Solution of Spin 1/2 Equation in $\Lambda$LTB Cosmology
with Factorized Parametric Representation and
the Problem of the Inner Product of States

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Abstract

The Dirac equation is considered in the ALTB cosmological model. The study makes use of the Newman Penrose formalism based on a previously defined null tetrad frame. The Dirac equation is completely separated within the class of ALTB cosmological models that are represented by factorized parametric solution and that still depend on an arbitrary integration function. The problem of orthogonal solutions of Dirac field is discussed on the base of a suitable definition of inner product of states. The scheme is specialized to an open Robertson Walker like metric and, except the separated time equation, it is explicitly solved. The discussion gives the occasion of correcting an error in a previous study of the Dirac equation. This allows to point out two different formulations of scalar product of Dirac states.

Keywords: Lemaître-Tolman-Bondi cosmology · Cosmological constant · Parametric solutions · Dirac equation · Variable separation · Robertson Walker like model

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1 Introduction

The Lemaitre-Tolman-Bondi model with cosmological constant Λ (ΛLTB) represents a spherically symmetric (inhomogeneous) comoving solution of the Einstein field equation for a universe filled with dust matter without pressure. The model has been originally formulated to possibly contain a cosmological constant Λ [8, 16, 2] and can be solved in general in terms of Weierstrass elliptic functions [8] (more recently see e. g., [17]).

In case Λ = 0 the model is an useful tool to study collapsing situations and trapped surfaces leading to the formation of black holes [4, 21], light propagation near a collapsing time, and visibility and nature of the physical singularities under formation ([21] and references therein).

Recently the ΛLTB model has attracted new interest. The scheme has been indeed proposed as an alternative to explain the cosmological data by means of inhomogeneity instead of invoking the notion of dark energy in the Standard Cosmological model (see, e.g., [5, 15]). Another theoretical aspect of interest of the model is the following. For ΛLTB model with vanishing Λ it has been explicitly shown that the spin 0, 1/2, 1, 3/2 field equations can be separated under a relation between two arbitrary independent integration functions of the form $M(r) \propto (2E(r))^{3/2}$ [19, 20, 24, 25]. Such results represent a preliminary step in view of a quantization of the field that has been already proposed for spin 1 field [24] in case Λ = 0. Another consequence of field quantization is the existence of particle creation in expanding universe [24] in analogy to what found in the Robertson Walker space-time [11, 12, 13], recently [26, 27, 28]. In turn this leads to include particle creation in the formulation of the ΛLTB cosmological model as it has been tentatively done, e. g., in [31, 32].

It seems therefore of interest to try to extend the last mentioned results to ΛLTB cosmological model with non trivial Λ. The corresponding simplest study, that concerns the scalar field equation, has been proposed in [29]: the separability of the scalar field equation has been there obtained by using parametric solutions of the Newton-like cosmological equation that are factorized in the parameter and radial dependence. Such solutions do indeed exist under the condition $M(r) \propto (2E(r))^{3/2}$ as it has been shown in [29, 30, 33] where examples of factorized solutions have been provided. In [29] it was supported, but not proved, that also the spin 1/2, 1 field equations can be separated in ΛLTB cosmological model admitting factorized parametric representation of the solution of the cosmological equation.

The object of the present paper is to prove those assertions. This will be done in a slightly different manner than what suggested in [29] for the spin 1/2 field equation.

The result is proved by using the Newman-Penrose formalism based on a previously defined null tetrad frame (e. g., [19]).
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equation, on account of the spherical symmetry, the angular part of the wave spinor factors out and one has to deal with two coupled differential equation in the directional derivatives in the \( r, t \) variables. The scheme is then further developed by passing to a new time variable \( \tau \) that in fact coincides with the parameter of the factorized parametric solution of the cosmological model. The \( r \) and \( \tau \) dependence can then be separated. One is then left a pair of coupled first order ordinary differential equation in the \( r \) variable and with pair of coupled first order ordinary differential equation in the \( \tau \) variable. In turn the radial and time equations can be disentangled to obtain two separated second order differential equation in the \( r \) variable and two separated second order differential equation in the \( \tau \) variable.

The separated time equations are in general very difficult to be solved. This depends on the nature of the cosmological time evolution that in general has a non elementary analytical form. They imply however the existence of a constant expression that, together with the existence of ortho-normal solutions of the radial equation, plays a central role for the determination of orthogonal solutions of the Dirac equation. To prove the existence of radial ortho-normal solutions is a difficult task in general. In the open Robertson-Walker like case, ortho-normal solutions of the Dirac equation are determined by explicitly solving the radial equations.

The discussion of the Robertson-Walker model gives the opportunity of pointing out an error contained in a previous study of the model. This in turn allows to discuss about the correct formulation of scalar product of solutions of the Dirac equation.

2 Parametric solutions of the ΛLTB model

The Lemaître-Tolman-Bondi (ΛLTB) model represents an inhomogeneous spherically symmetric universe filled with freely falling dust like matter without pressure. In spherically symmetric co-moving coordinates the time evolution of the Universe is governed by the the Einstein field equation

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = k T_{\mu\nu} - \Lambda g_{\mu\nu}
\]

\( T_{\mu\nu} = \eta U^\mu U^\nu, \quad U^0 = 1, \quad U^i = 0, \quad i = r, \theta, \varphi, \)

\[
g_{\mu\nu} = diag\{1; -\exp \gamma(r, t); -y^2(r, t); -y^2(r, t) \sin^2 \theta\}
\]

where \( \eta = \eta(r, t) \) is the proper energy density, \( k = 8\pi G \), \( \Lambda \) the cosmological constant. The equation can be completely integrated. Indeed the scheme (1)-(3) can be first reported to the equations (dot and prime means t, r-derivative):

\[
\exp \gamma = \frac{y^2}{1 + 2E}
\]
\[ \frac{y^2}{2} - \frac{M(r)}{y} + \frac{\Lambda}{6} y^2 = E(r) \]  
\[ M(r) = 4\pi G \int_0^r dr' y'(r, t) y^2(r, t) \eta(r, t) \]

where \( E, M \) are arbitrary integration functions. The function \( E \) comes from the integration of the \((t, r)\) component of (1), while the conservation of \( M \) follows from the vanishing of the divergence of the Einstein tensor (energy conservation). The equation (5) can then be explicitly integrated. The integral can be put both in closed and in (non factorized) parametric form. (See e.g. \[4, 7\] for \( \Lambda = 0 \) and \[17\] for \( \Lambda \neq 0 \)). If \( \Lambda = 0 \) the parametric solutions have the well known form:

\[
y(r, \eta) = \frac{M}{2E} \cosh \eta \eta - 1, \quad t(r, \eta) = \frac{M}{2(E)} \sinh \eta \eta, \quad \eta \in [0, \infty), \quad (E > 0) \quad (7)
\]
\[
y(r, \eta) = \frac{M}{(2|E|)^{3/2}} (1 - \cos \eta), \quad t(r, \eta) = \frac{M}{2|E|} (\eta - \sin \eta), \quad \eta \in [0, 2\pi] \quad (E < 0) \quad (8)
\]

(\( \eta \) has been set \( G = 1 \)). Both in (7) and (8) the solution for \( t \) is up to a further integration constant \( t_0(r) \) that has been neglected. The particular structure of the solution (7), (8) makes possible to separate spin field equations of interest. Indeed, under the proportionality condition \( M^2 \propto E^3 \), the separability of the scalar field equation [20], of the Dirac equation [19] and of the spin 1 field equation [24] is ensured.

To generalize the result to \( \Lambda \neq 0 \), one is led to consider parametric solutions of (5) of the form

\[
y(r, \eta) = g(r) a(\eta), \quad t(r, \eta) = f(r) b(\eta)
\]
\[
b'(\eta) = a(\eta), \quad f'(\eta) = 0 \quad (f = \text{const.}) \quad (9)
\]

Solutions of this kind do indeed exist. Explicit examples have been furnished through the study, e.g., of [29, 30, 33]. For \( E, \Lambda > 0 \), one has, e., g.,

\[
g(r) = \frac{\alpha}{3} \sqrt{\frac{8E}{\Lambda}}, \quad a(\eta) = \frac{1}{\alpha(\tan^2 \frac{\eta}{2} + \frac{1}{3})}, \quad (E, \Lambda > 0)
\]
\[
f(r) = \frac{2\alpha}{3\sqrt{\Lambda}}, \quad b(\eta) = \int_0^\eta a(\eta') d\eta', \quad 0 \leq \eta \leq \pi \quad (10)
\]

(\( \alpha \) a constant) for \( E, \Lambda \) both negative or of opposite sign, see e.g. [29, 33]. Those solutions have been determined under the proportionality condition \( M^2 \propto E^3 \). Therefore the ALTB cosmologies considered here finally depend on one arbitrary integration function \( E \), say. The ALTB models admitting factorized representation like (9) will be called ALTB factorized cosmologies. As shown in [29], the minimally coupled scalar field equation is separable in the ALTB cosmologies described by (9)). In the present paper the separability is proved for \( s = 1/2 \) in a slightly different manner than that suggested in [29].
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3 Dirac equation: separation in general ΛLTB model

The spinorial form of Dirac’s equation can be written, following [3], in the form

\[
\nabla_{AA'} P^A + i\mu_* \bar{Q}_{A'} = 0
\]
\[
\nabla_{AA'} Q^A + i\mu_* \bar{P}_{A'} = 0
\]  \hspace{1cm} (11)

(\sqrt{2}\mu_* = m_0). \ \nabla_{AA'} \ \text{is the covariant spinorial derivative and } \sqrt{2}\mu_* = m_0 \ \text{is the mass of the particle. The equation may be made explicit by using the Newmna Penrose formalism [10, 14] that is based on a suitable null tetrad frame \{l^i, n^i, m^i, m^*i\}. The tetrad used here has been originally introduced in [18] and it is such that the corresponding directional derivatives and non trivial spin coefficients are}

\[
D = l^i \partial_i = (\partial_t + e^{-\gamma/2} \partial_r) / \sqrt{2},
\]
\[
\Delta = n^i \partial_i = (\partial_t - e^{-\gamma/2} \partial_r) / \sqrt{2},
\]
\[
\delta = m^i \partial_i = (\partial_\theta + i \csc \theta \partial_\phi) / (y\sqrt{2}),
\]
\[
\delta^* = m^*i \partial_i = (\partial_\theta - i \csc \theta \partial_\phi) / (y\sqrt{2})
\]
\[
\rho = -(\dot{y} + y'e^{-\gamma/2}) / (y\sqrt{2}), \quad \mu = (\dot{y} - y'e^{-\gamma/2}) / (y\sqrt{2})
\]
\[
\beta = -\alpha = \cot \theta / (2y\sqrt{2}), \quad \epsilon = -\gamma = y' / (2\sqrt{2}y')
\]  \hspace{1cm} (12)

By developing the covariant spinor derivatives \(\nabla_{AA'}\) in terms of the directional derivatives and spin coefficients, the Dirac equation read

\[
(D + \epsilon - \rho)P^0 + (\delta^* - \alpha)P^1 - i\mu_* \bar{Q}^1 = 0
\]
\[
(\delta - \alpha)P^0 + (\Delta + \mu + \epsilon)P^1 + i\mu_* \bar{Q}^0 = 0
\]
\[
(D + \epsilon - \rho)Q^0 + (\delta^* - \alpha)Q^1 - i\mu_* \bar{P}^1 = 0
\]
\[
(\delta - \alpha)Q^0 + (\Delta + \mu + \epsilon)Q^1 + i\mu_* \bar{P}^0 = 0
\]  \hspace{1cm} (13)

By further setting

\[
P^A \equiv \frac{1}{y} \left( H_1(r,t)S_1(\theta), \ H_2(r,t)S_2(\theta) \right) \exp(im\varphi)
\]
\[
\bar{Q}^{A'} \equiv \frac{1}{y} \left( -H_1(r,t)S_2(\theta), \ H_2(r,t)S_1(\theta) \right) \exp(im\varphi), \quad m = 0, \pm 1, ..
\]  \hspace{1cm} (14)

Together with (12) into (13), one is finally left with the separated equations

\[
DH_1 + \epsilon H_1 = (i\mu_* + \frac{\lambda}{y\sqrt{2}})H_2
\]
\[
\Delta H_2 + \epsilon H_2 = (i\mu_* - \frac{\lambda}{y\sqrt{2}})H_1
\]  \hspace{1cm} (15)
\[
L^- S_2 = -\lambda S_1, \quad L^+ S_1 = \lambda S_2
\]  \hspace{1cm} (16)

\(\lambda\) the angular separation constant and \(L^\pm = \partial_\theta \mp m / \sin \theta + (1/2) \cot \theta\).
4 Solution of the separated angular equation

The angular equations (16) coincide with those of the Robertson Walker metric. A closed equation for $S_1, S_2$ can be easily obtained from (16). Such equations are the special case $s = 1/2$ of the general treatment of the angular problem for arbitrary spin solved in [22]. Under the boundary conditions $S_1(0) = S_1(\pi) = 0$ one has that $S_1(\theta) = S_{l'lm}(\theta)$ is essentially expressed by Jacobi polynomials [1]) and that is explicitly given in [22]. Moreover $S_{2lm} = S_{l'lm}$ and it holds

$$\lambda^2 = (l + 1/2)^2, \quad l = 0, 1, 2, 3, ..$$  \hspace{1cm} (17)

By multiplying by a suitable coefficient, the angular function satisfy the orthonormalization condition

$$\int d\Omega (e^{im\varphi} S_{jlm}(\theta) e^{-im\varphi} S_{j'l'm'} = \delta_{mm'}\delta_{ll'}, \quad (j = 1, 2)$$  \hspace{1cm} (18)

This is an helpful relation for the determination of the normal modes in view of a quantization of the Dirac field. The angular dependence of the wave function has been separated here by mimicking with some details the procedure considered in [19].

5 Time and radial equations in factorized $\Lambda$LTB cosmology

To separate (15) one can proceed by a slightly different method than what suggested in [19]. By setting

$$H_1 = U(r, t) + V(r, t) \quad H_2 = U(t) - V(r, t)$$  \hspace{1cm} (19)

into (15), by summing and subtracting the resulting equations and using the definitions of $D, \Delta$ one obtains the equations

$$\dot{U} + \sqrt{1 + 2E} \frac{\lambda}{y'} V' + \sqrt{2} (\epsilon - i\mu_*) U + \frac{\lambda}{y} V = 0$$

$$\dot{V} + \sqrt{1 + 2E} \frac{\lambda}{y'} U' + \sqrt{2} (\epsilon + i\mu_*) V - \frac{\lambda}{y} U = 0$$  \hspace{1cm} (20)

It is useful to consider a new time variable $\tau$ that in fact coincides with the parameter $\eta$:

$$\tau(r, t) = \frac{g(r)}{f} \int^t dt' \frac{dV'}{y(r, t')} = \int d\eta = \eta$$  \hspace{1cm} (21)
that follows from (9). Using \( \tau \) instead of \( t \), the equations (20) read
\[
\frac{1}{f} \dot{U} + \frac{\sqrt{1+2E}}{g'} V' + \left[ \frac{1}{2f} - i \mu_* \sqrt{2} a \right] U + \frac{\lambda}{g} V = 0
\]
\[
\frac{1}{f} \dot{V} + \frac{\sqrt{1+2E}}{g'} U' + \left[ \frac{1}{2f} + i \mu_* \sqrt{2} a \right] V - \frac{\lambda}{g} U = 0
\] (22)
where now dot means \( d/d\tau \). By finally setting
\[ U = F(r)T(\tau), \quad V = G(r)S(\tau) \] (23)
one obtains the coupled time equations
\[
\dot{T} + \left( \frac{1}{2a} - i \mu_* \sqrt{2} f a \right) T = k_1 S
\]
\[
\dot{S} + \left( \frac{1}{2a} + i \mu_* \sqrt{2} f a \right) S = k_2 T
\] (24)
and the coupled radial equations
\[
G' \sqrt{1+2E} + \frac{\lambda g'}{g} G = -k_1 \frac{g'}{f} F
\]
\[
F' \sqrt{1+2E} - \frac{\lambda g'}{g} F = -k_2 \frac{g'}{f} G
\] (25)
where \( k_1, k_2 \) are the separation constants of the first and second equation (21) respectively. The last equations can then be easily disentangled to obtain the time and radial equations in closed form
\[
\dot{T} + \frac{\dot{a}}{a} T + \left[ \frac{1}{2} \partial_\tau \left( \frac{\dot{a}}{a} \right) + \frac{1}{4} \left( \frac{\dot{a}}{a} \right)^2 + 2f^2 \mu_*^2 a^2 - i \mu_* \sqrt{2} f \dot{a} \right] T = k_1 k_2 T
\]
\[
\dot{S} + \frac{\dot{a}}{a} S + \left[ \frac{1}{2} \partial_\tau \left( \frac{\dot{a}}{a} \right) + \frac{1}{4} \left( \frac{\dot{a}}{a} \right)^2 + 2f^2 \mu_*^2 a^2 + i \mu_* \sqrt{2} f \dot{a} \right] S = k_1 k_2 S
\] (26)
\[
F'' + \left[ \frac{E'}{1+2E} - \frac{g''}{g^2} \right] F' + \frac{g''}{g^2} \left[ \frac{\lambda}{\sqrt{1+2E}} - \frac{\lambda^2}{1+2E} - \frac{k_1 k_2}{1+2E} \frac{g^2}{f^2} \right] F = 0
\]
\[
G'' + \left[ \frac{E'}{1+2E} - \frac{g''}{g^2} \right] G' - \frac{g''}{g^2} \left[ \frac{\lambda}{\sqrt{1+2E}} + \frac{\lambda^2}{1+2E} + \frac{k_1 k_2}{1+2E} \frac{g^2}{f^2} \right] G = 0
\] (27)
Note that the radial equations (27) follow one from the other by the substitution \( \lambda \rightarrow -\lambda \). As to the time equations, one can verify that they imply
\[
\frac{d}{dt} \left[ a(T - S) \right] = a [(\kappa_1 + k_2)T - (k_1 + \kappa_2)S] \] (28)
\[
k_1 = \kappa_2 \quad \Rightarrow \quad a(T - S) = \text{const.} = A_-
\] (29)
\[
k_1 = -\kappa_2 \quad \Rightarrow \quad a(T + S) = \text{const.} = A_+
\] (30)
Hereafter it will be assumed \( k_1 = -\kappa_2 = ik \) so that \( k_1 k_2 = -k^2 \), (30) is satisfied and the time equations (26) are the complex conjugate one of the other.
6 Dirac equation: orthogonal solutions

On account of the previous results some comments are possible on the existence of orthogonal solutions of Dirac equation. To that end the general definition of inner product of states considered in [9] are useful. From the solutions \( \psi \leftrightarrow (P, Q) \), \( \psi' \leftrightarrow (P', Q') \) of the Dirac equation one can construct the spinor

\[
J^{AA'}(\psi, \psi') = P^A P'^A + Q^A Q'^A
\]

that is divergence free \( \nabla AA' J^{AA'} = \nabla_\alpha J^{\alpha A'} = 0 \) (\( J^{A'A'}(\psi, \psi) \) is in fact the conserved current). Accordingly an inner product can be defined between solutions of Dirac equation by

\[
(\psi, \psi') = \int \Sigma J_\alpha(\psi, \psi') \left( -g_\alpha(x) \right)^{\frac{1}{2}} n^\alpha d\Sigma
\]

[6] where \( \Sigma \) is a space-like Cauchy surface, \( n^\alpha \) a future directed unit vector orthogonal to \( \Sigma \) and \( \sigma'_{AA'} \) the \( t \)-generalized Pauli matrix based on the null tetrad frame assumed in Section 3 such that \( \sigma'_{AA'} = (1/2) \text{diag}\{1, 1\} \).

When applied to factorized ALTB cosmologies, with obvious notations and by taking into account (9), (14), (18), (19), (23), (30), one obtains

\[
(\psi, \psi') = a \int dr \frac{g'}{\sqrt{1 + 2E}} (F_{ki} \bar{F}_{k'i} T_k T'_{k'} + G_{ki} \bar{G}_{k'i} S_k S'_{k'}) \delta_{m'm'} \delta_{l'l'}
\]

(35)

\[
= a (T_k T'_{k'} + S_k S'_{k'}) \delta(k - k') \delta_{m'm'} \delta_{l'l'}
\]

(36)

\[
= \delta(k - k') \delta_{m'm'} \delta_{l'l'}
\]

(37)

by a suitable choice of \( A_+ \) in (30) and provided

\[
\int dr \frac{g'}{\sqrt{1 + 2E}} F_{ki} \bar{F}_{k'i} = \int dr \frac{g'}{\sqrt{1 + 2E}} G_{ki} \bar{G}_{k'i} = \delta(k - k')
\]

(38)

The problem is then to establish whether solutions of (27) do indeed satisfy the last relation. In doing this one has to consider that, by the argument of [33], \( E \) and \( g \) are not independent, but there results \( g \propto \sqrt{E} \). In the next Section the relevant case of the Robertson Walker like space-time is studied and a class of ortho-normal solutions determined.

7 Open Robertson Walker like model

In this Section the previous scheme is applied to the special case

\[
E = \frac{r^2}{2}, \quad g = \frac{r}{\sqrt{2}}, \quad f = 1
\]

(39)
Accordingly, by setting \( s = \sinh^{-1} r \), the first equation in (27) reas

\[
F'' + \frac{\lambda \cosh s}{\sinh^2 s} - \frac{\lambda^2}{\sinh^2 s} = \frac{-k^2}{2} F
\]  
(40)

By further setting \( \xi = \cosh s \) in (40) gives

\[
F'' + \frac{\xi}{(\xi + 1)(\xi - 1)} F' + \frac{\lambda \xi - \lambda^2 + (k^2/2)(\xi^2 - 1)}{(\xi + 1)^2(\xi - 1)^2} F = 0
\]  
(41)

By the position \( F(\xi) = (\xi - 1)^{-\lambda/2}(\xi + 1)^{\lambda/2}Z(\xi) \) in (41) and then by passing to the variable \( x = (1 - \xi)/2 \) in the resulting equation one obtains for \( Z(x) \) the hypergeometric equation [1]

\[
Z'' + [c_o - (a_o + b_o + 1)x]Z' - a_ob_oZ = 0
\]  
(42)

\[
c_o = c_o(\lambda) = \lambda + 1/2 = l + 1, \quad a_o = -b_o = ik
\]  
(43)

where the substitution \( k \rightarrow k/\sqrt{2} \) has been done. A solution of (27) is therefore in terms of the hypergeometric function

\[
F_{kl}(s) \equiv F_{-kl}(s) \equiv F_{kl}(s) = \left(\frac{\cosh s - 1}{\cosh s + 1}\right)^{\lambda/2} F(a_o, b_o; c_o; 1 - \cosh s/2)
\]  
(44)

because \( F(a_o, b_o; c_o; z) = F(b_o, a_o; c_o; z) \). On account of that property we confine to consider only values \( k > 0 \). Note that [1]

\[
F_{kl}(s) \xrightarrow{s \rightarrow \infty} A(k)e^{-iks} + \overline{A}(k)e^{iks}
\]  
(45)

\[
A(k) = \frac{4ik(l + 1)!\Gamma(-2ik)}{\Gamma(-ik)\Gamma(l + 1 - ik)} \equiv \overline{A}(-k)
\]  
(46)

From (40) and (45) one has then

\[
\int_0^r \frac{F_{kl}F_{k'l}}{\sqrt{1 + r^2}} dr \equiv \int_0^s F_{kl}F_{k'l} ds = \frac{1}{k'^2 - k^2} \left[ F_{kl}'(s)F_{k'l}(s) - F_{k'l}'(s)F_{kl}(s) \right]_0^s
\]  
(47)

\[
\xrightarrow{s \rightarrow \infty} -i \left[ \frac{1}{k'-k} \left( A(k)\overline{A}(k')e^{is(k'-k)} - A(k')\overline{A}(k)e^{-is(k'-k)} \right) + \frac{1}{k' + k} \left( \overline{A}(k)\overline{A}(k')e^{is(k'+k)} - A(k)A(k')e^{-is(k'+k)} \right) \right]
\]  
(48)

If one writes (see also [1])

\[
A(k)\overline{A}(k') = |A(k)|^2 + O(k' - k) \equiv \frac{(l)!^2}{4\pi} \frac{k}{(k^2)_{l+1}} \tanh(\pi k) + O(k' - k)
\]  
(49)
and consider $k, k' > 0$ then

$$\int_0^{s} F_{kl} \overline{F}_{k'l} ds \xrightarrow{s \to \infty} -\frac{i}{k' - k} |A(k)|^2 \left( e^{is(k' - k)} - e^{-is(k' - k)} \right)$$

(50)

$$s \to \infty \quad 2|A(k)|^2 \frac{\sin(k' - k)s}{k' - k}$$

(51)

$$s \to \infty \quad \frac{(l!)^2}{k^2} \frac{k}{(k^2)_{l+1}} \tanh(\pi k) \delta(k' - k)$$

(52)

because the other terms in (48) go to zero in the weak sense for $s \to \infty$. Therefore, by multiplying $F_{kl}$ by $(2\pi)^{-1/2} |A(k)|^{-1}$ one has the orthonormal condition (38).

For what concerns the solution $G_{kl}$ one can choose, as mentioned, $G_{kl}(r, \lambda) \equiv F_{kl}(r, -\lambda)$. In this case however the hypergeometric function $F(a_o, b_o; c_o; x)$ is not defined because $c_o = c(-\lambda) = -l$. In this case one has then [1]

$$G_{kl}(s) = \frac{(\xi - 1)}{(\xi + 1)} \lim_{c_o \to -l} \frac{1}{\Gamma(c_o)} F(a_o, b_o; c_o; x)$$

(53)

$$= \frac{(\xi + 1)}{(\xi - 1)} \frac{(l!)^2}{(l + 1) (l + 1)!} F(ik + l + 1, -ik + l + 1; l + 2; 1 - \frac{\xi}{2})$$

(54)

$$s \to \infty \quad B(k)e^{-iks} + \overline{B}(k)e^{iks}$$

(55)

$$B(k) = \frac{(-1)^{l+1}(k^2)_{l+1} \Gamma(-2ik)}{\Gamma(1 - ik) \Gamma(l + 1 - ik)}, \quad |B(k)|^2 = \frac{(k^2)_{l+1}}{4\pi k} \tanh(\pi k)$$

(56)

where $\xi = \cosh s$. Therefore on account of (55), (56) one can proceed as for $F_{lk}$ to obtain the orthogonality relation (38) for the solutions $G_{kl}(r)$. Hence the condition (37) is explicitly satisfied in the RW like space time for the class of solutions with $k > 0$.

8 Correction of a previous error and comments.

The present study gives the occasion to point out an error in a previous study of the Dirac equation in the RW metric. This in turn gives indication of the choice of the definition of inner product of Dirac states. More precisely one can check that equation (8) in Ref.[23] is not correct, the right expression being the present one in (28) in correspondence to the upper signs. There follows that eq. (9) of Ref. [23] holds for $k_1 = k_2$, that is not satisfied by the choice $k_1 = k_2 = ik$ there assumed. On the other hand, if a correct choice is done like $k_1 = -k_2 = ik, \ k \in \mathbb{R}$, then $k_1k_2 = k^2$ and the arguments following equation (8) in [23] should be accordingly corrected. In the present scheme such choice implies that the corresponding asymptotic behavior of $F_{kl}$ is like (45), but
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in terms of real exponential \(\exp\pm(k s)\). There are therefore two situation in correspondence to the choice of the integrals (29), (30) of the time equation. The choice (29) is compatible with the scalar product of Dirac given in Ref. [23], eq. (17)), in order to have time independent scalar products and selects highly divergent solutions for \(s \to \infty\). The choice (30) is compatible with the present scalar product (32) but not with the one given in [23] eq.(17), and selects solutions (see (45) and (55)) that are oscillating for \(s \to \infty\).

In this paper the Dirac equation is studied within a class of LTB cosmological model with cosmological constant. The model still depends on an arbitrary integration function \(E(r)\). The Dirac equation is completely separated by using the Newman Penrose formalism. The problem of orthogonal solutions is also discussed. It is finally reduced to the solution of the separated radial equation that is in general difficult. The goal is obtained in the special case of a Robertson Walker like metric. A set of orthogonal solutions is explicitly determined. (The problem of the determination of normal modes, in view of a quantization of the Dirac field in RW metric, is beyond the object of the present paper).

The general solution of the Dirac equation for arbitrary \(E(r)\) is an open problem. It depends on the solution of the separated radial equation for general expression of \(E(r)\). Establishing conditions on \(E(r)\) under which acceptable solutions exist is an interesting problem.

Also the problem of the solution of the separated time equation is left open. Those equations heavily depend on the underlying cosmological evolution that, as it appears in the examples reported at the beginning of this paper, is quite complicated for cosmologies with parametric factorized representation.

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