Exact Solution of Dark Energy and a Primordial and Undisturbed Magnetic Field at an Anisotropic Cosmological Space-Time of Petrov Type D

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Abstract

In the present paper it is obtained and analyzed an exact solution of dark energy and a primordial magnetic field, so that the magnetic field does not induce currents or electric fields. It is determined that the main role in relation to the present anisotropy, in said solution, is played by the magnetic field. The solution is analyzed in points where, in appearance, some sort of singularity could exist or change to a complex regimen \( g_{\mu\nu} \in \mathbb{C} \), using the Kretschmann invariant and analyzing the metric’s functions and it is established that this is free of singularities and is not complex, especially when \( t \to 0 \); a point that in appearance is singular and the solution tends to be a flat world (without being one). It is determined that the solution has such a behavior that, for large time values, tends to become isotropic and equivalent to the solution of dark energy that is obtained for the flat model of Friedmann, Robertson, Walker and Lemaitre (FRWL).

Keywords: cosmology, magnetic field, exact, solution, einstein

1 Introduction

The cosmology has become more dynamic from the data in the microwave background radiation, obtained by the satellite COBE, WMAP AND PLANCK,
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so they have stoked series of problems which have been discussed in [1]. The problem of the existence of magnetic fields in interstellar and intergalactic spaces is not new, but it is still a current issue [2], [3]. Reviewing those fields produce values of $2 \mu G$ in structures of matter such as clusters and superclusters, and among those what appears to be a considerable value and a potential influence on cosmic phenomena; for example, the radiation of the microwave background is of $\sim 3 \mu G$ [4]. The dynamo mechanism, that has been used in the explanation of magnetic and galactic fields, does not generate a satisfactory explanation of the magnetic fields in the intergalactic spaces nor how the mechanism arises; therefore, it is concluded that it is an amplifier mechanism of a primordial magnetic field in the process in which galaxies are formed [3]. Recently, it has also been considered the possibility that a primordial magnetic field, generated during the inflation, may explain the results obtained from the project BICEPS2 [5] (assuming those are confirmed) in relation to the modes B of the power spectrum of the gravitational waves [6], concluding that the prior is possible. For all the above reasons, it is obvious that there is an interest in studying cosmological models that better represent those observations, for example models anisotropic homogeneous or heterogeneous isotropic anisotropic heterogeneous. For example, if the Universe was created with a field magnetic paramount, the nature of the field itself, forced to consider at least the anisotropic space in the process of determining a solution, that is why in this work analyzes a magnetic field and dark energy, under an anisotropic symmetry but homogeneous.

2 The Symmetry, Einstein’s Tensor, the Electromagnetic Field and the Dark Energy

2.1 The Symmetry and Einstein’s Tensor

The anisotropic symmetry of the Petrov Type D has been considered in [1] as follows

$$ds^2 = F dt^2 - t^{2/3} K (dx^2 + dy^2) - \frac{t^{2/3}}{K^2} dz^2,$$

(1)

where $F$ and $K$, are functions of $t$.

The components of Einstein’s tensor [7] ($G^0_\alpha = R^0_\alpha - 1/2 \delta^0_\alpha R$) different from zero, of (1), are

$$G^0_0 = \frac{4 K^2 - 9 t^2 \dot{K}^2}{12 t^2 K^2 F},$$

(2)

$$G^1_1 = - \frac{3 K t \dot{K} \left(2 F - \dot{F} t\right) + 3 F t^2 \left(2 K \dot{K} - 5 \dot{K}^2\right) + 4 K^2 \left(\dot{F} t + F\right)}{12 t^2 K^2 F^2},$$

(3)
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\[ G_2^2 = G_1^1, \quad (4) \]

\[ G_3^3 = \frac{-6Kt\dot{K} \left( 2F - \dot{F}t \right) - 3Ft^2 \left( 4K\dot{K} - \dot{K}^2 \right) + 4K^2 \left( \dot{F}t + F \right) }{12t^2K^2F^2}, \quad (5) \]

where the points over the functions represent derivatives by the time.

### 2.2 The Magnetic Field

The magnetic field is considered in such way that the only components of the tensor of the electromagnetic field \([7]\) \(F_{\mu\nu}\) different from zero are \(F_{12} = -F_{21} = B_{0z} = \text{const}\), where it can be noticed that the invariant \(F_{\mu\nu}F^{\mu\nu} = 2B(t)^2 = 2B_{0z}^2\pi/(t^{4/3}K^2)\), where \(B(t) = B_{0z}\pi^{1/2}/(t^{2/3}K)\) is the magnitude of the effective magnetic field. The effective magnetic field does not generate currents or induced electric fields, since the flow of the magnetic field \(\Phi\) does not change with time,

\[ d\Phi = B(t)dA(t) = B(t)\sqrt{g_{11}g_{22}}dxdy = B_{0z}\pi^{1/2}dxdy. \quad (6) \]

The choice of the field, in the given way, enables the compliance of field equations \(F_{\mu\nu}^{\mu\nu} = 0\), besides the equality to zero of the divergence of the energy momentum tensor of the electromagnetic field \(\text{em}\, T_{\mu\nu} = 0\), where the tensor \(\text{em}\, T_{\mu\nu}\) is the energy momentum tensor of the electromagnetic field, whose only components, different from zero, for \(\text{em}\, T_{\mu\nu}\), are

\[ \text{em}\, T_0^0 = -\text{em}\, T_1^1 = -\text{em}\, T_2^2 = \text{em}\, T_3^3 = \frac{B_{0z}^2}{8t^{4/3}K^2}. \quad (7) \]

### 2.3 The Dark Energy Model

The dark energy model can be considered as a type of perfect fluid whose energy momentum tensor has the form \([7]\)

\[ T_{\alpha\beta} = (\mu + P) u_\alpha u_\beta - g_{\alpha\beta}P, \quad (8) \]

where \(T_{\alpha\beta}\) is the energy momentum tensor of the perfect fluid, \(u_\alpha\) the tetradimensional speed, \(g_{\alpha\beta}\) the metric tensor, \(\mu\) and \(P\) the energetic density and the fluid pressure respectively.

The dark energy model has a state equation the relation \(\mu = -P \sim \Lambda = \text{const}\), where \(\Lambda\) is usually considered the cosmological constant (in this work it is considered that \(\mu = \Lambda\)). The energy momentum tensor of dark energy takes the following form

\[ d\, T_\alpha^\alpha = \delta^\alpha_\nu \mu = \delta^\alpha_\nu \Lambda, \quad (9) \]

where \(\delta^\alpha_\nu\) is the delta of Kronecker.
3 The Einstein’s Equations and the solution of the model of the magnetic field and the dark energy

The equations of Einstein has the form [7] $G^\beta_\alpha = \kappa T^\beta_\alpha$, where $T^\beta_\alpha = T^\beta_\alpha^{em} + T^\beta_\alpha^{de}$.

De (2-5, 7, 9) there is the following system of equations independent from each other

\[
\frac{4 K^2 - 9 t^2 \dot{K}^2}{12 t^2 K^2 F} - \frac{B_{0z}^2 + 8 \Lambda \ t^{4/3} K^2}{8 t^{4/3} K^2} = 0, \quad (10)
\]

\[
- \frac{3 K t \dot{K} \left( 2 F - \dot{F} t \right) + 3 F t^2 \left( 2 K \ddot{K} - 5 \dot{K}^2 \right) + 4 K^2 \left( \dot{F} t + F \right)}{12 t^2 K^2 F^2} + \frac{-t^{2/3} B_{0z}^2 + 8 \Lambda \ t^2 K^2}{8 K^2 t^2} = 0, \quad (11)
\]

\[
- \frac{-6 K t \dot{K} \left( 2 F - \dot{F} t \right) - 3 F t^2 \left( 4 K \ddot{K} - \dot{K}^2 \right) + 4 K^2 \left( \dot{F} t + F \right)}{12 t^2 K^2 F^2} + \frac{-t^{2/3} B_{0z}^2 + 8 \Lambda \ t^2 K^2}{8 K^2 t^2} = 0, \quad (12)
\]

From the equation (10), it is obtained that

\[
F = \frac{2 \left( 4 K^2 - 9 t^2 \dot{K}^2 \right)}{3 t^{2/3} \left( B_{0z}^2 + 8 t^{4/3} \Lambda \ K^2 \right)}. \quad (13)
\]

Considering (13) in (11) and (12), it is obtained that both equations are satisfied with each other

\[
\left( -27 t^3 \dot{K}^3 + 48 K^2 t \dot{K} + 36 \ddot{K} K^2 t^2 - 16 K^3 \right) B_{0z}^2 + 72 t^{4/3} K^2 \Lambda \left( -9 t^3 \dot{K}^3 - 4 t^2 K \ddot{K}^2 + 4 \ddot{K} K^2 t^2 + 8 K^2 t \dot{K} \right) = 0. \quad (14)
\]

The found solution, not complex, of the equation (14) has the form

\[
K = \frac{A(t)}{4 \Lambda \ t^{2/3}} + \frac{B_{0z}^2}{2 A(t) \ t^{2/3}}, \quad (15)
\]

where $A(t)$ is
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\[ A(t) = A = \sqrt[3]{4 t^2 + 2 \sqrt{\frac{-2 B_{0z}^6 + 4 t^4 \Lambda}{\Lambda}}} \Lambda^2, \quad (16) \]

or of the form

\[ K = \frac{1}{4 \sqrt{\Lambda}} t^{-2/3} \sqrt{4 t^2 + 2 \sqrt{\frac{-B_{0z}^6 + 2 t^4 \Lambda}{\Lambda}}} + \frac{1}{4 \sqrt{\Lambda}} t^{-2/3} \sqrt{4 t^2 - 2 \sqrt{\frac{-B_{0z}^6 + 2 t^4 \Lambda}{\Lambda}}}, \quad (17) \]

the solution (17), can be rewritten, for being used in a more convenient way in an interval where \( |t| < B_{0z}^{3/2}/(2\Lambda)^{1/4} \), as

\[ K = 1/2 B_{0z} \sqrt{2} \cos \left( \frac{1}{3} \arctan \left( \frac{1}{2} \sqrt{\frac{B_{0z}^6 - 2 t^4 \Lambda}{\Lambda}} t^{-2} \right) \right) \frac{1}{\sqrt{\Lambda}} t^{-2/3}. \quad (18) \]

From (16) and (15), the function \( F \) in (13) can be written as

\[ F = \frac{2 \left( t^2 (B_2^2 - A(t)^2)^2 + \sqrt{(-2 B_{0z}^6 + 4 t^4 \Lambda)/(\Lambda)} (-A(t)^4 + B_2^4) \right)}{3 (B_{0z}^6 - 2 t^4 \Lambda) (3 B_2^2 A(t)^2 + A(t)^4 + B_2^4)} \quad (19) \]

where \( B_2 = B_{0z} \sqrt{2\Lambda} \).

4 Analysis of the Solution

The function \( F \) in (19), has real and positive values, for all the values of \( t \), some temporal points, or intervals, that may appear inconsistent with the prior affirmation and that could generate doubts related to the validity of the solution, are \( t = 0 \), o \( |t| \rightarrow B_{0z}^{3/2}/(2\Lambda)^{1/4} \). Because of the above it is convenient to analyze the solution for this case more closely. For example, when \( |t| \rightarrow B_{0z}^{3/2}/(2\Lambda)^{1/4}, F \rightarrow \frac{4}{27} \sqrt{\frac{2}{\Lambda B_1^3}} \) and if \( t \rightarrow 0, F \rightarrow 1/(B_{0z}^3 \sqrt{6\Lambda}) \).

The function \( K \), tends to \( B_{0z} \sqrt{6}/(4\sqrt{\Lambda} t^{2/3}) \), when \( t \rightarrow 0 \), so this diverges when \( t \rightarrow 0 \), but no so the metric (1), which tends to be

\[ ds^2 \rightarrow dt'^2 - (dx'^2 + dy'^2) - t'^2 dz'^2, \quad (20) \]

where \( t' = t/(B_{0z}^{3/2}(6\Lambda)^{1/4}), x' = B_{0z}^{1/2} 6^{1/4} x/(2\Lambda^{1/4}), \) analogous with \( y' \), and \( z' = 4\Lambda^{3/4} B_{0z}^{1/2} z/6^{1/4} \), therefore the metric tends to be flat in proximity with
$t = 0$ similar to what was obtained for one of the studied cases of the dark energy model, with the same symmetry (1), in a previous work [1]. For the analysis of the solutions in proximity with $t = 0$ or $t \to \infty$, it will be used the Kretschmann invariant. The condition $Krets < \infty$ is necessary and sufficient [8] for the finitude of all the invariants of algebraic curvature, this is defined as $Krets = R^\mu\nu\alpha\beta R_{\mu\nu\alpha\beta}$ and has the form, for the metric (1),

$$Krets = \frac{2}{81} \frac{(-K + 3t\dot{K})^2}{t^4K^4F^2} \left(2K + 3t\dot{K}\right)^2 + \frac{1}{324} \frac{(2K + 3t\dot{K})^4}{t^4K^4F^2} + \frac{1}{81} \frac{(R_0 + 4FK^2 + 3\dot{F}tK^2 - 36Ft^2\dot{K}^2)^2}{t^4K^4F^2} + \frac{1}{162} \frac{(R_0 - 8FK^2 - 6\dot{F}tK^2 - 9Ft^2\dot{K}^2)^2}{t^4K^4F^2}, \quad (21)$$

where $R_0 = 12FKt\dot{K} + 18FKt^2\ddot{K} - 9\dot{F}t^2KK$. The Kretschmann invariant (21) tends to $\frac{3\pi}{6} \Lambda^2$ (value that depends only on $B_{0z} \neq 0$), when $|t| \to B_{0z}^{3/2}/(2\Lambda)^{1/4}$, at $80\Lambda^2/9$, when $t \to 0$ and at $Krets = 8\Lambda^2/3$ when $|t| \to \infty$ so it is not singular at those points. The values of the mixed energy momentum tensor $T^\beta_\alpha = \epsilon_m T^\beta_\alpha + \epsilon_\alpha T^\beta_\alpha$, are not singular when $|t| = B_{0z}^{3/2}/(2\Lambda)^{1/4}$, nor complex when $|t| < B_{0z}^{3/2}/(2\Lambda)^{1/4}$.

If the magnetic field is void $B_{0z} = 0$, the Kretschmann invariant is constant and equal to $Krets = 8\Lambda^2/3$, and the functions of the solution take the form $\dot{K} = \dot{K_0} = 1/(8\Lambda)^{1/3}$ and $F = 1/(\Lambda t^2)$, where, when making the change of the temporal coordinate $t = e^{\eta/\sqrt{3\Lambda}}$, it is obtained that

$$ds^2 = d\eta^2 - e^{2\eta/\sqrt{3\Lambda}} (dx'^2 + dy'^2 + dz'^2), \quad (22)$$

where $x' = \sqrt{K_0} x, y' = \sqrt{K_0} y$ and $z' = z/K_0$, which is the solution of the dark energy model for the solution of the flat model of the type FRWL. From the prior, in the found anisotropic solution, the magnetic field plays a main role, or generator, in relation to the existent anisotropy.

The anisotropy in the solution is diluted with the increase in time, so for large values of $t$, the above is noticed by making a change of temporal coordinate $t = e^{\eta/\sqrt{3\Lambda}} - (3B_{0z}^2 e^{-\eta/\sqrt{3\Lambda}})/(16\Lambda^{1/3})$ where

$$ds^2 \to d\eta^2 - e^{2\sqrt{\frac{\eta}{3\Lambda}}} (dX^2 + dY^2 + dZ^2) - ds_p^2, \quad (23)$$

where $X = x/(\sqrt{2\Lambda^{1/3}})$, analogous for $Y, Z = 2\Lambda^{1/3} z$ and $ds_p$ is

$$ds_p^2 = \frac{3B_{0z}^2}{8\Lambda^{1/3}} e^{-2\sqrt{\frac{\eta}{3\Lambda}}} (dX^2 + dY^2 - 3dZ^2). \quad (24)$$
from (24) it is noticed that $ds_p \to 0$, when $t \to \infty$ and the metric (23) has asymptote at (22), which is the solution of the model with dark energy of flat FRWL, in general the obtained solution is transformed in an isotropic solution when $t^4 \gg B_{0z}^6/(2\Lambda)$, which is easily noticeable since it is analogous to consider $B_{0z} = 0$.

5 Conclusions

In a mixture of dark energy and magnetic field, for which the flux of the magnetic field is kept constant, that is to say that neither currents nor electric fields are induced, the magnetic field is crucial in relation to the anisotropy presented in the space, so that the space expands faster on the perpendicular plane to the magnetic field direction. The presence of the magnetic field does not generates singular points in the metric or temporal intervals where this is complex. The solution tends to become isotropic with the time increase, and it is equivalent to the one of the dark energy model of the FRWL flat universe, so the influence of the magnetic field is diluted with the time increase. In the proximity with $t = 0$, the metric tends to be the flat world, without being it; in an analogous manner to what was obtained for one of the two possible solutions of the dark energy model with the same symmetry in other work [1].

References


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