Theoretical Study of Diffusion Process
Around a Slowly Rotating Neutron Star

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Abstract
Relativistic diffusion process around a slowly rotating neutron star has been studied. In this article, we formulated the relativistic Fokker-Planck equation to describe the dynamic of particles undergoing diffusion around slowly rotating neutron star. This equation is derived both
in the parametrization of phase space proper time and the coordinate time. The relativistic Fokker-Planck equation is expressed in the hyperbolic coordinate system since the observation that the velocity space in special relativity is a noncompact hyperbolic 3-dimensional Riemannian manifold embedded into the 4-dimensional velocity Minkowski space.

Keywords: diffusion process, neutron star, slowly rotating neutron star

1 Introduction

Diffusion process is the motion of particles from a region of higher concentration to a region of lower concentration. In other words, diffusion is defined as the motion of a substance down a concentration gradient such as gradient of concentration, pressure, and temperature. Diffusion process is a solution of a stochastic differential equation, that is a continuous-time Markov process with almost surely continuous sample paths, for examples Brownian motion, reflected Brownian motion, and Ornstein-Uhlenbeck process. The most fundamental diffusion process is Brownian motion. Brownian motion is a sample path of a diffusion process models the trajectory of a particle embedded in a flowing fluid and subjected to random displacements due to collisions with fluid molecules. The position of the particle is random and governed by an advection-diffusion equation. Its probability density function is a function of space and time.

There are many research on relativistic diffusion, within both special and general relativity framework. Debbasch et al. [1] have introduced a relativistic diffusion in Minkowski space to describe the motion of a point of particle surrounded by a heat bath with respect to the rest-frame of the fluid, called relativistic Ornstein-Uhlenbeck process. Furthermore, the relativistic Ornstein-Uhlenbeck process has been studied more by [3], [4], [5], and [2]. Debbasch in [6] has generalized the relativistic Ornstein-Uhlenbeck process to a process on the curved manifold. Then Dunkel and Hanggi [7], [8] explained relativistic Brownian motion in Minkowski space. Independently, Frachi and Le [10] has analyzed a relativistic Diffusion on any Lorentz manifold. They have investigated the case of the Schwarzschild-Kruskal-Szekeres manifold. The case of Godels universe was recently considered in [9]. Debbasch and Rivet [2] argued qualitatively that the so-called hydrodynamical limit of relativistic Ornstein-Uhlenbeck process should behave in a Brownian way. They stress that a mathematical rigourous proof remains needed, to confirm such not much intuitive statement. In [7] and [8], Dunkel and Hanggi asked the question of the asymptotic behaviour of the variance of their diffusion. Indeed, comparing to the nonrelativistic case, and after numerical computations, they guess that this variance, normalised by time, should converge, to some constant for which
they conjecture an empirical formula. We answer here these two questions, asked by Debbasch and Rivet in [2], and by Dunkel and Hanggi in [7], [8], and indeed a more general one. Angst and Franchi in [18] established a central limit theorem for a class of Minkowskian diffusions, to which the two above mentioned ones, relativistic Ornstein-Uhlenbeck process and Dunkel-Hanggi [7] diffusion, belong.

Herrmann has studied diffusion process of both special [13] and general relativity [15]. In general relativity framework, he has derived the diffusion equation or the general relativistic Kramers equation. That equation was applicated to explain the diffusion in expanding universe. Whereas, different from Herrmann’s work [15], Calogero [16] proposed a new model to describe the dynamics of particles undergoing diffusion in general relativity framework. Calogero has derived a Fokker-Planck equation without friction on the tangent bundle spacetime to describe the evolution of the particle system. Calogero has shown the incompatibility of Fokker-Planck equation and Einstein field equation. To solve this problem he did two alternatives, that are to assume the existence of additional matter fields in spacetime which absorb the energy lost by the particles due to the action of the diffusion forces and to add a cosmological scalar field term in the left hand side of Einstein field equation.

By refering to Herrmann’s work [15], Andra and Rosyid [17] have studied the general relativistic of diffusion process around neutron star, for the case non-rotating neutron star. They have constructed the relativistic Fokker-Plack equation that describes the particles dynamic undergoing diffusion around non-rotating neutron star. Now, in this article we will continue our work to study the diffusion process around neutron star, for the case slowly rotating neutron star. Different from Calogero’s work, we supposed the diffusion materials don’t change the spacetime metric under consideration. In this work we only derived the differential equation without general solution. Furthermore (in the next article), we will apply this equation for realistic specific case via numeric computational to get the general solution of this equation.

2 Diffusion Process

Historically, the heat equation is firstly proposed by Fourier in 1822. He has applied it to investigating the temperature distribution in materials. Some years later, Brownian motion was found in 1827. Brownian motion is visualized by pollen micro particles motion. Nevertheless, the Brownian motion had not been recognized as a diffusion problem until the Einstein theory of Brown motion in 1905, although it was a typical diffusion problem. In 1855, Fick applied the heat equation to diffusion phenomena as it had been.

In probability theory, a diffusion process is a solution of a stochastic differential equation. It is a continuous-time Markov process with almost surely
continuous sample paths. Some examples of diffusion process are Brownian motion, reflected Brownian motion, and Ornstein-Uhlenbeck processes. A sample path of a diffusion process models the trajectory of a particle embedded in a flowing fluid and subjected to random displacements due to collisions with molecules, which is called Brownian motion. The position of the particle is then random; its probability density function as a function of space and time is governed by diffusion equation.

Mathematically, diffusion processes is described by diffusion equation. The diffusion equation is a partial differential equation. In physics, it describes the behavior of the collective motion of micro-particles in a material resulting from the random motion of each micro-particle. In mathematics, it is applicable in common to a subject relevant to the Markov process as well as in various other fields, such as the material sciences, information science, life science, social science, and so on. These subjects described by the diffusion equation are generally called Brown problems.

The equation of non relativistic diffusion is given by

$$\frac{\partial \Phi(r, t)}{\partial t} = \nabla \cdot \left[ D(\Phi, r) \nabla \Phi(r, t) \right],$$  

where $\Phi(r, t)$, $D(\Phi, r)$, and $\nabla$ are the density of the diffusing material at location $r$ and time $t$, the collective diffusion coefficient for density $\Phi$ at location $r$, and the vector differential operator, respectively. If the diffusion coefficient depends on the density then the equation is nonlinear, otherwise it is linear. More generally, when $D$ is a symmetric positive definite matrix, the equation describes anisotropic diffusion, which is written (for three dimensional diffusion) as:

$$\frac{\partial \Phi(r, t)}{\partial t} = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial}{\partial x_i} \left[ D_{ij}(\Phi, r) \frac{\Phi(r, t)}{x_j} \right].$$  

If $D$ is constant, then the equation reduces to the following linear differential equation:

$$\frac{\partial \Phi(r, t)}{\partial t} = D \nabla^2 \Phi(r, t),$$  

also called the heat equation.

**Fokker-Planck Equation**

Fokker-Planck equation was first proposed by Fokker and Planck to describe the Brownian motion of particles. Fokker-Planck equation is a partial differential equation that describes the time evolution of the probability density function of the velocity of particle under the influence of drag forces and random forces. The Fokker-Planck equation is just an equation of motion for the
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distribution function of fluctuating macroscopic variables. For a deterministic treatment we neglect the fluctuations of the macroscopic variables. The Fokker-Planck equation is one of the simplest equations for continuous macroscopic variables. It usually appears for variables describing a macroscopic but small subsystem, like the position and velocity for the Brownian motion of a small particle, a current in an electrical circuit, the electrical field in a laser. If the subsystem is larger the fluctuations may then usually be neglected and thus one has a deterministic equation. In these cases, however, where the deterministic equations are not stable, a stochastic description is then necessary even for large systems.

The general Fokker-Planck equation for one variable $x$ is given by (see [22])

$$\frac{\partial \Phi}{\partial t} = \left[ -\frac{\partial D^1(x)}{\partial x} + \frac{\partial^2 D^2(x)}{\partial x^2} \right] \Phi,$$

where $D^1(x)$ is the drift coefficient and $D^2(x) > 0$ is the diffusion coefficient. The drift and diffusion coefficients may also depend on time. If the drift coefficient is linear and the diffusion coefficient is constant, the equation (4) is reduced to be a special Fokker-Planck equation. Equation (4) is an equation of motion for the distribution function $\Phi(x,t)$. Mathematically, it is a linear second-order partial differential equation of parabolic type. Roughly speaking, it is a diffusion equation with an additional first-order derivative with respect to $x$. Equation (4) is also called a forward Kolmogorov equation.

3 Neutron Star

Neutron stars were first hypothesized by Landau in 1932. Landau calculated the maximum mass of white dwarfs and speculated on a possible existence of stars more compact than white dwarfs, containing matter of nuclear density. The actual theoretical prediction of neutron stars was made Baade and Zwicky in 1934. They have analyzed observations of supernova explosions and proposed an explanation of an enormous energy release in these explosions. The idea was that there exist stars of very high density and small radius. In 1939, Oppenheimer and Volkoff constructed general relativistic models for such objects, assuming that the stars are composed of degenerate free neutrons at high densities. After these ideas were proposed, physicists turned their attention to other fields of physics and neutron stars were mostly forgotten. The discovery of pulsars excited new interest for neutron stars. Neutron stars play a unique role in physics and astrophysics. On the one hand, they contain matter under extreme physical conditions, and their theories are based on risky and far extrapolations of what we consider reliable physical theories of the structure of matter tested in laboratory. On the other hand, their observations offer the unique opportunity to test these theories.
Neutron stars contain the matter of density ranging from a few \( g/cm^3 \) at their surface, where the pressure is small, to more than \( 10^{15} g/cm^3 \) at the center, where the pressure exceeds \( 10^{36} \text{dyn/cm}^2 \). To calculate neutron star structure, one needs the dependence of the pressure on density, the so called equation of state (EOS), in this huge density range, taking due account of temperature, more than \( 10^9 \text{K} \) in young neutron stars, and magnetic fields, sometimes above \( 10^{15}G \) \[19\].

Neutron stars are compact stars which contain matter of supranuclear density in their interiors. They have typical masses \( M \sim 1.4M_\odot \), where \( M_\odot \) is the Solar mass and radii \( R \sim 10 \text{km} \). Thus, their masses are close to the solar mass \( M_\odot = 1.989 \times 10^{33}g \), but their radii are \( \sim 10^5 \) times smaller than the solar radius \( R_\odot = 6.96 \times 10^5 \text{km} \). Because of its small size and high density, a neutron star possesses a surface gravitational field about \( 2 \times 10^{11} \) times that of Earth. Accordingly, neutron stars possess an enormous gravitational energy \( E_{\text{grav}} \sim GM^2/R \sim 5 \times 10^{53}\text{erg} \sim 0.2M^2c^2 \) and surface gravity \( g \sim GM/R^2 \sim 2 \times 10^{14} \text{cm/s}^2 \), where \( G \) is the gravitational constant and \( c \) is the speed of light. Clearly, neutron stars are very dense. Their mean mass density is \( \bar{\rho} \simeq 3M/(4\phi R^3) \simeq 7 \times 10^{14}g/cm^3 \sim (2-3)\rho_0 \), where \( \rho_0 = 2.8 \times 10^{14}g/cm^3 \) is the so called normal nuclear density, the mass density of nucleon matter in heavy atomic nuclei. The central density of neutron stars is even larger, reaching \( (10-20)\rho_0 \). By all means, neutron stars are the most compact stars known in the Universe \[19\].

The interior of a neutron star can be divided into four main internal regions, that are the outer crust, the inner crust, the outer core, and the inner core. Neutron star also has atmosphere around it. The atmosphere is a thin plasma layer, where the spectrum of thermal electromagnetic neutron star radiation is formed. The outer crust (the outer envelope) extends from the atmosphere bottom to the layer of the density \( \rho = 4 \times 10^{11}g/cm^3 \). Its thickness is some hundred meters. Its matter consists of ions \( Z \) and electrons \( e \). The inner crust (the inner envelope) may be about one kilometer thick. The density \( \rho \) in the inner crust varies from \( \rho_{ND} \) at the upper boundary to \( \sim 0.5\rho_0 \) at the base. The matter of the inner crust consists of electrons, free neutrons, and neutron-rich atomic nuclei. The outer core occupies the density range \( 0.5\rho_0 \lesssim \rho \lesssim 2\rho_0 \) and is several kilometers thick. Its matter consists of neutrons with several per cent admixture of protons, electrons, and possibly muons. The inner core, where \( \rho \gtrsim 2\rho_0 \), occupies the central regions of massive neutron stars. Its radius can reach several kilometers, and its central density can be as high as \( (10-50)\rho_0 \). Its composition and the EOS are very model dependent. Several hypotheses have been put forward, predicting the appearance of new fermions and boson condensates \[19\].
4 Results and Discussion

Mathematical description of gravitation was firstly proposed by Sir Isaac Newton in 1687. Newton’s theory of gravitation was widely accepted until the beginning of the 20th century. In 1915, Albert Einstein introduced the theory of general relativity (GR). It was a revolutionary theory that changed the way physicists think about gravitation. Today, GR is considered a well established theory. Unlike Newton’s, where gravitation is described as a force from a distance, in GR gravitation is not force, but something arises from curvature of spacetime that affects the motion of particles. On the other hand, massive bodies and energy sources cause the spacetime to curve. In GR we study how the curvature of spacetime acts on matter to manifest itself as gravitation, and how energy and momentum influence spacetime to create curvature.

To study the curvature we employ a mathematical description of the squared length of an infinitesimal displacement in an arbitrary direction. This squared length is given by

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \]

where \( x^\mu \) are the coordinates describing the spacetime and \( g_{\mu\nu} \) is the metric tensor. The metric tensor is one of the fundamental quantities in GR. Given the metric tensor the geometry of the spacetime can be fully described. If the geometry of the spacetime is known, the motion of test particles can be predicted.

4.1 Gravitational Force Around SlowlyRotating Neutron Star

The metric around a slowly rotating neutron star is given by [21]

\[ ds^2 = e^{2\nu(r)} dt^2 - e^{2\lambda(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2 + 2 \omega r^2 \sin^2 \theta d\varphi dt. \]

Based on this metric, the components of covariant metric tensor can be written as

\[
g_{\alpha\beta} = \begin{pmatrix}
e^{2\nu(r)} & 0 & 0 & \omega r^2 \sin^2 \theta \\
0 & -e^{2\lambda(r)} & 0 & 0 \\
0 & 0 & -r^2 & 0 \\
\omega r^2 \sin^2 \theta & 0 & 0 & -r^2 \sin^2 \theta \end{pmatrix}.
\]

The non-zero covariant basis, \( e^\mu_M(x) \), and the contravariant ones, \( e^\nu_N(x) \), for
metric (6) are
\[
\begin{align*}
  e_0^0 &= e^{\nu(r)}, & e_0^0 &= e^{-\nu(r)}, \\
  e_1^0 &= e^{\lambda(r)}, & e_1^0 &= e^{-\lambda(r)}, \\
  e_2^2 &= r, & e_2^2 &= \frac{1}{r}, \\
  e_3^3 &= r \sin \theta, & e_3^3 &= \frac{1}{r} \sin \theta, \\
  e_0^3 &= -\omega(r) r \sin \theta, & e_0^3 &= \omega(r) e^{-\nu(r)}.
\end{align*}
\]

Then, we calculate the Christoffel symbols for metric (6) by
\[
\Gamma^\alpha_{\mu\sigma} = \frac{1}{2} g^{\alpha\eta}(g_{\eta\mu,\sigma} + g_{\eta\sigma,\mu} - g_{\mu\sigma,\eta}).
\]

The non-zero Christoffel symbols are given by
\[
\begin{align*}
  \Gamma^0_{01} &= \Gamma^0_{10} = \frac{\nu' e^{2\nu} + (\omega r \sin \theta)^2 \left(\frac{\omega'}{2} + \frac{1}{r}\right)}{e^{2\nu} + (\omega r \sin \theta)^2}, & \Gamma^2_{03} &= \Gamma^2_{30} = \frac{1}{2} \omega \sin 2\theta, \\
  \Gamma^0_{02} &= \Gamma^0_{20} = \frac{\omega^2 r^2 \sin 2\theta}{e^{2\nu} + (\omega r \sin \theta)^2}, & \Gamma^2_{12} &= \Gamma^2_{21} = \frac{1}{r}, \\
  \Gamma^0_{13} &= \Gamma^0_{31} = \frac{\omega' r^2 \sin^2 \theta}{e^{2\nu} + (\omega r \sin \theta)^2}, & \Gamma^1_{00} &= \nu' e^{2(\nu - \lambda)}, \\
  \Gamma^1_{30} &= \Gamma^1_{03} = \frac{1}{2} r (\omega' r + 2\omega) e^{-2\lambda} \sin^2 \theta, & \Gamma^1_{11} &= \lambda', \\
  \Gamma^3_{01} &= \Gamma^3_{10} = \frac{\omega e^{2\nu} (\nu' - \frac{\omega'}{2} + \frac{1}{r})}{e^{2\nu} + (\omega r \sin \theta)^2}, & \Gamma^1_{22} &= -re^{-2\lambda}, \\
  \Gamma^3_{02} &= \Gamma^3_{20} = \frac{\omega e^{2\nu} \cot \theta}{e^{2\nu} + (\omega r \sin \theta)^2}, & \Gamma^1_{33} &= -re^{-2\lambda} \sin^2 \theta, \\
  \Gamma^3_{13} &= \Gamma^3_{31} = \frac{\omega^2 r^2 \sin^2 \theta (\frac{\omega'}{2} + \frac{1}{r}) + 2e^{2\nu}}{e^{2\nu} + (\omega r \sin \theta)^2}, & \Gamma^3_{23} &= -\frac{1}{2} \sin 2\theta, \\
  \Gamma^3_{23} &= \Gamma^3_{32} = \cot \theta.
\end{align*}
\]

Furthermore, the gravitational force components are determined by
\[
F^a_g = -\Omega^a_{\mu B}(x) e^\mu_C(x) u^B u^C,
\]

where $\Omega^a_{\mu B}(x)$ is spin connection coefficients and $u^B$ is 4-velocity. Spin connection is a connection on a spinor bundle that is induced from the affine connection. It can also be regarded as the gauge field generated by local Lorentz transformations. In some canonical formulations of general relativity, a spin connection is defined on spatial slices and can also be regarded as the
gauge field generated by local rotations. To calculate spin connection we use equation

$$\Omega_{\mu\nu}(x) = \epsilon^\nu_{\mu}(x)\Gamma_{\mu\rho}^\nu e^\rho_N(x) + \epsilon^\mu_{\nu}(x)\partial_\mu e^\nu_N(x).$$  \hspace{1cm} (12)$$

Based on (12), then we get non-zero components of spin connection coefficient as

$$\Omega^0_{10} = \frac{(\omega r \sin \theta)^2(\frac{\omega'}{2} + \frac{1}{r} - \nu')}{e^{2\nu} + (\omega r \sin \theta)^2}, \quad \Omega^0_{01} = \frac{\nu e^{-\lambda} + \frac{1}{2} e^{-\nu-\lambda} \sin^2 \theta(\omega' r + 2\omega)}{e^{2\nu} + (\omega r \sin \theta)^2},$$

$$\Omega^0_{02} = \frac{\frac{1}{2}\omega^2 r^2 e^\nu \sin 2\theta}{e^{2\nu} + (\omega r \sin \theta)^2}, \quad \Omega^1_{00} = \nu' e^{-\nu-\lambda} + \frac{1}{2} e^{-\nu-\lambda (\nu+\lambda)} \sin^2 \theta(\omega' r + 2\omega),$$

$$\Omega^1_{22} = -e^{-\lambda}, \quad \Omega^1_{21} = e^{-\lambda} \cos \theta,$$

$$\Omega^1_{33} = -e^{-\lambda} \sin \theta, \quad \Omega^2_{21} = e^{-\lambda}, \quad \Omega^2_{33} = -\cos \theta,$$

$$\Omega^2_{01} = -\frac{1}{2} e^{-\lambda} \sin \theta(\omega' r + 2\omega), \quad \Omega^3_{02} = -\omega \cos \theta,$$

$$\Omega^3_{01} = \frac{1}{2} e^{-\lambda} \sin \theta(\frac{\omega'}{2} + \omega \omega' - \omega^2 \nu'), \quad \Omega^3_{32} = \cos \theta,$$

whereas the other components are zero.

The components of gravitational force around a slowly rotating neutron star are determined by (11). Finally, we get the gravitational force around a slowly rotating neutron star as
\[
F_g^1 = \frac{e^{-3\lambda - 2\nu}}{2r} \left[ -ru^0 e^{2\lambda} \left( r^2 u^3 \sin^2 \theta \omega' + r^2 u^0 \omega' \sin^2 \theta + 2u^0 \nu' e^{2\nu} \ight.ight.
\]
\[
+ 2ru^0 \omega' \sin^2 \theta + (ru^0 \omega' + 2u^0 \omega - 2u^3) \omega e^{\nu} \sin \theta \bigg] 
\]
\[
- \left( ru^0 \omega' + 2u^0 \omega - 2u^3 \right) u^3 e^{2\lambda + 2\nu} + 2(u^2)^2 e^{2\lambda + 2\nu} \right] 
\]
\[
(14a)
\]
\[
F_g^2 = -\frac{e^{-2\lambda - \nu}}{r \sin \theta} \left[ ru^0 \left( u^3 \omega e^{2\lambda} \sin 2\theta + ru^1 u^2 e^{\nu} \sin \theta \ight. \right.
\]
\[
+ u^3 (u^0 \omega - u^3) e^{2\lambda + \nu} \cos \theta \bigg] 
\]
\[
(14b)
\]
\[
F_g^3 = r^2 u^0 u^1 \omega^3 e^{-2\lambda - 2\nu} \sin^3 \theta - ru^0 u^1 \omega^3 e^{-3\nu} \sin^2 \theta 
\]
\[
- \frac{ru^0}{2} u^1 e^{-2\nu} \sin \theta \omega' + \frac{ru^0}{2} u^1 e^{-2\lambda} \sin \theta \omega' + u^0 u^1 \omega e^{-2\lambda} \sin \theta 
\]
\[
- u^1 u^3 e^{-2\lambda} \sin \theta + \frac{u^0 u^2}{r} \omega \cos \theta - \frac{u^2 u^3}{r} \cos \theta \right] 
\]
\[
(14c)
\]

The gravitational force govern the dynamics of particle that is moving around the slowly rotating neutron star. Finally, to get more insight of gravitational force around a slowly rotating neutron star we assumed that the \( \lambda(r) = -\frac{1}{2} \ln \left( 1 - \frac{2M}{r} \right) \), \( \nu(r) = \frac{1}{2} \ln \left( 1 - \frac{2M}{r} \right) \), \( M = 2 \), \( u^0 = 2e^{-\nu(r)} \), \( u^1 = \sqrt{2}e^{-\lambda(r)} \), \( u^2 = 1/r \), \( u^3 = 0 \), and \( \omega(r) = 0.01r \). Furthermore, by substituting those assumption to the equation (14) we obtained

\[
F_g^1 = \frac{1}{2r} \sqrt{1 - \frac{2m}{r}} \left[ -\frac{2r}{(1 - \frac{2m}{r})^{\frac{3}{2}}} \left( \frac{4m}{r^2} + \frac{0.0002}{r} \sin^2 \theta \ight. \right.
\]
\[
+ \frac{0.0002}{r^2} \sin \theta \right) + \frac{2}{r^2} \bigg] 
\]
\[
(15a)
\]
\[
F_g^2 = -\frac{\sqrt{1 - \frac{2m}{r}}}{r \sin \theta} \left[ 1.414 \left( 1 - \frac{2m}{r} \right) \sin \theta + \frac{0.0002 \sin 2\theta}{r \left( 1 - \frac{2m}{r} \right)^2} \right] 
\]
\[
(15b)
\]
\[
F_g^3 = \frac{0.014 \sin \theta}{r} \left( 1 - \frac{2m}{r} \right) + 2.828 \cdot 10^{-6} \frac{\sin^3 \theta}{r} + \frac{0.014 \sin \theta}{r \left( 1 - \frac{2m}{r} \right)} 
\]
\[
- \frac{2.828 \cdot 10^{-6} \sin^2 \theta}{r^2 \left( 1 - \frac{2m}{r} \right)^{\frac{3}{2}}} + \frac{0.02 \cos \theta}{r^3 \sqrt{1 - \frac{2m}{r}}} 
\]
\[
(15c)
\]
Figure 1: Components of gravitational force for slowly rotating neutron star

Then we plot the gravitational force (equations 15) around a slowly rotating neutron star as showed by Figure 1. Based on this graph, we can conclude that the gravitational forces become smaller when the source becomes farther away. This corresponds to the classical intuition. Furthermore, the gravitational force as one of tools that will be used to construct a differential diffusion equation of particles around the neutron star.

4.2 Diffusion Process Around Slowly Rotating Neutron Star

In this section we will investigate the relativistic diffusion process around a slowly rotating neutron star. We will derive the corresponding Fokker-Planck equation in phase space (general relativistic Kramers equation) within the frame of general relativity. These equation is a stochastic differential equation which describes the diffusive particles dynamic around a slowly rotating neutron star. The derivation of these equation refers to Herrmann [15] and Andra [17]. Herrmann has generalized the Markovian diffusion theory in the phase space within the general theory of relativity framework. He has derived the general relativistic Fokker-Planck equation in the phase space both in the parametrization of phase space proper time and the coordinate time. Whereas Andra has constructed the general relativistic Fokker-Planck equation to describe the diffusion process around non-rotating neutron star. Based on Herrmann’s work, the general relativistic Kramers equation in the parametrization of observer time is given by

$$ N^{-1} v^0 \frac{\partial \Phi}{\partial x^0} = -v^M div_v(e_M(x)\Phi) - div_v(F\Phi) + \frac{D}{2} \Delta_v \Phi, \quad (16) $$

where $N = 1/\sqrt{g_{00}}$ and $\Phi$ is the probability density function.
The important point in relativistic diffusion is the observation that the velocity space in special relativity is a noncompact hyperbolic 3-dimensional Riemannian manifold embedded into the 4-dimensional velocity Minkowski space. Using normalized velocity variables \( u^\mu (\mu = 0, 1, 2, 3) \) the hyperbolic metric structure for a relativistic system of massive particles is described by the relation

\[
(u^0)^2 - (u^1)^2 - (u^2)^2 - (u^3)^2 = 1. \tag{17}
\]

Therefore relativistic Markovian diffusion processes can be described in a rigorous way by using the mathematical stochastic calculus on Riemannian manifolds, but adopted to the velocity space. The 4-velocity are expressed in the hyperbolic coordinate system as below

\[
\begin{align*}
  u^0 &= \cosh \alpha \\
  u^1 &= \sinh \alpha \sin \beta \cos \phi \\
  u^2 &= \sinh \alpha \sin \beta \sin \phi \\
  u^3 &= \sinh \alpha \cos \beta.
\end{align*} \tag{18}
\]

Therefore, by substituting 4-velocity in the hyperbolic coordinate system to equation (14a), (14b), and (14c), then we obtained the gravitational force components in hyperbolic coordinate system are

\[
\begin{align*}
  F^1_g &= \frac{\sinh \alpha}{2re^\lambda} \left[ r\omega' \cos \beta \cosh \alpha + 2\omega \cos \beta \cosh \alpha - 2\cos^2 \beta \sinh \alpha - 2\sin^2 \beta \sin^2 \phi \sinh \alpha \right] \\
  &\quad + \frac{\cosh \alpha}{2e^{\lambda+2\nu}} \left[ r^2\omega' \sin^2 \theta \cos \beta \sinh \alpha + r^2\omega \omega' \sin^2 \theta \cosh \alpha + 2r\omega^2 \sin^2 \theta \cosh \alpha \\
  &\quad + 2e^{2\nu} \omega' \cosh \alpha + \omega e^\nu \sin \theta \left( r\omega' \cosh \alpha + 2\omega \cosh \alpha - 2 \cos \beta \sinh \alpha \right) \right] \tag{19a}
\end{align*}
\]

\[
\begin{align*}
  F^2_g &= \frac{\cos \beta \sinh \alpha}{r \sin \theta} \left[ \omega \cos \theta \cosh \alpha - \cos \beta \cos \theta \sinh \alpha \right] + e^{-2\lambda} \sin^2 \beta \sin \phi \cos \phi \sinh^2 \alpha \\
  &\quad + \omega e^{-\nu} \cos \alpha \left[ \omega \cos \theta \cosh \alpha - \cos \beta \cos \theta \sinh \alpha \right] + \frac{r}{2} \omega^2 e^{-2\nu} \sin (2\theta) \cosh^2 \alpha \tag{19b}
\end{align*}
\]

\[
\begin{align*}
  F^3_g &= -\frac{\sin \beta \sinh \alpha}{2r} \left[ - r^2\omega' e^{-2\nu} \sin \theta \cosh \alpha - re^{-2\lambda} \sin \theta \cos \phi \left[ r\omega' \cosh \alpha \\
  &\quad + 2\omega \cosh \alpha - 2 \cos \beta \sinh \alpha \right] + 2 \sin \phi \cos \theta \left[ \omega \cosh \alpha - \cos \beta \sinh \alpha \right] \right] \tag{19c}
\end{align*}
\]
Whereas the gravitational force components in the new coordinate are given by

\[
F^\alpha_g = (\cosh \alpha)^{-1} \left[ \sin \beta (\cos \phi F^1_g + \sin \phi F^2_g) + \cos \beta F^3_g \right] \tag{20a}
\]

\[
F^\beta_g = (\sinh \alpha)^{-1} \left[ \cos \beta (\cos \phi F^1_g + \sin \phi F^2_g) - \sin \beta F^3_g \right] \tag{20b}
\]

\[
F^\phi_g = (\sin \alpha)^{-1} (\sin \beta)^{-1} \left[ -\sin \phi F^1_g + \cos \phi F^2_g \right] \tag{20c}
\]

Explicitly, they are:

\[
F^\alpha_g = \frac{\sin \beta \cos \phi}{2e^{\lambda+2\nu}} \left[ r^2 \omega' \sin^2 \theta \cos \beta \sinh \alpha + r^2 \omega' \sin^2 \theta \cos \phi \sinh \alpha + 2r^2 \omega' \sin^2 \theta \cosh \alpha + 2e^{2\nu} \cos \alpha \sin \theta \left( r \omega' \cosh \alpha + 2 \omega \cosh \alpha - 2 \cos \beta \sinh \alpha \right) \right.
\]

\[
+ \frac{\omega \sin \beta \sin \phi}{2e^{2\nu}} \left[ r \omega' \sin 2 \theta \cosh \alpha + 2e^{2\nu} \cos \theta \left( \omega \cosh \alpha - \cos \beta \sinh \alpha \right) \right]
\]

\[
- \frac{\sin 2 \beta \sin \theta \cos \phi \tanh \alpha}{4e^{2\lambda}} \left[ r \omega' \cosh \alpha + 2 \omega \cosh \alpha - 2 \cos \beta \sinh \alpha \right]
\]

\[
+ \frac{\sin 2 \beta \cos \phi \tanh \alpha}{4re^{\lambda}} \left[ r \omega' \cosh \alpha + 2 \omega \cosh \alpha - 2 \cos \beta \sinh \alpha \right]
\]

\[- \frac{e^{-\lambda}}{\sinh \alpha} \sin^3 \beta \sin^2 \phi \cos \phi \sin^2 \alpha + \frac{e^{-2\lambda}}{\sinh \alpha} \sin^3 \beta \sin^2 \phi \cos \phi \sin^2 \alpha
\]

\[
+ \frac{1}{2r} \sin \phi \sin 2 \beta \cosec \theta \tanh \alpha \left[ \omega \cosh \alpha - \cos \beta \sinh \alpha \right]
\]

\[- \frac{1}{2r} \sin \phi \cos \theta \sin 2 \beta \tanh \alpha \left[ \omega \cosh \alpha - \cos \beta \sinh \alpha \right]
\]

\[
+ \frac{\omega' \omega}{4e^{2\nu}} \sin \theta \cos \phi \sin 2 \beta \sinh \alpha \tag{21}
\]

\[
F^\beta_g = \frac{e^{-2\lambda-3\nu}}{2r \sin \theta \sinh \alpha} \cos \beta \left[ e^{-\lambda+\nu} \sin \theta \cos \phi \left[ r e^{2\lambda} \cosh \alpha \left[ r^2 \omega' \sin^2 \theta \cos \beta \sinh \alpha \right.ight.ight.
\]

\[
+ \omega' \sin \theta \left( r \omega' \cosh \alpha + 2 \omega \cosh \alpha - 2 \cos \beta \sinh \alpha \right) + \cos \alpha \left( r^2 \omega' \sin^2 \theta \right.
\]

\[
+ 2r \omega' \sin^2 \theta + 2e^{2\nu} \cos \beta \sinh \alpha \left[ r \omega' \cosh \alpha + 2 \omega \cosh \alpha
\]

\[- 2 \cos \beta \sinh \alpha \right) - 2e^{2\lambda+2\nu} \sin^2 \beta \sin \phi \sin^2 \alpha \right] + \sin \phi \left[ r \omega e^{2\lambda+\nu} \sin \theta \cosh \alpha \right.
\]

\[
\times \left[ r \omega \sin 2 \theta \cosh \alpha + 2e^{2\nu} \cos \theta \left( \omega \cosh \alpha - \cos \beta \sinh \alpha \right) \right]
\]

\[+ 2r e^{3\nu} \sin^2 \beta \sin \phi \sin \theta \cos \phi \sin^2 \alpha + 2e^{2\lambda+3\nu} \cos \beta \cos \theta \sinh \alpha \left[ \omega \cosh \alpha - \cos \beta \sinh \alpha \right]
\]

\[+ \cos \beta \sinh \alpha \right) \right] \right]
\]

\[
+ e^{2\nu} \sin \theta \cos \phi \left[ r \omega' \cosh \alpha + 2 \omega \cosh \alpha - 2 \cos \beta \sinh \alpha \right]
\]

\[+ 2e^{2\lambda+2\nu} \sin \phi \cos \theta \left[ \omega \cosh \alpha - \cos \beta \sinh \alpha \right] \right] \tag{22}
\]
\[ F^\phi_y = -\frac{e^{-2\lambda-3\nu}}{2r \sin \beta \sin \theta \sinh \alpha} \left[ e^{-\lambda-\nu} \sin \phi \sin \theta \left[ re^{2\lambda} \cosh \alpha \left[ r^2 \omega' \sin^2 \theta \cos \beta \sinh \alpha \\
+ \omega e^{\nu} \sin \theta \left[ r \omega' \cosh \alpha + 2 \omega \cosh \alpha - 2 \cos \beta \sinh \alpha \right] + \cosh \alpha \left[ r^2 \omega \sin^2 \theta \\
+ 2r \omega^2 \sin^2 \theta + 2e^{\nu} \omega' \right] \right] + e^{2\lambda+2\nu} \cos \beta \sinh \alpha \left[ r \omega' \cosh \alpha + 2 \omega \cosh \alpha \\
- 2 \cos \beta \sinh \alpha \right] - 2e^{2\lambda+2\nu} \sin^2 \beta \sinh^2 \alpha \right] - \cos \phi \\
\times \left[ r \omega e^{2\lambda+\nu} \sin \theta \cosh \alpha \left[ r \omega \sin 2\theta \cosh \alpha + 2e^{\nu} \cos \theta \left[ \omega \cosh \alpha - \cos \beta \sinh \alpha \right] \right] \\
+ 2re^{3\nu} \sin^2 \beta \sin \phi \sin \theta \cos \phi \sinh^2 \alpha + 2e^{2\lambda+3\nu} \cos \beta \cos \theta \sinh \alpha \\
\times \left[ \omega \cosh \alpha - \cos \beta \sinh \alpha \right] \right] \] (23)

The random impact of particles around of neutron star give rise to a frictional force. In the nonrelativistic theory, frictional force is formulated by
\[ F_i^f = -\kappa u_i, \]
where \( \kappa \) and \( u^i \) are the friction coefficients and the components of nonrelativistic velocity, respectively. The generalization of relativistic friction force lead us to introduce a friction tensor \( \kappa^i_\alpha \). This tensor is similar to pressure tensor in relativity theory, so that the friction force is given by
\[ F_i^f = \kappa^i_\alpha [u^\alpha - V^\alpha], \]
where \( V^\alpha \) is the 4-velocity of heat bath and \( \kappa^i_\alpha \) is a tensor of friction coefficient. In the observer frame the heat bath is at rest described by \( V^\alpha = (1, 0, 0, 0) \).
For an isotropic homogeneous heat bath the friction tensor is given by [7],[15]
\[ \kappa^i_\alpha = \kappa (\eta^i_\alpha + u^i u_\alpha). \]
(25)
where \( \kappa \) denoting the scalar friction coefficient measured in the rest frame of the particles. Therefore, the friction force is given by \( F_i^f = -\kappa u^i u^0 \) or in the hyperbolic coordinate system is expressed by
\[ F^\alpha_i = -\kappa \sinh \alpha; F^\beta_i = 0; F^\phi_i = 0, \]
(26)

The total force components are the summation of gravitational force and friction force. Then, the total force components are expressed by
\begin{align*}
F^\alpha &= F^\alpha_y - \kappa \sinh \alpha \\
F^\beta &= F^\beta_y \\
F^\phi &= F^\phi_y,
\end{align*}
(27)
where $F^\alpha_y$, $F^\beta_y$, and $F^\phi_y$ are given by equation (21), (22), and (23), respectively.

Based on the equation (16), the Fokker-Planck equation or the diffusion differential equation that describes the diffusion process of diffusant around a slowly rotating neutron star can be constructed by calculating the Laplace-Beltrami operator and the divergence operator in the position and velocity space. Then, the divergence operator in the position space, $\text{div}_x (e_M(x)\Phi)$, is given by

$$\text{div}_x (e_M(x)\Phi) = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\sqrt{g} e^i_M(x)\Phi),$$

that is:

$$\text{div}_x (e_M(x)\Phi) = r \sin \theta \cos \beta \sinh \alpha \frac{d\Phi}{d\varphi} + r \sin \beta \sin \phi \sinh \alpha \frac{d\Phi}{d\theta} + e^\lambda \sin \beta \cos \phi \sinh \alpha \frac{d\Phi}{dr} + \Phi e^\lambda \lambda' \sin \beta \cos \phi \sinh \alpha$$

+ $r \Phi \cot \theta \sin \beta \sin \phi \sinh \alpha + \Phi \lambda' e^\lambda \sin \beta \cos \phi \sinh \alpha$

+ $2\Phi e^\lambda \sin \beta \cos \phi \sinh \alpha + \frac{r^3 \omega^2 \sin 2\theta \sin \beta \sin \phi \sinh \alpha}{2(r^2 \omega^2 \sin^2 \theta + e^{2\nu})} \Phi$

+ $\frac{e^\lambda \sin \beta \cos \phi \sinh \alpha}{2(r^2 \omega^2 \sin^2 \theta + e^{2\nu})} (2r^2 \omega \omega' \sin^2 \theta + 2r \omega^2 \sin^2 \theta + 2e^{2\nu} \nu') \Phi$.  

(29)

Whereas the divergence operator in the velocity space, $\text{div}_v (F\Phi)$, is given by

$$\text{div}_v (F\Phi) = \frac{1}{G} \frac{\partial}{\partial v^m} (\sqrt{G} F^m \Phi),$$

where $G = \det G_{ij}, g = \det g_{ij}$. In the hyperbolic coordinate system the divergence operator in the velocity space is expressed by

$$\text{div}_v (F\Phi) = (\sinh \alpha)^{-2} \frac{\partial}{\partial \alpha} ((\sinh \alpha)^2 F^\alpha \Phi) - (\sinh \alpha)^{-1} (\sin \beta)^{-1} \frac{\partial}{\partial \beta} (\sin \beta F^\beta \Phi)$$

$$- (\sinh \alpha)^{-1} (\sin \beta)^{-1} \frac{\partial}{\partial \varphi} (F^\varphi \Phi),$$

(31)

where $F^\alpha, F^\beta$, and $F^\phi$ are given by equation (27). Laplace-Beltrami operator
can be determined via below equation

\[
\frac{D}{2} \Delta_u \Phi = \frac{D}{2} \left\{ \frac{\partial^2 \Phi}{\partial \alpha^2} + 2 \coth \alpha \frac{\partial \Phi}{\partial \alpha} - \frac{1}{(\sinh \alpha)^2} \left( \frac{\partial^2 \Phi}{\partial \beta^2} + \cot \beta \frac{\partial \Phi}{\partial \beta} \right) + \frac{J(J+1)}{(\sinh \alpha)^2 \partial \varphi^2} \right\},
\]

(32)

where \( J \) is discrete index from 0 to \( \infty \).

Finally, by substituting equation (29), (31), and (32) to equation (16), then we get the diffusion equation that describes diffusion process of particles around slowly rotating neutron star in the parametrization of the phase-space proper time within the frame of general relativity for the probability density function as below

\[
e^\nu \cosh \alpha \frac{\partial \Phi}{\partial t} = \frac{D}{2} \left\{ \frac{\partial^2 \Phi}{\partial \alpha^2} + 2 \coth \alpha \frac{\partial \Phi}{\partial \alpha} + J(J+1) \Phi \right\} + (\sinh \alpha)^{-1}(\sin \beta)^{-1} \frac{\partial}{\partial \varphi}(F^\varphi \Phi)
- (\sinh \alpha)^{-2} \frac{\partial}{\partial \alpha}((\sinh \alpha)^2 F^\alpha \Phi) + (\sinh \alpha)^{-1}(\sin \beta)^{-1} \frac{\partial}{\partial \beta}(\sin \beta F^\beta \Phi)
- \frac{e^\lambda \sin \beta \cos \phi \sinh \alpha}{2(r^2 \omega^2 \sin^2 \theta + e^{2\nu})} \left( 2r^2 \omega' \sin^2 \theta + 2r \omega^2 \sin^2 \theta + 2e^{2\nu} \nu' \right) \Phi
- \frac{2 \Phi e^\lambda}{r} \sin \beta \cos \phi \sinh \alpha - \frac{r^3 \omega^2 \sin 2\theta \sin \beta \sin \phi \sinh \alpha}{2(r^2 \omega^2 \sin^2 \theta + e^{2\nu})} \Phi
- r \Phi \cot \theta \sin \beta \sin \phi \sinh \alpha - \Phi \lambda' e^\lambda \sin \beta \cos \phi \sinh \alpha
- \Phi e^\lambda \lambda' \sin \beta \cos \phi \sinh \alpha - e^\lambda \sin \beta \cos \phi \sinh \alpha \frac{d\Phi}{dr}
- r \sin \beta \sin \phi \sinh \alpha \frac{d\Phi}{d\theta} - r \sin \theta \cos \beta \sinh \alpha \frac{d\Phi}{d\varphi},
\]

(33)

where \( F^\alpha, F^\beta, \) and \( F^\phi \) are given by equation (27).

Similar with [17], it is very difficult to get a general solution of equation (33), the probability density function \( \Phi(t, x, u) \). In addition to long differential equations, the probability density function also consists of seven parameters. However, to get a glimpse of it, we will consider asymptotic case, \( r \rightarrow \infty \) and \( e^{\nu(r)} = e^{\lambda(r)} = 1 \). For asymptotic case, we found that there are new terms of \( \theta \) and \( \varphi \). It differences from the asymptotic case for the diffusion process around non-rotating neutron star [17]. In the non-rotating neutron case, the diffusion depends only on the radial coordinate as well as the 4-velocity. The terms containing \( \theta \) and \( \varphi \) can be intuitively explained by the fact that there is a new term in metric tensor of (6) which contains \( \omega \) as angular frequency of
the rotating neutron star. Besides that, from equation (33), we found terms as the source of diffusion due to the curvature of spacetime which depend on 4-velocity and angular frequency.

5 Conclusion

Analytically, we have constructed the relativistic diffusion equation or the Fokker-Planck equation to describe the dynamics of particles undergoing diffusion around slowly rotating neutron star given by equation (33). This equation is expressed in the hyperbolic coordinate system. The Fokker-Planck equation is also derived both in the parametrization of phase space proper time and the coordinate time.

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