have shown elsewhere [17], using a Newtonian approximation to the potential, the exact Schwarzschild metric obtains in a simple "derivation" due to a cancellation of two errors. Therefore, it is not sufficient to get the right answer in a claim to rigor, the correctness of intermediate steps must be verified, too. In strong fields, deviations from Newton's law, expressible as higher powers of  $GM/rc^2$ , have to be expected. The assumption that all of these are zero is unjustified, even though it may be true by accident that a particular metric function is already given exactly by the first-order expansion.

Sacks and Ball as well as Rindler pointed out the failure of Schiff's argument with different lines of reasoning [28, 22]. Rindler's argument proceeded via a counterexample, based on a static metric, going by his name nowadays. More recently, the subject of simple derivations of the Schwarzschild metric was resumed by Gruber et al. [15]. They gave detailed arguments why such a derivation is impossible. From their analysis, it becomes clear that by *simple* they mean that only the mentioned ingredients SR, NL and EP are used. With this restriction to the meaning of *simple*, they prove their point.

Einstein's field equations constituted, at the time of their inception, a *new law of nature*, going beyond and not contained in, the combination of the EP (including SR) with the Newtonian limit. There are alternative theories of gravity, such as the Brans-Dicke (BD) theory [2], with *different* field equations and the *same* Newtonian limit, and with spherically symmetric solutions different from the Schwarzschild one. Since all three ingredients, SR, EP, and NL form part of the BD theory as well, it is obviously logically impossible to derive the Schwarzschild solution from these constituents *only*.

Even if we completely concur with the conclusions of Ref. [15], it would be premature to claim that one cannot do without the *field equations*. – Basically, either the field equations or their generating action are a set of *postulates* within GR. However, postulates or axioms are *not unique*. In thermodynamics, we have different formulations of the second law, a postulate of the theory. It is sufficient to require one of them, then the others are derivable as theorems. We do not rack our brains about this fact, because the different forms of the second law have similar complexity and are easily shown to be equivalent. Things are

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<sup>1</sup> $G$  is Newton's gravitational constant,  $c$  the speed of light,  $M$  the gravitating mass and  $r$  the radial coordinate at which the metric is considered.

much more thorny in set theory, where the axiom of choice, Zorn's Lemma and the well-ordering theorem are all interchangeable. Again, it suffices to postulate one of them to make the other two derivable, but their plausibility levels as axioms seem very distinct, ranging from almost self-understood (axiom of choice) to outright difficult to believe (well-ordering theorem).

Returning to GR, if we restrict consideration to the stationary spherically symmetric case, what is needed to get by without the field equations is one or several postulates that can stand in for them in this particular situation. These postulates need not be of geometric nature. Nor do they have to be powerful enough to replace the field equations altogether. If our intuition permits a good guess at the most important aspects of the physics of that simple case, refined differential geometry may not be necessary. Of course, the so-found postulates must be *compatible* with the field equations, even *derivable* from them, whereas the converse need not be true. The special case must follow from the general one but not vice versa. It may however happen that our physically motivated postulates reveal properties of the theory that were not easy to see before on the basis of the geometric maxim alone.

This kind of approach is not only logically possible, it has even been discussed favorably by Sacks and Ball [28] with regard to Tangherlini's postulational approach to the Schwarzschild metric [33]. Unfortunately, Rindler later showed one of the two Tangherlini postulates to be unconvincing [23]. But clearly, Tangherlini's approach is not subject to the criticism (nor the impossibility proof) offered by Gruber *et al.* [15].

However, Tangherlini's argument may be criticized on the simple grounds that it is coordinate dependent: he equates a coordinate acceleration with a "Newtonian acceleration" without clarifying what makes his coordinate so special that this identification is possible, whereas it is not for other coordinates. And while his choice of radial coordinate may seem plausible in the case of the Schwarzschild metric, the coordinate choice needed to make the postulate work with the Rindler metric looks highly implausible [23]. Similar arguments can be raised against a proposition by Dadhich [5], in which he suggests that a photon on a radial trajectory cannot be accelerated:  $d^2r/d\lambda^2 = 0$  in terms of an affine time parameter  $\lambda$ , from which he concludes  $g_{tt}g_{rr} = -1$ . This is true for the radial coordinate of the standard form of the Schwarzschild metric, because that coordinate is an affine parameter itself [16], but not for all conceivable radial coordinates. Photons on radial trajectories *are* accelerated in affine time, if the radial coordinate of the *isotropic* form of the Schwarzschild metric is taken instead of the circumferential coordinate  $r$ . So any postulate based on independence of coordinate acceleration of the energy of the particle (Tangherlini) or absence of coordinate acceleration for photons (Dadhich) should give a reason, why it is the particular coordinate considered that has this property. Needless to say, the argument should not have recourse to the

field equations, from which the property (and the metric) could be derived.

Note that if *simple* is taken to just mean *technically simple*, there are ways to obtain the Schwarzschild metric with little effort, as has been shown by Deser [8]. His approach is based on the field equations and requires both understanding and control of sophisticated mathematical tools. Whoever masters these is likely to have done the more tedious standard derivation before.

In order to move from an abstract level of discussion to more concrete ideas, let us briefly consider how Coulomb's law could be deduced on the basis of an appropriate postulate. Historically, it was of course obtained inductively, following experimental work [6, 7]. Modern intuition about fields however opens a pretty direct theoretical route. Let us define the electric field, as usual, as the force on a small test body per unit electrical charge. Fields are visualized via field lines. Charges are sources and sinks of the field, so the number of field lines is proportional to the charge density emitting or absorbing them. Then, the *field strength* must be proportional to the *density* of field lines.<sup>2</sup> Our basic postulate will simply be that *no field lines can begin or end in empty space*. This is sufficient to derive the field of a point charge, assumed to be spherically symmetric. Consider two concentric spherical surfaces with the point charge at their center and no other charges present, so there is vacuum between the two spheres. For symmetry reasons, all the field lines must converge radially on (or diverge radially from) the point charge. Any solid angle cutting out pieces of the spherical surfaces will contain a fixed number of field lines that must pierce both surfaces. Since no field lines are lost or added, their density must be inversely proportional to the cut-out part of the two surfaces, i.e., to their surface area. The field strength must then be inversely proportional to the square of the radius  $r$ . In a more mathematical formulation, since the field strength is proportional to the density of field lines, the surface integral of the field, i.e., its flux through the surface  $\int \mathbf{E} d\mathbf{f}$  must be the same for both surfaces. This of course immediately leads us to Gauss's law which in this symmetric configuration is sufficient to determine the functional dependence  $\mathbf{E} \propto \mathbf{e}_r/r^2$  ( $\mathbf{e}_r$  being unit vector in the radial direction). The charge factors in Coulomb's law are then more or less a consequence of the definition of the electric field and an overall constant prefactor is determined by the choice of units for the electrical charge. Clearly, the postulate that lead us here may be reformulated as saying that the electric field is divergence free in vacuum. Obviously, it does not exhaust Maxwell's equations reduced to the electrostatic case, but it is sufficient to determine the field in a spherically symmetric situation.

The purpose of this paper is to obtain the vacuum metric about a spherically symmetric mass distribution in a similar fashion, i.e., without reference to Einstein's field equations. Since the situation with gravity is a bit more

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<sup>2</sup>If we double the number of charges on a fixed surface, both the number of field lines and the field strength will double.

complex than with electrostatics, it will be necessary to invoke more than a single postulate. As it turns out, *two* postulates that I shall call P1 and P2 will be sufficient, and P1 will be very similar to what we used for Coulomb's law. Moreover, P1 will lead us to P2 which takes the form of a dynamical postulate. It will be argued that the two postulates are sufficiently self-evident to be required prior to any knowledge of the field equations. We will therefore pretend to know nothing about the latter throughout Secs. 2 and 3.

However, since the full theory is known already, we can immediately check, without waiting for experiments to be done, whether our postulates are satisfied, by deriving them from the field equations. This will be done in Sec. 4, where we will “remember” those equations again. While this derivation proves the truth of the postulates (at least if we believe in GR), it is not a premise when requiring them in the process of building a simplified theory of gravity outside a spherically symmetric mass distribution.

Moreover, the two postulates will shed light on certain properties of the Schwarzschild solution, having to do with spacetime curvature. One of these properties seems to be pretty familiar, whereas the author has not seen the other in the literature so far. For readers who feel that no additional exploration into the foundations of the Schwarzschild metric is needed, the main purpose of the paper may be seen in its discussing interesting properties of the metric that hitherto have not found much attention, if any.

Besides the new postulates, the three ingredients SR, EP, and NL will all be employed in the following deductions. Since we use the Einstein form of the EP stating that the outcome of non-gravitational experiments in sufficiently local freely falling systems is governed by the laws of special relativity, SR is already ingrained in the EP. The NL is used in determining the asymptotic behavior of the metric at infinity. SR is also prominent in motivating some ideas by reference to the Rindler metric, describing a rigidly accelerating frame. A few facts about it are collected in Sec. 2. They are fully derivable within SR, but derivations will only be given for a few less well-known properties. Section 3 describes the postulates P1 and P2 and gives the central result of the paper, while Sec. 4 establishes the connection of the two postulates with the field equations. In Sec. 5, some conclusions are presented. Two appendices discuss how to apply the postulates to two different coordinate systems, yielding alternative forms of the Schwarzschild metric.

## 2 The Rindler metric

The Rindler metric may be obtained from the Minkowski metric

$$ds^2 = -c^2dT^2 + dX^2 + dY^2 + dZ^2 \quad (1)$$

by a coordinate transformation

$$cT = x \sinh \frac{\bar{g}}{c} t, \quad X = x \cosh \frac{\bar{g}}{c} t, \quad Y = y, \quad Z = z, \quad (2)$$

and reads

$$ds^2 = g_{ij} dx^i dx^j = -f(x) c^2 dt^2 + dx^2 + dy^2 + dz^2, \quad f(x) = \frac{\bar{g}^2 x^2}{c^4} \equiv \frac{x^2}{x_0^2}, \quad (3)$$

where we have adopted, and will use from now on, Einstein's summation convention. The Rindler metric is the form of the Minkowski spacetime adapted to the description of a set of linearly accelerated observers, each of which is subject to constant proper acceleration.<sup>3</sup>  $\bar{g}$  is the proper acceleration of the observer at position  $x_0$ , where  $f(x_0) = 1$ . The ensemble of observers perform Born rigid motion, i.e., in the frame of each observer<sup>4</sup> the distance to any other observer of the set remains constant. As a consequence, observers at different  $x$  positions experience different proper accelerations  $a = c^2/x$ . We may introduce an acceleration potential  $\Phi$ , requiring  $a(x) = d\Phi/dx$ , which can be integrated immediately:

$$\Phi(x) = c^2 \ln \frac{x}{x_0}. \quad (4)$$

The constant of integration has been chosen so that  $f(x) = e^{2\Phi/c^2}$ , i.e., the prefactor of the exponential is one. The proper time  $\tau$  of a coordinate stationary observer (CSO) at position  $x$  is related to the global time  $t$  by

$$d\tau = \sqrt{f(x)} dt = \frac{\bar{g}x}{c^2} dt. \quad (5)$$

Next, we try to characterize the *force* field resulting from the acceleration of the Rindler frame. Suppose an observer at a fixed position  $x_O$  wishes to measure this field, perceived by him as gravity. To obtain the force at distant positions, he slowly lowers or raises a test mass  $m$ , fastened to the end of a massless rigid pole, in the field and determines how strong the mass will pull against, or push down, the pole at its end. For a finite accuracy of the measurement, we need only approximate rigidity and masslessness, which are compatible with relativity, so the experiment is feasible in principle [17].

The resulting force can be calculated using energy conservation. If the pole is shifted down or up by a piece  $d\ell$  at its near end, the far end will move down or up by the same amount  $d\ell$  in terms of the local proper length. On being

<sup>3</sup>*Adapted to* essentially means that the accelerated observers are coordinate stationary in the metric.

<sup>4</sup>We may assign an *extended* momentarily comoving inertial frame to each observer, because the spacetime is flat.

lowered, the mass is doing work, on being raised, work must be done on it, so if its energy in the field is  $E(x)$ , the force exerted by it will be

$$F = -\frac{dE(x)}{d\ell} = -\frac{dE(x)}{dx}. \quad (6)$$

Now locally, the mass always has the energy  $mc^2$ , as it does not acquire kinetic energy – the experiment is performed quasistatically. But the observer at  $x_O$  will not assign this local value to its energy, because to him everything at  $x$  happens at a slower or higher rate due to time dilation. This changes the energy of photons by the time dilation factor. Clearly, all other energies must be affected the same way, otherwise we would run into problems with energy conservation [17]. This leads to

$$E(x) = \sqrt{\frac{f(x)}{f(x_O)}} mc^2 \quad \Rightarrow \quad F(x) = -mc^2 \frac{f'(x)}{2\sqrt{f(x_O)}\sqrt{f(x)}}, \quad (7)$$

which evaluates to  $F = -m\bar{g}/\sqrt{f(x_O)}$ , i.e., the force is *constant* in space, a fact that has been noted by Grøn before [13]. It is in this sense that the inertial field described by the Rindler metric may be called a *uniform gravitational field* – the force on an object is homogeneous in each CSO's frame, even though the proper acceleration (the local force per unit mass) is not. A detailed discussion of the issue of field uniformity in GR is given in Ref. [19]. Thus, the force field defined by the discussed measuring procedure behaves as the Newtonian force in a homogeneous gravitational field. However, observers at different  $x_O$  positions will measure *different* forces for the same mass.

In Newtonian gravity, the potential satisfies a Laplace equation at points where the mass density vanishes. It is then natural to ask what kind of field equation is satisfied by the potential in (4). An immediate conspicuity is that there are two inequivalent ways to define the Laplacian, given the metric (3).

The first is to start from the spatial part  $\gamma_{ij}$  of the metric and take the general expression for the Laplacian in curvilinear coordinates

$$\Delta_s = \frac{1}{\sqrt{\gamma}} \partial_i \sqrt{\gamma} \gamma^{ij} \partial_j \quad (i, j = 1 \dots 3), \quad (8)$$

where  $\gamma = \det(\gamma_{ij})$ . This is not unique, since the decomposition of spacetime into space and time is not, but if the metric is stationary, we can decompose spacetime into the proper time of CSOs and the proper space orthogonal to it, which both are unique. A clean way to define proper space as a congruence of world lines of test particles is presented in Ref. [26]. Then  $\gamma_{ij} = g_{ij} - g_{0i}g_{0j}/g_{00}$ , where the subscript 0 refers to the time coordinate, as usual. In the case of the Rindler metric,  $\gamma_{ij}dx^i dx^j = dx^2 + dy^2 + dz^2$  ( $i, j = 1 \dots 3$ ), so the spatial Laplacian



$\Delta_s$  is equal to the ordinary flat-space Laplacian in 3D. Obviously, the potential from Eq. (4) does *not* satisfy a Laplace equation with this Laplacian.

The second definition of a Laplacian uses the *full* spacetime metric to define a four-space Laplacian or d'Alembertian

$$\square = \frac{1}{\sqrt{|g|}} \partial_i \sqrt{|g|} g^{ij} \partial_j \quad (i, j = 0 \dots 3), \quad (9)$$

where  $g = \det(g_{ij})$ , and then takes as three-space Laplacian  $\Delta_w$  the time independent part of the d'Alembertian. Whereas the d'Alembertian is unique, its decomposition into spatial and temporal parts is not, in principle. Again, the situation simplifies for stationary metrics, where the decomposition becomes invariant under coordinate transformations that leave the metric stationary.

For the Rindler metric, we find

$$\square = -\frac{c^2}{g^2 x^2} \partial_t^2 + \frac{1}{x} \partial_x x \partial_x + \partial_y^2 + \partial_z^2, \quad (10)$$

hence  $\Delta_w = \frac{1}{x} \partial_x x \partial_x + \partial_y^2 + \partial_z^2$ , and it is easy to verify that

$$\square \Phi(x) = \Delta_w \Phi(x) = 0, \quad (11)$$

i.e., with this *wave Laplacian*, the potential does satisfy a Laplace equation. Alternatively, we may simply consider it a *time independent* solution of the wave equation.

This then suggests to have a closer look at *time dependent* solutions to the wave equation, which turns out to be solved by plane waves of the form

$$\chi(x, t) = \psi \left( t \mp \frac{c}{g} \ln x \right) = \tilde{\psi} \left( x e^{\mp \frac{g}{c} t} \right), \quad (12)$$

where  $\psi$  or  $\tilde{\psi}$  is an *arbitrary* function, required only to be twice continuously differentiable. The *temporal* wave form  $\psi$  is the same for arbitrary fixed position  $x$ ; different values of  $x$  just correspond to different phases. The *spatial* wave form has the similarity property that for a fixed value of  $t$ , it is a squeezed or stretched version of the shape described by  $\tilde{\psi}(x)$ . The discussion of properties of solutions to the wave equation in a spherically symmetric spacetime will become important in the next section.

## 3 The metric outside a spherically symmetric mass distribution

### 3.1 Symmetry considerations

One of the simplest gravitating systems is a time-independent spherically symmetric mass distribution, describable by a stationary metric. The line element

may be written as

$$ds^2 = -\tilde{f}(\tilde{r}) c^2 d\tilde{t}^2 + 2\tilde{k}(\tilde{r}) c d\tilde{t} d\tilde{r} + \tilde{h}(\tilde{r}) d\tilde{r}^2 + \tilde{n}(\tilde{r}) \tilde{r}^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) . \quad (13)$$

Herein,  $\vartheta$  and  $\varphi$  are the usual angular coordinates which, due to spherical symmetry, may only appear in the combination  $d\Omega^2 = d\vartheta^2 + \sin^2 \vartheta d\varphi^2$  but not in any of the coefficient functions. Because the metric is assumed time independent, none of the coefficients may depend on the time  $\tilde{t}$ . Thus, all of them must be functions of the radial coordinate  $\tilde{r}$  only. Only two of these four functions are fixed by physics, whereas two can be chosen with a great degree of arbitrariness, amounting to a choice of the time and radial coordinates.

For example, the prefactor  $\tilde{n}(\tilde{r})$  of  $d\Omega^2$  may be chosen equal to one, meaning we define  $\tilde{r} = r$  so that the surface area of a sphere about the coordinate center, described by  $r = \text{const.}$  becomes  $4\pi r^2$ .<sup>5</sup> Instead, we might require  $\tilde{h}(\tilde{r}) = \tilde{n}(\tilde{r})$ , which leads to isotropic coordinates. Further, a coordinate transformation of the form  $\tilde{t} = t + w(\tilde{r})$  may be used to remove the term  $\propto d\tilde{t} d\tilde{r}$ .<sup>6</sup>

For now, we will set  $\tilde{n}(\tilde{r}) = 1$ , renaming the radial coordinate chosen this way to  $r$ , and choose a time coordinate  $t$  so that the metric becomes diagonal. The line element then takes the form

$$ds^2 = -f(r) c^2 dt^2 + h(r) dr^2 + r^2 d\Omega^2 . \quad (14)$$

Alternative coordinate choices are considered in the appendix.

At large radii, gravitation becomes negligible, so the line element should approach that of the Minkowski metric, hence we require

$$\lim_{r \rightarrow \infty} f(r) = 1 , \quad \lim_{r \rightarrow \infty} h(r) = 1 . \quad (15)$$

So far, just symmetry has been exploited. In order to determine  $f(r)$  and  $h(r)$ , we need to invoke physical ideas.

## 3.2 Equivalence principle and potential

First, we make use of the EP. Instead of translating the physics in a gravitating system into terms of an accelerating one, which requires to visualize two different but equivalent systems in parallel, let us consider a freely falling observer in the *actual* system under consideration. The prescription then is to describe local physics in the frame of that inertial observer by SR. For the freely falling observer, there *is* no gravitational field and everything that the gravitational field does to CSOs<sup>7</sup> must be due to the fact that they are accelerating with respect to his local inertial frame.

<sup>5</sup>This is equivalent to the circumference of a circle about the origin being given by  $2\pi r$ .

<sup>6</sup>With the choice  $w(y) = \int^y dx \tilde{k}(x)/\tilde{f}(x)c$ .

<sup>7</sup>Here, a CSO is an observer satisfying  $dr = d\vartheta = d\varphi = 0$ .

As has been discussed in Ref. [17], by describing the observed frequency change of a photon sent from a CSO at position  $r$  to one at  $r + dr$  in a freely falling frame that is momentarily at rest with respect to the CSO at  $r$ , we can establish a relationship between the potential  $\Phi(r)$ , linked to the local proper acceleration  $a(r)$  via

$$a(r) = \frac{1}{\sqrt{h(r)}} \frac{d\Phi}{dr}, \quad (16)$$

and the function  $f(r)$ , reading  $\sqrt{f(r + dr)}/f(r) = 1 + [\Phi(r + dr) - \Phi(r)]/c^2$  and generating the differential equation

$$\frac{1}{2} \frac{f'(r)}{f(r)} = \frac{1}{c^2} \Phi'(r). \quad (17)$$

This is solved by

$$f(r) = e^{2\Phi/c^2}, \quad (18)$$

where the standard boundary condition  $\lim_{r \rightarrow \infty} \Phi(r) = 0$  of Newtonian physics has been used. Equation (18) can be found in textbooks [25] and merely reformulates the metric function  $f(r)$  in terms of a more readily interpretable quantity that must approach the Newtonian potential as  $r \rightarrow \infty$ . Hence,  $\Phi(r) \sim -GM/r$  ( $r \rightarrow \infty$ ) and we may infer the asymptotic behavior of  $f(r)$  at large  $r$ :

$$f(r) \sim 1 - \frac{2GM}{rc^2}, \quad r \rightarrow \infty. \quad (19)$$

### 3.3 Global force field: absence of sources in vacuum

Let us now consider the gravitational force exerted on a mass  $m$  at the end of a massless rigid pole, felt by an observer at radius  $r_O$  holding the pole at its other end. We have considered a similar situation in the Rindler metric. Then, reasoning as in Sec. 2, we obtain

$$F(r) = -\frac{dE(r)}{dr} \frac{dr}{d\ell} = -\frac{1}{\sqrt{h(r)}} \frac{dE(r)}{dr} = -\frac{mc^2}{\sqrt{f(r_O)}} \frac{f'(r)}{2\sqrt{f(r)h(r)}}. \quad (20)$$

For simplicity, we let  $r_O \rightarrow \infty$ , implying  $f(r_O) = 1$ .

At this point, we introduce postulate P1.  $F(r)$  is a global force field (measured by an observer at infinity), and we require *its flux  $\int F dS$  through the surface  $S$  of a sphere about the center of gravity to remain constant outside the mass distribution*, in keeping with the idea that this mass distribution is the *only* source of gravity. If we visualize the force in terms of field lines, then every field line must end in a mass element for static fields, so all field lines that enter a spherical shell through its outer surface must exit through its inner surface, if the shell does not contain any mass, i.e., in vacuum. Since  $r$  was

chosen so that the surface area of such a sphere is  $S = 4\pi r^2$ , this means that  $F(r) = A/r^2$  with some constant  $A$  that may be determined by comparison with the Newtonian limit, hence

$$F(r) = -G \frac{mM}{r^2}, \quad (21)$$

which gives us a first equation for the two functions  $f(r)$  and  $h(r)$ :

$$\frac{f'(r)}{\sqrt{f(r)h(r)}} = \frac{2GM}{r^2 c^2}. \quad (22)$$

### 3.4 Local formulation – vanishing divergence

The main disadvantage of P1 in this formulation is that we need a *global* description of the force field. The global force  $F(r)$  is not what a local observer measures as gravitational force. Instead, the *local* force on the mass reads:

$$\mathbf{F}_{\text{loc}}(r) = -ma(r) \mathbf{e}_r = -m \frac{1}{\sqrt{h(r)}} \frac{\partial \Phi}{\partial r} \mathbf{e}_r = -\frac{mc^2}{2} \frac{f'(r)}{f(r)\sqrt{h(r)}} \mathbf{e}_r = \frac{F(r)}{\sqrt{f(r)}} \mathbf{e}_r. \quad (23)$$

Noting that  $d\Phi = \nabla\Phi \cdot d\mathbf{s}$ , where  $\nabla$  is the four-gradient, we obviously have  $\partial\Phi/\partial r = \nabla\Phi \cdot \partial\mathbf{s}/\partial r = \nabla\Phi \cdot \sqrt{h} \mathbf{e}_r$ , which shows that the local force is the spatial part of the four-gradient of  $\Phi$  and its temporal part vanishes, i.e., with a slight abuse of notation<sup>8</sup>  $\mathbf{F}_{\text{loc}}(r) = -m\nabla\Phi$ . Our requirement that the field is source free in vacuum should then take the form  $\nabla \cdot \mathbf{F}_{\text{loc}} = 0$ , with an appropriately defined divergence operator, valid in the curved spacetime. Experience with the Rindler metric suggests that this divergence is not the three-divergence of the restriction of spacetime to its spatial part but rather the four-divergence of a four-vector with zero time component:

$$\nabla \cdot \mathbf{F}_{\text{loc}} = \nabla^j F_{\text{loc}j} = \frac{1}{\sqrt{|g|}} \partial_i \sqrt{|g|} g^{ij} F_{\text{loc}j} = -m \nabla^j (\nabla\Phi)_j = -m \square \Phi. \quad (24)$$

Hence, postulate P1 has an explicit expression in terms of the potential, reading

$$\square \Phi = \Delta_w \Phi = \frac{1}{\sqrt{|g|}} \partial_i \sqrt{|g|} g^{ij} \partial_j \Phi = 0, \quad (25)$$

where  $\Delta_w$  is the wave Laplacian again, obtained from the d'Alembertian by simply dropping the summand(s) with time derivatives. For a purely  $r$  dependent potential, by use of  $\sqrt{|g|} = c\sqrt{f(r)h(r)}r^2 \sin\vartheta$ , this reduces to

$$\frac{1}{\sqrt{f(r)h(r)}r^2} \partial_r \sqrt{f(r)h(r)}r^2 \frac{1}{h(r)} \partial_r \Phi = 0, \quad (26)$$

<sup>8</sup>Because we write  $\mathbf{F}_{\text{loc}}$  both for a three-force and a four-force with temporal part zero.

from which we obtain  $\sqrt{f(r)/h(r)r^2}f'(r)/f(r) = \text{const.}$  using Eq. (17), hence  $f'(r)/\sqrt{f(r)h(r)} = \tilde{A}/r^2$ . The constant  $\tilde{A}$  can be determined from the asymptotic behaviors (19) and  $h(r) \sim 1$  ( $r \rightarrow \infty$ ), yielding  $\tilde{A} = r_s \equiv 2GM/c^2$ , whence we recover (22). This shows that postulate P1 may be reformulated as the requirement that *the potential must satisfy a Laplace equation obtained as the time independent limit of the wave equation constructible from the full metric.* Since the potential itself is expressible by the metric functions, this is a constraint on the metric. Note that even though physically P1 is essentially the same postulate for the gravitational field as the one we used for the electric field in deriving Coulomb's law, its formulation gets more complicated than in the electrostatic case, due to the necessity of introducing a potential. It takes the form of a second-order equation instead of a first-order one. Moreover, since this potential is not the only independent function arising in the metric, a single postulate (of this simple type) is not enough.

Postulate P1 should be true for all stationary forms of the metric. While the potential  $\Phi$  is not a four-scalar, it behaves as a scalar under coordinate transformations leaving the metric stationary.

That P1 is not powerful enough to determine the metric completely may alternatively be understood from its referring to a static aspect of the field only, which is not likely to provide enough information in a spacetime picture. The idea then immediately suggests itself that a second postulate ought to be a requirement on fully time dependent solutions of the wave equation, involving dynamic aspects, i.e., the relationship between space and time.

### 3.5 Wave equation and Huygens' principle

Before we proceed with metric considerations, a diversion on wave properties in *flat space* may be in place. The flat-space vacuum wave equation has some remarkable properties, if the space has an *odd* number of dimensions exceeding one. In a three-dimensional space, these properties take an even more fascinating form. The main property of interest here is expressed by the (strong) Huygens' principle.<sup>9</sup> It states that the wave solution at some event will only be influenced by other events precisely *on* its past light cone, not by events *inside* it,<sup>10</sup> i.e., the wave does not have a *tail* [1].

Mathematically, this property follows from the retarded Green's function for the wave equation in  $d = 2n + 1$  dimensions ( $n \geq 1$ ) being proportional to

$$\frac{d^{n-1}}{(R dR)^{n-1}} \left[ \frac{1}{R} \delta \left( t - t' - \frac{R}{c} \right) \right],$$

<sup>9</sup>This is related to but not the same as the Huygens-Fresnel principle, allowing the reconstruction of a wave front from a set of elementary waves.

<sup>10</sup>By causality, it cannot, of course, be influenced by events *outside* the light cone either.

where  $R = |\mathbf{r} - \mathbf{r}'|$  [11]. Since the  $\delta$  function and its derivatives are zero whenever their argument is not, an observer at  $\mathbf{r}$  is influenced, at time  $t$ , by an event at  $(t', \mathbf{r}')$  only, if the time difference  $t - t'$  it takes the wave to travel from  $\mathbf{r}'$  to  $\mathbf{r}$  is *exactly*  $|\mathbf{r} - \mathbf{r}'|/c$  but not larger. In contrast, this is *not* true for an *even* number of spatial dimensions,  $d = 2n$ , where the Green's function behaves as

$$\left[ \frac{d^{n-1}}{(R dR)^{n-1}} \left( (t - t')^2 - R^2/c^2 \right)^{-1/2} \right] \Theta \left( t - t' - \frac{R}{c} \right)$$

or in 1D, where

$$G(t, \mathbf{r}, t', \mathbf{r}') = \frac{c}{2} \Theta \left( t - t' - \frac{R}{c} \right), \quad (27)$$

i.e., it is just a Heaviside function. Therefore, if a sufficiently short light pulse is created at the origin in three-dimensional space at time  $t = 0$ , an observer at a distance  $R$  will see it at time  $t = R/c$  and then no more, whereas a similar flash sent out from the origin of two-dimensional space will be seen at  $t = R/c$  and *forever after*, albeit with continuously decreasing intensity.

Huygens' principle holds for all odd space dimensions greater than one. If the dimension exceeds three, due to the appearance of derivatives of the  $\delta$  function in the Green's function ( $n > 1$  in the formula above), the wave form will be distorted as the wave moves along. In three dimensions ( $n = 1$ ), however, the Green's function is just a  $\delta$  function multiplied by  $1/R$ , so a wave originating from a point source will travel at constant shape, only being damped due to the factor of  $1/R$  as the distance  $R$  from the source increases. This does not imply that *all* wave solutions keep their temporal wave form. Obviously, a superposition of waves from two point sources cannot remain undistorted as its constituent waves will decay with different spatial prefactors. Also, if the "point source" has internal structure, as is the case, e.g., with a Hertzian dipole, the total wave need not have a strictly preserved shape. We know that the electric field of such a dipole has a near-field component decaying as the sum of a  $1/R^2$  and a  $1/R^3$  term, and a far-field component, carrying the energy to infinity, that behaves as  $1/R$ . Any oscillating multipole has a leading field decaying as  $1/R$  and this field will propagate at constant shape, whereas the superposition of near and far fields cannot of course be the same function of time at arbitrary distances. Clearly, *shape-preserving* solutions of the wave equation satisfy Huygens' principle, but the converse is not true. These solutions (or rather a slight generalization, the so-called *similarity solutions*) are called *relatively undistorted* or *simple progressing* waves in Ref. [1].

What is interesting about similarity solutions of *arbitrary* waveform is that their existence can be verified from the form of the wave equation in a pretty straightforward way, much more easily than whether Huygens' principle is

satisfied or not. Moreover, shape preservation has a very direct interpretation that can be easily visualized. Mathematically, we may express the similarity property by saying that the wave equation has solutions of the form  $\chi(r, t) = B(r)\psi(u)$  with arbitrary functions  $\psi$ , where  $u$  is some composite variable of  $t$  and  $r$ , e.g.  $t - r/c$ . Here, we restrict ourselves to scalar waves, because they are most readily made to have spherical symmetry. This keeps the mathematical discussion simple. We would of course be hard pressed to point out physical scalar waves traveling at the vacuum speed of light.

One way to demonstrate that a wave equation has this property is to show that it is possible to choose a function  $B(r)$  so that the operator product of the d'Alembertian and  $B(r)$  can be factorized according to

$$\square B(r) = (\bar{a}\partial_t + \bar{b}\partial_r + \bar{d})(a\partial_t + b\partial_r) + B(r) \times \text{angular derivatives} , \quad (28)$$

where  $\bar{a}$ ,  $\bar{b}$ ,  $\bar{d}$ ,  $a$ , and  $b$  are functions of  $r$  and  $t$ . The important point here is that the second factor in the (first) operator product does not contain a term without a derivative. A similarity property is then implied, because we may obtain a solution by requiring  $\psi(u)$  to satisfy

$$(a\partial_t + b\partial_r)\psi(u) = 0 \quad (29)$$

with  $u$  independent of  $\vartheta$  and  $\varphi$ . This is a first-order equation that can be solved by the method of characteristics, i.e., by finding a coordinate  $v$  satisfying

$$\frac{\partial t}{\partial v} = a , \quad \frac{\partial r}{\partial v} = b \quad \Rightarrow \quad \partial_v \psi(u) = 0 \quad (30)$$

and hence  $\psi(u)$  is constant as  $v$  varies.  $u$  is the second characteristic coordinate and determined in the solution process. Obviously, to verify whether the wave equation has this factorization property, it is sufficient to consider the operator  $\square^{(tr)}$  obtained from the d'Alembertian by dropping the terms containing angular derivatives (the action of which on  $\psi(u)$  will always give zero,  $u$  being a function of  $t$  and  $r$  only).

In flat spacetime, we have  $\square^{(tr)} = \frac{1}{r^2}\partial_r r^2 \partial_r - \frac{1}{c^2}\partial_t^2$  and setting  $B(r) = \frac{1}{r}$ , we may easily verify that

$$\square^{(tr)} \frac{1}{r} = \frac{1}{r} \left( \partial_r - \frac{1}{c} \partial_t \right) \left( \partial_r + \frac{1}{c} \partial_t \right) , \quad (31)$$

which proves the similarity property. The characteristic equations  $\partial t / \partial v = \frac{1}{c}$ ,  $\partial r / \partial v = 1$  with initial conditions  $t(v = 0) = u$  and  $r(v = 0) = 0$  are solved by  $r = v$  and  $u = t - \frac{r}{c}$ , so  $\psi(u) = \psi\left(t - \frac{r}{c}\right)$ . Because the last two factors in Eq. (31) commute, we may even infer the general spherically symmetric solution to the flat-space scalar wave equation, which is given by  $\chi(r, t) = \frac{1}{r} \left[ \psi\left(t - \frac{r}{c}\right) + \tilde{\psi}\left(t + \frac{r}{c}\right) \right]$ .

### 3.6 Dynamical postulate

The demonstration of (31) requires that the wave speed  $c$  is constant, i.e., the medium in which the wave propagates must either be *vacuum* or at least *homogeneous* (have a spatially constant index of refraction). In a curved spacetime, the wave speed normally varies as a function of position, so we do not expect the existence of general shape-preserving solutions to survive. In fact, Ref. [1] gives as one reason for the appearance of wave tails, i.e., the invalidation of Huygens' principle "backscattering off potentials and/or spacetime curvature".

On the other hand, Eq. (12) tells us that in the Rindler metric one-dimensional shape-preserving waves<sup>11</sup> exist in spite of the fact that the wave speed varies locally.<sup>12</sup> Moreover, we know that in a *local freely falling* frame, if it is small enough, light waves will behave according to SR, i.e., as in flat spacetime, so the shape-preservation property of spherically symmetric waves should hold in vacuum. It must get lost in curved spacetime in general, but what about a globally spherically symmetric situation?

Consider a spherical wave front outside of our mass distribution, moving outward from the center of gravity. For any local inertial system, i.e., sufficiently small freely falling system, passed by the wave, it will be either locally planar (if the radius of curvature of the front is large compared with the local system) or a section of a spherical wave, propagating without distortion. This is required by the EP. Now consider the *spherically symmetric extension* of this local inertial system, i.e., the union of all local inertial systems obtained from it by rotations about the center of symmetry. A frame of reference is defined by (the non-intersecting worldlines of) a collection of (point-like) observers [26], so the union of the freely falling observers in the spherical shell obtained by these symmetry operations constitutes a *new frame of reference*, but one that obviously is no longer inertial. Observers a large angular distance apart will not perceive themselves at rest with respect to each other.<sup>13</sup>

The many pieces of the wave seen by these observers combine into a single spherical wave in their common non-inertial frame of reference. It then seems difficult to conceive of this spherical wave as *not* being shape-preserving at least for a short time interval during which neighboring observers consider each other inertial. After all, the wave travels in a shape-preserving manner in each local inertial system. Therefore, we might expect it to be true for *any*

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<sup>11</sup>In one space dimension, the existence of undistorted waves is related to conditions with velocity or time-derivative initial data. Via integration by parts the derivative may be shifted over to the Green's function, so the Heaviside function becomes a  $\delta$  function, and the waveform is just the function multiplying that  $\delta$  function.

<sup>12</sup>From Eq. (3) we find, setting  $ds^2 = 0$ , that photons traveling parallel to the  $x$  axis have velocities  $\pm gx/c$ .

<sup>13</sup>Two observers near the equator with angular coordinates differing by  $180^\circ$  will fall in opposite directions.



metric that the considered union of wave sections into a single wave will, by symmetry, produce a shape-preserving solution to the wave equation in the extended freely falling system. This would be satisfactory but of course not constrain the metric. As it turns out, it is *not* true for all spherically symmetric metrics. But how can the symmetry argument fail?

An underlying cause for its failure could be that the particular  $r$  dependent prefactor of the solution needed for the factorization property of the d'Alembertian is not realized. For reasons of energy conservation, we would expect the prefactor of a small-amplitude shape-preserving wave to be inversely proportional to the square root of the surface of the spherical shell being traversed. In Euclidean space (a flat spatial section of Minkowski spacetime) this gives the  $1/r$  prefactor required by Eq. (31). How will Euclidean space be modified, if we put a localized spherically symmetric mass distribution at the center of the wave?

On the one hand, since  $f(r)$  and  $h(r)$  will depend on that mass, the radial proper distance from the symmetry center to the wave front will be altered.<sup>14</sup> On the other hand, the proper surface of the spherical wave as measured by CSOs (and also by radially freely falling observers), would be *unchanged* by insertion of mass at the center of the wave. Then, while shape distorting effects due to the modification of radial proper distance should go away in a freely falling system, no distortions would be expected anyway from the azimuthal geometry. The wave, as observed in the freely falling spherical shell should essentially behave as a wave in Euclidean space, at least for a short time and in a small radial interval.

Therefore, given that the existence of shape-preserving waves is not assured *mathematically* in spherically symmetric metrics, it seems reasonable to *postulate* it to be true for the *physical* metric. To be precise, our second postulate – P2 – is that *in a frame describing a freely falling thin spherical shell*, which is local in the time and radial directions but global in the angular directions (so the frame is non-inertial), *the wave equation has shape-preserving solutions of arbitrary wave form*. Differently stated, we require a class of local-inertial-frame wave solutions that are valid during a short time interval, to be extensible to solutions global in  $\vartheta$  and  $\varphi$  by spherical symmetry.

### 3.7 Application of the dynamical postulate

The recipe to apply this postulate then is to first transform the metric to a freely falling frame local in  $r$  and  $t$  and to calculate the d'Alembertian in this frame. The transformation being local, we need not care about integrability conditions. The equations of motion of the freely falling observers can be

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<sup>14</sup>We are not obliged to take the mass distribution so concentrated that an event horizon arises, which would render this proper distance meaningless.

obtained from the Lagrangian

$$L = \frac{1}{2} \left( -f(r) c^2 \dot{t}^2 + h(r) \dot{r}^2 + r^2 \dot{\vartheta}^2 + r^2 \sin^2 \vartheta \dot{\varphi}^2 \right) \quad (32)$$

associated with the metric (14), with overdots denoting derivatives with respect to the proper time  $\tau$ . The Lagrangian does not depend on  $t$  explicitly, so

$$\frac{\partial L}{\partial \dot{t}} = -f(r) c^2 \dot{t} \quad (33)$$

is an integral of the motion (describing energy conservation). We set

$$f(r) c \dot{t} = c_l \quad (34)$$

with a constant  $c_l$  having the dimension of a velocity. This implies  $f(r) c dt = c_l d\tau$  along the trajectory of a freely falling observer, and we require the constant  $c_l$  to be the same for all of them, due to symmetry. This defines a spherical freely falling shell in some sufficiently small  $r$  interval, say, between  $r$  and  $r + \Delta r$ . It is convenient to keep the spatial coordinates unchanged in specifying the shell. As has been mentioned before, a frame of reference is defined by the non-intersecting world lines of a collection of test particles or observers [26], so the particular choice of spatial coordinates is unimportant.<sup>15</sup> P2 then means we assume any spatiotemporal distortions to the wave shape to be removable at fixed  $r$  by measuring the shape in the proper time variable of an appropriate set of freely falling observers.

The local coordinate transformation expressing  $dt$  by  $d\tau$  produces the line element

$$ds^2 = -\frac{c_l^2}{f(r)} d\tau^2 + h(r) dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) . \quad (35)$$

The inverse and the determinant of this local metric read

$$\left( g_{(l)}^{ij} \right) = \text{diag} \left( -\frac{f(r)}{c_l^2}, \frac{1}{h(r)}, \frac{1}{r^2}, \frac{1}{r^2 \sin^2 \vartheta} \right), \quad g_{(l)} = -c_l^2 \frac{h(r)}{f(r)} r^4 \sin^2 \vartheta, \quad (36)$$

and the d'Alembertian becomes

$$\begin{aligned} \square_{(l)} &= \frac{1}{\sqrt{|g_{(l)}|}} \partial_i \sqrt{|g_{(l)}|} g_{(l)}^{ij} \partial_j \\ &= -\frac{f(r)}{c_l^2} \partial_\tau^2 + \sqrt{\frac{f}{h}} \frac{1}{r^2} \partial_r \sqrt{\frac{h}{f}} r^2 \frac{1}{h} \partial_r + \frac{1}{r^2 \sin \vartheta} \partial_\vartheta \sin \vartheta \partial_\vartheta + \frac{1}{r^2 \sin^2 \vartheta} \partial_\varphi^2. \end{aligned} \quad (37)$$

<sup>15</sup>Moreover, if we imagine our observers to be close to the apex of their free fall, all of them will have small velocities with respect to CSOs, and the spatial coordinates of the static frame will be almost constant in the falling shell frame during the short time interval considered.

Again, we may restrict attention to the time and radial derivatives

$$\square_{(l)}^{(\tau r)} = -\frac{f(r)}{c_l^2} \partial_\tau^2 + \frac{1}{h} \partial_r^2 + \frac{1}{h} \left( \frac{2}{r} - \frac{(fh)'}{2fh} \right) \partial_r, \quad (38)$$

where a prime denotes a derivative with respect to  $r$ .

The second step is to require  $\square_{(l)}^{(\tau r)}$  to have the factorization property

$$\square_{(l)}^{(\tau r)} B(r) = \frac{B(r)}{h(r)} (\partial_r + \bar{a} \partial_\tau + \bar{b}) (\partial_r - a \partial_\tau). \quad (39)$$

It is obvious that the functions  $\bar{a}$ ,  $\bar{b}$ , and  $a$  cannot depend on  $\tau$ . Moving all the derivatives to the right on both sides of Eq. (39), we obtain the following set of equations for the coefficient functions:

$$a(r) = \bar{a}(r), \quad (40)$$

$$\bar{a}(r) a(r) = \frac{f(r)h(r)}{c_l^2}, \quad (41)$$

$$a'(r) + \bar{b}(r)a(r) = 0, \quad (42)$$

$$\bar{b}(r) = \frac{2}{r} - \frac{(fh)'}{2fh} + \frac{2B'(r)}{B(r)}, \quad (43)$$

$$0 = \frac{2B'(r)}{rB(r)} - \frac{(fh)'}{2fh} \frac{B'(r)}{B(r)} + \frac{B''(r)}{B(r)}. \quad (44)$$

The first two equations are solved by  $a = \bar{a} = \pm\sqrt{f\bar{h}}/c_l$ , which on insertion in the third produces  $\bar{b} = -(fh)'/2fh$ . This leads to a major simplification of the fourth equation, giving  $B'/B = -1/r$  and  $B = c_1/r$  with some constant  $c_1 \neq 0$ . Then the first and third terms on the right-hand-side of the fifth equation cancel and we obtain

$$(f(r)h(r))' = 0 \quad (45)$$

as a condition that the metric must satisfy. This is a second equation for the two functions  $f(r)$  and  $h(r)$ . Together with Eq. (22) and the boundary conditions (15), we have enough information to determine both  $f$  and  $h$ . Eq. (45) implies  $f(r)h(r) = \text{const.}$  and the constant must be 1, due to Eqs. (15). Therefore,  $h(r) = 1/f(r)$ . Plugging this into (22), we find

$$f'(r) = \frac{2GM}{r^2 c^2}, \quad (46)$$

which is easily integrated. The constant of integration follows from the boundary conditions (15) once more and we end up with

$$f(r) = 1 - \frac{2GM}{rc^2}, \quad h(r) = \left( 1 - \frac{2GM}{rc^2} \right)^{-1}. \quad (47)$$

This completes the determination of the metric, which turns out to produce the standard form

$$ds^2 = - \left( 1 - \frac{2GM}{rc^2} \right) c^2 dt^2 + \frac{1}{1 - \frac{2GM}{rc^2}} dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \quad (48)$$

of the Schwarzschild line element. Note that this is the correct line element outside of *any* bounded spherically symmetric mass distribution, losing its validity only where the mass density is non-zero. For a “point mass”, it holds everywhere except at  $r = 0$  (and  $r = r_s$ , which is just a coordinate singularity).

It may be useful to have a short look at the functional form of our spherical wave. The equation  $(\partial_r - a\partial_\tau)\psi(u) = 0$  is solved by any function  $\psi(t + r/c_l)$  [ $\psi(t - r/c_l)$ ], for  $a = 1/c_l$  [ $a = -1/c_l$ ], hence  $dr/d\tau = \pm c_l$  is the constant radial velocity of light in our freely falling frame. That it is not  $\pm c$  is simply due to the fact that we did not rescale  $r$  in the coordinate transformation.  $dr$  is not a radial proper length increment. The amplitude of the wave decays as  $1/r$ , as the solution for  $B(r)$  displays, and this is due to the fact that the proper surface of a sphere with radius  $r$  is indeed  $4\pi r^2$ .

While our task of obtaining the Schwarzschild metric without using the field equations has been accomplished, doubts might remain that we have utilized some particular property of standard Schwarzschild coordinates inadvertently, producing a fortuitous agreement. Then again, our postulates are of a physical nature and should therefore work in *any* appropriate coordinate system. To somewhat solidify this argument, I use the same approach in the appendix to derive the Schwarzschild metric in two further coordinate systems, one with a different radial, the other with a different time coordinate.

In the main text, we will stick to standard Schwarzschild coordinates and briefly discuss how P1 and P2 are related to the field equations.

## 4 Relationship between the field equations and the postulates

For the general metric described by Eq. (14), the mixed components of the Ricci tensor take the form<sup>16</sup>

$$R_t^t = \frac{1}{fh} \left( \frac{f'h'}{4h} - \frac{f''}{2} + \frac{f'^2}{4f} - \frac{f'}{r} \right), \quad (49)$$

$$R_r^r = \frac{1}{fh} \left( \frac{f'h'}{4h} - \frac{f''}{2} + \frac{f'^2}{4f} + \frac{h'f}{hr} \right), \quad (50)$$

$$R_\vartheta^\vartheta = \frac{1}{rh} \left( \frac{h'}{2h} - \frac{f'}{2f} + \frac{h-1}{r} \right), \quad (51)$$

$$R_\varphi^\varphi = R_\vartheta^\vartheta, \quad (52)$$

with all other components being equal to zero. For a vacuum solution, the field equations reduce to the statement that the Ricci tensor equals zero. Hence, the nontrivial elements in Eqs. (49) through (52) must also be zero, and this determines the functions  $f(r)$  and  $h(r)$ . At first sight, three of these four equations might be independent, so solvability by two functions is not evident.

According to Eq. (26), combined with the representation of  $\partial_r \Phi$  in terms of  $f$  [Eq. (17)], postulate P1 may be written as

$$0 = \partial_r \sqrt{fhr^2} \frac{1}{h} \frac{f'}{f} = \partial_r \frac{f'r^2}{\sqrt{fh}} = -\frac{2r^2}{\sqrt{fh}} \left( -\frac{f''}{2} - \frac{f'}{r} + \frac{f'^2}{4f} + \frac{f'h'}{4h} \right) = -2r^2 \sqrt{fh} R_t^t. \quad (53)$$

Clearly,  $R_t^t = 0$  implies postulate P1, which therefore follows from the field equations. Moreover, the difference of Eqs. (49) and (50) is

$$R_t^t - R_r^r = -\frac{(fh)'}{fh^2r}, \quad (54)$$

so if both  $R_t^t = 0$  and  $R_r^r = 0$ , then  $(fh)' = 0$ , which is Eq. (45), the condition that guarantees the factorization property (39) to hold. Hence, vanishing of the difference of these two elements of the Ricci tensor implies postulate P2, which means that P2 is also a consequence of the vacuum field equations.

Conversely, this also shows that if P1 and P2 hold, then we will certainly have  $R_t^t = 0$  and  $R_t^t - R_r^r = 0$ , wherefrom  $R_r^r = 0$ , so two out of the three vacuum field equations are satisfied. It is not immediately obvious that  $R_\vartheta^\vartheta$  must vanish, too. In fact, in order to show  $R_\vartheta^\vartheta = 0$  we need, besides P1 and P2, the boundary conditions at infinity for one of the functions  $h$  and  $f$  at least,

<sup>16</sup>The Ricci tensor was calculated via computer algebra, employing the differential geometry package of Maple 17.

so the field equations do not follow from P1 and P2 alone. To see this, we first note that P2 implies

$$\frac{f'}{f} = -\frac{h'}{h}, \quad (55)$$

so  $R_r^r$  and  $R_{\vartheta}^{\vartheta}$  simplify

$$R_r^r = \frac{1}{fh} \left( -\frac{f''}{2} + \frac{h'f}{hr} \right), \quad (56)$$

$$R_{\vartheta}^{\vartheta} = \frac{1}{rh} \left( \frac{h'}{h} + \frac{h-1}{r} \right) = \frac{1}{r^2} \frac{d}{dr} \left[ \left( 1 - \frac{1}{h} \right) r \right]. \quad (57)$$

Taking the derivative of Eq. (55), we may express  $R_r^r$  in terms of  $h$  alone

$$\frac{f''}{f} = -\frac{h''}{h} + 2\frac{h'^2}{h^2}, \quad (58)$$

$$R_r^r = \frac{h''}{2h^2} - \frac{h'^2}{h^3} + \frac{h'}{h^2 r} = \frac{1}{2r^2} \frac{d}{dr} \frac{h'r^2}{h^2}. \quad (59)$$

Since P1 and P2 imply that  $R_r^r$  vanishes, we have  $h'/h^2 = A_1/r^2$  with some constant  $A_1$ . Integration yields  $1/h = A_2 + A_1/r$ . Using the boundary condition for  $h$  at infinity we have  $A_2 = 1$  and  $1 - 1/h = -A_1/r$ , so that  $(1 - 1/h)r$  is a constant and Eq. (57) shows that  $R_{\vartheta}^{\vartheta}$  is zero. Hence, P1 and P2 together with the boundary condition for  $h$  at infinity imply Einstein's vacuum field equations in the spherically symmetric case.

## 5 Conclusions

What has been shown here is that the role of the field equations in the derivation of the Schwarzschild metric can be taken by appropriate postulates instead. This may be viewed in two different ways.

One is to just emphasize that *some* additional element beyond SR, EP, and NL is needed in order to calculate a true gravitational field. Of course, that much was known already from the impossibility proofs mentioned in the introduction [30, 15]. However, since there are still a few indefatigable seekers of simpler ways to arrive at spacetime curvature and to explain everything from little more than SR,<sup>17</sup> it may be useful to show by way of an explicit example what it actually takes to obtain a fundamental result that otherwise is provided by the field equations.

In this view, the precise nature of the postulates needed to go beyond the three ostensibly necessary ingredients may not appear important. But then

<sup>17</sup>In the question and answers threads of ResearchGate (<http://www.researchgate.net/topics>), extensive discussions of GR can be found demonstrating strong interest in, and poor understanding of, how the theory extends its scope beyond SR.

the whole exercise would seem unnecessary, because postulational approaches to the Schwarzschild metric have been given before [33, 5]. Instead, one of the motivations of this work was to develop postulates that are physically plausible, are not *ad hoc* and could be found without prior knowledge of the field equations. That is, they might have been used as foundations, on which the deduction of elements of the theory was based, rather than as assertions that *have* to be derived from more fundamental axioms.<sup>18</sup>

Clearly, these requirements do not lead to a unique set of postulates. However, *none* of the other three ingredients used is an absolute necessity in the development of GR either. For SR and EP, this is nicely demonstrated in a pedagogical paper by Rindler [24], where he muses how Riemann might have developed GR in 1854, at least to the level of the vacuum field equations. Special relativity could then have been obtained before Einstein, simply by considering the flat-spacetime limit of the general theory.

A second way to view what has been achieved here is that an answer is provided to the question posed in the first paragraph of the introduction. Einstein aimed high, he wanted to develop a general theory of gravity from the start. He managed to do so, but the process was laborious. Suppose he had attempted a step-by-step approach, trying to build the theory for a point mass first, in order to gain intuition for the field theory he was after. Would he have succeeded and would this simpler theory have been useful?

The first of these questions refers to whether P1 and P2 are well-motivated enough to find them without the field equations as a guide. It may be helpful to remember that in the years between 1907 and 1915, Einstein tried out various assumptions in kind of a “tinker’s approach” [24]. Postulates, before they can be tested experimentally, are mostly based on beliefs about the properties a theory should have.

P1 expresses the belief that the theory of gravitation should not have gravitational sources or sinks in vacuum, i.e., that field lines do not end in empty space.<sup>19</sup> The only way this may *not* be satisfied in a *classical* theory<sup>20</sup> is that an *additional* field (creating sources/sinks) permeates vacuum, which is indeed the case in the BD theory [2]. Einstein would have discarded this possibility

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<sup>18</sup>Of course, the restriction to spherical geometry reduces their fundamentality. But P1 is definitely valid beyond the spherically symmetric case, and for P2 it seems at least possible to extend the argument concerning the correctness of the prefactor to more general geometries.

<sup>19</sup>The field line concept is applicable to GR in the weak-field limit, since it is applicable to the NL. Moreover, there is no obvious way to introduce some threshold value of the field strength, beyond which it might become inapplicable, so field lines may be used to visualize strong fields as well, the nonlinearity and geometric foundation of the theory notwithstanding.

<sup>20</sup>Quantum mechanics is a different matter, where we may have the vision of virtual particles popping in and out of existence and conferring medium-like properties to vacuum. In a classical theory, vacuum is just empty space.

for reasons of simplicity and so would have found P1 without any doubt.

P2 expresses the belief that, given waves to exist (due to the EP) which travel distortion-free in a local freely falling frame, this property may be continued to the spherical extension of such a frame by means of spherical symmetry. That is, we assume that the spherical continuation of a local solution allows us to predict the short-time wave behavior in a particular non-inertial system. Whereas P1 is grounded in solid physics, P2 is motivated geometrically in part, but above all, it has a certain esthetic appeal. Einstein believed in symmetry and beauty, so he would have found P2 or something similar.

Would it have been useful? Most certainly. With the Schwarzschild metric at hand, he could have postdicted Mercury's perihelion precession, which would have convinced him of the correctness of the solution. Then he might have found a less contorted path to the correct field equations than he actually did, being able to recognize erroneous results more easily by checking them against the spherically symmetric case. And of course, we would not talk of the "Schwarzschild" metric nowadays. . .

The approach to the gravitational field of a spherically symmetric mass distribution presented here lays emphasis on tradition rather than revolution. I have tried to use, in the postulates, physical ideas mostly, and no more than the absolutely necessary minimum of geometry. For someone fully acquainted with Riemannian geometry, Einstein's postulates leading to the field equations can hardly be beaten as regards their beauty and simplicity.<sup>21</sup> But one has to embrace the geometrical point of view to begin with. For some researchers, it may be easier to visualize force fields and wave phenomena in space than to emphasize concepts of non-Euclidean geometry. As long as there are no dynamical changes of topology, thinking of GR in terms of a tensor field on a fixed background could have its uses, removing a thought obstacle to quantizing the theory (but of course not the technical difficulties). The methods employed here may have useful generalizations.

Note that most of this exposition consists in motivating and developing the two postulates. Once they are accepted, they provide a pretty fast calculational approach to the Schwarzschild metric, much faster than the calculation in the old tensor formalism and still competitive when compared with a modern differential geometric calculation using the Cartan formalism [14]. This may be seen by examining Appendix B that shows how the Schwarzschild metric is obtained for Gullstrand-Painlevé coordinates, in barely three pages. The calculation in standard Schwarzschild coordinates is yet more concise. The brevity of the modern calculation [14] is purchased by acquiring sufficient dif-

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<sup>21</sup>Even though Einstein himself later may have believed that he got to the field equations via an action principle in the first place, which was not the case. The action principle approach is even more elegant than Einstein's postulates applying directly to the field tensor. It is a little more distant to physical intuition as well.



ferential geometry first, whereas the present method does not need any tools beyond standard calculus.

Finally, the two postulates teach us something about GR itself, and at least some of these results seem new. Whereas the validity of P1 is pretty well-known – it has found use in the application of concepts such as the “force at infinity” and is already implicit in the approaches of Refs. [33, 5] – I have not seen a formulation of P2 before. The truth of this postulate gives us new insights about the behavior of waves in spherically symmetric gravitational fields. In particular, it shows that Einstein’s GR is distinguished among all metrical theories of gravitation in yet another way: it permits certain energy-carrying spherical waves to propagate, in a sense, in the least distorted way compatible with spacetime curvature. Knowledge about P2 could also be useful as the starting point of a post-Newtonian perturbative calculation of wave phenomena in the Schwarzschild metric.

## A Isotropic coordinates

Isotropic coordinates are characterized by the condition  $\tilde{h}(\tilde{r}) = \tilde{n}(\tilde{r})$  in Eq. (13), which we impose after having removed the off-diagonal term containing  $\tilde{k}(\tilde{r})$ . We rename  $\tilde{r}$  to  $\rho$ ,  $\tilde{f}$  to  $F$  and  $\tilde{h}$  to  $H$ , whence Eq. (13) turns into

$$ds^2 = -F(\rho)c^2 dt^2 + H(\rho) [d\rho^2 + \rho^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)] \quad (\text{A1})$$

and we take  $\lim_{\rho \rightarrow \infty} F(\rho) = 1$ ,  $\lim_{\rho \rightarrow \infty} H(\rho) = 1$ , requiring the metric to become Minkowskian at infinity. We are entitled to assume that  $\rho$  approaches the standard flat space radial coordinate  $r$  as  $\rho \rightarrow \infty$ .<sup>22</sup> The inverse and the determinant of the metric are

$$(g^{ij}) = \text{diag} \left( -\frac{1}{F(\rho)c^2}, \frac{1}{H(\rho)}, \frac{1}{H(\rho)\rho^2}, \frac{1}{H(\rho)\rho^2 \sin^2 \vartheta} \right), \quad (\text{A2})$$

$$g = -c^2 F(\rho) H(\rho)^3 \rho^4 \sin^2 \vartheta. \quad (\text{A3})$$

Again, we set  $F(\rho) = e^{2\Phi(\rho)/c^2}$  and may interpret, by virtue of the EP,  $\Phi(\rho)$  as a potential allowing us to calculate the proper acceleration of CSOs, once  $H(\rho)$  is known. The Newtonian limit then provides us with the asymptotic behavior  $F(\rho) \sim 1 - 2GM/\rho c^2$  ( $\rho \rightarrow \infty$ ), because  $\rho \sim r$  ( $\rho \rightarrow \infty$ ).

The radial part of the d’Alembertian is given by

$$\square^{(\rho)} = \frac{1}{\sqrt{F(\rho)H(\rho)^3\rho^2}} \partial_\rho \sqrt{F(\rho)H(\rho)^3\rho^2} \frac{1}{H(\rho)} \partial_\rho \quad (\text{A4})$$

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<sup>22</sup> $\rho$  can be rescaled by an arbitrary constant factor in the metric without destroying its isotropy, so we might require  $\rho \sim \alpha r$  ( $\rho \rightarrow \infty$ ) with some positive constant  $\alpha$  instead.

and postulate P1 requires  $\square \Phi(\rho) = \square^{(\rho)} \Phi(\rho) = 0$ , leading to

$$\sqrt{F(\rho)H(\rho)}\rho^2 \frac{F'(\rho)}{F(\rho)} = \text{const.} \quad (\text{A5})$$

Using the limiting behaviors of  $F(\rho)$  and  $H(\rho)$  for large  $\rho$ , we evaluate the constant to be  $2GM/c^2$  and arrive at

$$\frac{F'(\rho)}{\sqrt{F(\rho)}} = \frac{2GM}{c^2 \rho^2 \sqrt{H(\rho)}}. \quad (\text{A6})$$

To apply postulate P2, we first have to transform to the proper time of a freely falling frame, using energy conservation

$$F(\rho) c \dot{t} = c_l = \text{const.} \quad (\text{A7})$$

The transformed metric, its inverse and determinant read

$$(g_{(l)ij}) = \text{diag} \left( -\frac{c_l^2}{F(\rho)}, H(\rho), H(\rho)\rho^2, H(\rho)\rho^2 \sin^2 \vartheta \right), \quad (\text{A8})$$

$$(g^{(l)ij}) = \text{diag} \left( -\frac{F(\rho)}{c_l^2}, \frac{1}{H(\rho)}, \frac{1}{H(\rho)\rho^2}, \frac{1}{H(\rho)\rho^2 \sin^2 \vartheta} \right), \quad (\text{A9})$$

$$g_{(l)} = -c_l^2 \frac{H(\rho)^3}{F(\rho)} \rho^4 \sin^2 \vartheta, \quad (\text{A10})$$

and the temporal plus radial parts of the d'Alembertian in the freely falling system are obtained as

$$\begin{aligned} \square_{(l)}^{(\tau\rho)} &= -\frac{F(\rho)}{c_l^2} \partial_\tau^2 + \sqrt{\frac{F}{H^3}} \frac{1}{\rho^2} \partial_\rho \sqrt{\frac{H^3}{F}} \rho^2 \frac{1}{H} \partial_\rho \\ &= -\frac{F(\rho)}{c_l^2} \partial_\tau^2 + \frac{1}{H} \partial_\rho^2 + \frac{1}{H} \left( \frac{2}{\rho} - \frac{F'}{2F} + \frac{H'}{2H} \right) \partial_\rho. \end{aligned} \quad (\text{A11})$$

We require the factorization

$$\square_{(l)}^{(\tau\rho)} B(\rho) = \frac{B(\rho)}{H(\rho)} (\partial_\rho + \bar{a} \partial_\tau + \bar{b}) (\partial_\rho - a \partial_\tau) \quad (\text{A12})$$

to hold, which gives

$$\bar{a} = a, \quad (\text{A13})$$

$$\bar{a}a = \frac{FH}{c_l^2} \Rightarrow a = \pm \frac{\sqrt{FH}}{c_l}, \quad (\text{A14})$$

$$a' + a\bar{b} = 0 \Rightarrow \bar{b} = -\frac{(FH)'}{2FH}, \quad (\text{A15})$$

$$\bar{b} = \frac{2}{\rho} - \frac{F'}{2F} + \frac{H'}{2H} + \frac{2B'}{B}, \quad (\text{A16})$$

$$0 = \left( \frac{2}{\rho} - \frac{F'}{2F} + \frac{H'}{2H} \right) \frac{B'}{B} + \frac{B''}{B}. \quad (\text{A17})$$

Combining Eqs. (A15) and (A16) and simplifying Eq. (A17) we have

$$0 = \frac{2}{\rho} + \frac{H'}{H} + \frac{2B'}{B}, \quad (\text{A18})$$

$$0 = \frac{2}{\rho} - \frac{F'}{2F} + \frac{H'}{2H} + \frac{B''}{B'}. \quad (\text{A19})$$

Either equation can be integrated once to yield  $B = c_1/\rho\sqrt{H}$ ,  $B' = c_2\sqrt{F}/\rho^2\sqrt{H}$ , and the large  $\rho$  asymptotics leads to  $B \sim c_1/\rho$  and  $B' \sim c_2/\rho^2$  from which we may immediately conclude that  $c_2 = -c_1$ , hence  $B'/B = -\sqrt{F}/\rho$ , which allows us to eliminate  $B'/B$  from Eq. (A18) and to write down a second equation for the two functions  $F$  and  $H$ :

$$\sqrt{F(\rho)} = 1 + \frac{\rho H'(\rho)}{2H(\rho)}. \quad (\text{A20})$$

The system of equations (A6) and (A20) is much more difficult to solve than the corresponding equations in standard Schwarzschild coordinates. Nevertheless, an analytic solution can be found by a series of clever transformations. First, we note that taking the derivative of (A20), we generate half the left-hand side of (A6) and therefore can eliminate  $F$ :

$$\frac{d}{d\rho} \frac{\rho H'}{2H} = \frac{GM}{c^2 \rho^2 \sqrt{H}} = \frac{r_s}{2\rho^2 \sqrt{H}}, \quad (\text{A21})$$

which is a nonlinear second-order differential equation for  $H$ . Next, we introduce a new variable  $x$  and a function  $Y(x)$  by the requirements

$$x = \rho\sqrt{H}, \quad \sqrt{H}d\rho = \sqrt{Y}dx. \quad (\text{A22})$$

The asymptotic behavior of  $H$  for  $\rho \rightarrow \infty$  then implies  $x \sim \rho$  ( $\rho \rightarrow \infty$ ) and  $Y(x) \sim 1$  ( $x \rightarrow \infty$ ). Taking the derivative of  $x$  w.r.t.  $\rho$ , we get

$$\frac{dx}{d\rho} = \sqrt{H} \left( 1 + \frac{\rho H'}{2H} \right) \Rightarrow 1 + \frac{\rho H'}{2H} = \frac{1}{\sqrt{Y}} \quad (\text{A23})$$

and Eq. (A21) turns into

$$\frac{1}{\sqrt{Y}} \frac{d}{dx} \frac{1}{\sqrt{Y}} = -\frac{Y'(x)}{2Y^2} = \frac{r_s}{2x^2}, \quad (\text{A24})$$

which may be integrated, using  $\lim_{x \rightarrow \infty} Y(x) = 1$ , to give

$$Y(x) = \frac{1}{1 - r_s/x}. \quad (\text{A25})$$

From Eq. (A22), we get

$$\frac{x}{\rho} d\rho = \sqrt{Y} dx \quad \Rightarrow \quad \frac{d\rho}{\rho} = \frac{dx}{\sqrt{x^2 - r_s x}}, \quad (\text{A26})$$

which can be integrated immediately

$$\ln \rho + \tilde{A} = \operatorname{arcosh} \left( \frac{2x}{r_s} - 1 \right). \quad (\text{A27})$$

$\tilde{A} \equiv \ln A$  is a constant of integration. Inversion is achieved by simply applying the cosh function to both sides

$$\frac{2x}{r_s} - 1 = \frac{1}{2} \left( e^{\ln(\rho A)} + e^{-\ln(\rho A)} \right) = \frac{1}{2} \left( \rho A + \frac{1}{\rho A} \right). \quad (\text{A28})$$

Taking the limit  $\rho \rightarrow \infty$ , we see that  $A = 4/r_s \equiv 1/\rho_s$ . Replacing  $x$  with  $\rho\sqrt{H}$  again, we find

$$\rho\sqrt{H} = \frac{r_s}{2} + \rho + \frac{r_s^2}{16\rho} = \rho \left( 1 + 2\frac{\rho_s}{\rho} + \frac{\rho_s^2}{\rho^2} \right), \quad (\text{A29})$$

hence

$$H(\rho) = \left( 1 + \frac{\rho_s}{\rho} \right)^4. \quad (\text{A30})$$

Noting that  $F = 1/Y = 1 - r_s/\rho\sqrt{H}$  [from Eqs. (A23) and (A20)], we finally obtain

$$F(\rho) = \frac{(1 - \rho_s/\rho)^2}{(1 + \rho_s/\rho)^2}. \quad (\text{A31})$$

The resulting line element

$$ds^2 = - \left( \frac{1 - \rho_s/\rho}{1 + \rho_s/\rho} \right)^2 c^2 dt^2 + \left( 1 + \frac{\rho_s}{\rho} \right)^4 \left[ d\rho^2 + \rho^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \right] \quad (\text{A32})$$

agrees with the form of the isotropic Schwarzschild line element found in the literature [18]. Note that this metric does not cover the part of spacetime inside the event horizon.

## B Painlevé-Gullstrand coordinates

Here we consider yet another choice of the independent functions in the metric (13), requiring  $\tilde{h}(\tilde{r}) = \tilde{n}(\tilde{r}) = 1$ . Since this reduces the number of independent

functions to two, we have no freedom left to set  $\tilde{k}(\tilde{r})$  equal to zero. Hence, our time coordinate is different from the one(s) in the previous examples, because we have not performed the transformation to a time-orthogonal frame. We rename  $\tilde{r} = r$ ,  $\tilde{t} = T$ ,  $\tilde{f} = f$ , and  $\tilde{k} = k$ . Because  $\tilde{n}(r) = 1$ ,  $r$  is indeed the same radial coordinate as in Sec. 3. The metric, its inverse and its determinant are

$$(g_{ij}) = \begin{pmatrix} -f(r)c^2 & k(r)c & 0 & 0 \\ k(r)c & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \vartheta \end{pmatrix}, \quad (\text{B1})$$

$$(g^{ij}) = \begin{pmatrix} -\frac{1}{(f+k^2)c^2} & \frac{k}{(f+k^2)c} & 0 & 0 \\ \frac{k}{(f+k^2)c} & \frac{f}{(f+k^2)} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2 \sin^2 \vartheta} \end{pmatrix}, \quad (\text{B2})$$

$$g = -(f + k^2) c^2 r^4 \sin^2 \vartheta. \quad (\text{B3})$$

The boundary conditions at infinity read  $\lim_{r \rightarrow \infty} f(r) = 1$ ,  $\lim_{r \rightarrow \infty} k(r) = 0$ . Now, the d'Alembertian contains mixed terms involving  $\partial_T \partial_r$ , but these do not play any role in the application of postulate P1, where the presence of any other derivative than  $\partial_r$  in a term makes it disappear. We only need  $\square^{(r)}$ . Setting  $f(r) = e^{2\Phi(r)/c^2}$  and requiring

$$\square \Phi(r) = \square^{(r)} \Phi(r) = \frac{1}{\sqrt{f + k^2} r^2} \partial_r \frac{r^2 f}{\sqrt{f + k^2}} \partial_r \Phi(r) = 0, \quad (\text{B4})$$

we find

$$\partial_r \Phi(r) = \frac{c^2 f'}{2 f} = \frac{A \sqrt{f + k^2}}{r^2 f}. \quad (\text{B5})$$

Taking advantage of the limiting behavior of  $f$  and  $k$  at large  $r$  as well as the Newtonian limit, we determine the constant  $A = GM = c^2 r_s / 2$ . Thus we obtain

$$f' = \sqrt{f + k^2} \frac{r_s}{r^2}, \quad (\text{B6})$$

a first relationship between the functions  $f$  and  $k$ .

To transform the metric using the proper time of freely falling observers, we note that stationarity again implies a conservation law, but now the Lagrangian involves the product  $\dot{T} \dot{r}$ , so the law takes a slightly different form

$$f c \dot{T} - k \dot{r} = c_l = \text{const.} \quad \Rightarrow \quad dT = \frac{c_l}{f c} d\tau + \frac{k}{f c} dr. \quad (\text{B7})$$

We then find for the line element in the local metric

$$ds^2 = -\frac{c_l^2}{f(r)} d\tau^2 + \left(1 + \frac{k(r)^2}{f(r)}\right) dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (\text{B8})$$

i.e., transforming to a frame of freely falling observers diagonalizes the metric, not unexpectedly. The inverse and determinant of the local metric read

$$(g_{(l)}^{ij}) = \text{diag} \left( -\frac{f(r)}{c_l^2}, \frac{f(r)}{f(r) + k(r)^2}, \frac{1}{r^2}, \frac{1}{r^2 \sin^2 \vartheta} \right), \quad (\text{B9})$$

$$g_{(l)} = -c_l^2 \frac{f(r) + k(r)^2}{f(r)^2} r^4 \sin^2 \vartheta, \quad (\text{B10})$$

and the temporal plus radial parts of the local d'Alembertian become

$$\begin{aligned} \square_{(l)}^{(\tau r)} &= -\frac{f}{c_l^2} \partial_\tau^2 + \frac{f}{r^2 \sqrt{f+k^2}} \partial_r \frac{r^2}{\sqrt{f+k^2}} \partial_r \\ &= -\frac{f}{c_l^2} \partial_\tau^2 + \frac{f}{f+k^2} \partial_r^2 + \frac{f}{f+k^2} \left( \frac{2}{r} - \frac{f'+2kk'}{2(f+k^2)} \right) \partial_r. \end{aligned} \quad (\text{B11})$$

The factorization ansatz

$$\square_{(l)}^{(\tau r)} B(r) = B f (\bar{a} \partial_\tau + \bar{b} \partial_r + \bar{d}) (-a \partial_\tau + b \partial_r) \quad (\text{B12})$$

produces the following equations for the coefficient functions (with the abbreviation  $\xi = f + k^2$ )

$$\bar{a} a = \frac{1}{c_l^2} \quad \Rightarrow \quad \bar{a} = \frac{1}{c_l^2 a}, \quad (\text{B13})$$

$$\bar{a} b - \bar{b} a = 0, \quad (\text{B14})$$

$$a' \bar{b} + \bar{d} a = 0, \quad (\text{B15})$$

$$\bar{b} b = \frac{1}{\xi} \quad \Rightarrow \quad \bar{b} = \frac{1}{\xi b}, \quad (\text{B16})$$

$$\bar{b} b' + \bar{d} b = \frac{1}{\xi} \left( \frac{2}{r} - \frac{\xi'}{2\xi} + \frac{2B'}{B} \right), \quad (\text{B17})$$

$$0 = \frac{2}{r} - \frac{\xi'}{2\xi} + \frac{B''}{B'}. \quad (\text{B18})$$

Using Eqs. (B13) and (B16) in Eq. (B14), we obtain

$$\frac{b}{a} = \pm \frac{c_l}{\sqrt{\xi}}. \quad (\text{B19})$$

Equation (B15) may then be used to eliminate  $\bar{d}$  from Eq. (B17)

$$\frac{1}{\xi} \left( \frac{2}{r} - \frac{\xi'}{2\xi} + \frac{2B'}{B} \right) = \bar{b}b' - \frac{a'}{a} \bar{b}b = \frac{1}{\xi} \left( \frac{b'}{b} - \frac{a'}{a} \right), \quad (\text{B20})$$

and we can drop the factor  $1/\xi$  from this equation. Multiplying Eq. (B19) through with  $a$  and taking the derivative with respect to  $r$ , we find

$$\frac{b'}{b} = \frac{a'}{a} - \frac{\xi'}{2\xi}, \quad (\text{B21})$$

which after insertion on the right-hand side of (B20) produces

$$\frac{2}{r} + \frac{2B'}{B} = 0. \quad (\text{B22})$$

This can be integrated and yields  $B = c_1/r$ . Then  $B' = -c_1/r^2$  and  $B'' = 2c_1/r^3$ , which gives us  $B''/B' = -2/r$ . Inserting this into (B18), we end up with  $\xi' = 0 \Rightarrow \xi = f + k^2 = \text{const.}$ , and the constant is determined taking the limit  $r \rightarrow \infty$ , so we have as second equation for  $f$  and  $k$

$$f(r) + k(r)^2 = 1. \quad (\text{B23})$$

Plugging this into Eq. (B6), we get the simple equation

$$f'(r) = \frac{r_s}{r^2}, \quad (\text{B24})$$

which can be immediately integrated using the boundary condition at infinity once again, and we finally obtain:

$$f(r) = 1 - \frac{r_s}{r}, \quad k(r) = \pm \sqrt{\frac{r_s}{r}}. \quad (\text{B25})$$

The more useful ingoing Painlevé-Gullstrand coordinates are obtained for positive  $k(r)$ . The resulting line element

$$ds^2 = - \left( 1 - \frac{r_s}{r} \right) c^2 dT^2 + 2\sqrt{\frac{r_s}{r}} cdT dr + dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \quad (\text{B26})$$

agrees with the standard form of the Painlevé-Gullstrand line element found in the literature [18]. Note that had we chosen  $\tilde{h}(r) = 0$  and  $\tilde{n}(r) = 1$  at the beginning of this section, we would have obtained, by an almost identical calculation, the line element in Eddington-Finkelstein coordinates [18].

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**Received: December 15, 2016; February 17, 2017**