Fractional Quantization of Holonomic Constrained Systems Using Fractional WKB Approximation

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Abstract

In this work, the fractional canonical quantization for holonomic constrained systems is examined using the fractional WKB approximation. The fractional Hamilton-Jacobi function is obtained. The solutions of the equations of motion are derived from this function. It is shown that these solutions are in exact agreement with using the fractional Euler-Lagrange equations and fractional Hamilton’s equations. Also, this function enables us to construct the suitable wave function and then to quantize these systems using the fractional WKB approximation. One example is examined to illustrative the formalism.

PACS: 11. 10. Ef, 45. 20. –j, 45. 20. Jj, 45. 10. Hj, 04. 60. Ds, 04.20.Fy

Keywords: Holonomic Constraints; Fractional Derivatives; Canonical Quantization; Fractional WKB Approximation

1. Introduction

Most references of classical mechanics have discussed the Euler-Lagrange equations for the holonomic constraints as regular systems [3, 6]. These systems describe dynamic systems with constraints depend only on the coordinates. Authors have examined formalism for singular systems within the framework of the canonical method [8, 14, 15, 17]. Rabei has used this method for regular Lagrangians with holonomic constraints as singular systems [11, 22]. In this formalism, the Lagrange multipliers for these systems were introduced as generalized
coordinates. The equations of motion were written as total differential equations, and then these systems were quantized using the WKB approximation [22].

The fractional derivatives have reached a great status in various branches of science, applied mathematics, physical systems and engineering [9, 10, 21]. Riewe has used fractional derivatives to construct a Lagrangian and a Hamiltonian for non-conservative systems. In Riewe formalism, one can obtain the Lagrangian and the Hamiltonian equations of motion for these systems [19, 20]. A powerful formalism for investigating the fractional variational problem of Lagrange was discussed by Agrawal’s [1, 2]. In this formalism, the fractional Euler-Lagrange equations were derived. Besides, the generalization of Lagrangian and Hamiltonian fractional mechanics with fractional derivatives were extended and discussed in details in [12, 18]. The Hamilton’s equations of motion were obtained in a similar manner to the usual mechanics.

Recently, the Hamilton-Jacobi partial differential equation and WKB approximation have been studied for systems containing fractional derivatives using the canonical method [13, 16]. Authors have proposed a general formalism to solve the HJIPDE for these systems. In this formalism, the Hamilton-Jacobi function was determined. The equations of motion and the wave function were determined from this function. By constructing the wave function, the quantization has been carried out using this approximation.

This paper is organized as follow. In section 2, basic definitions of fractional derivatives are briefly reviewed. In section 3, the fractional Hamilton-Jacobi formulation for holonomic constraints has been discussed. In section 4, the Hamilton-Jacobi treatment is investigated. In section 5, fractional WKB approximation for holonomic constraints is discussed. In section 6, one illustrative physical example is examined. The work closes with some concluding remarks in section 7.

2. Mathematical Backgrounds

In this section, we will give the definitions of fractional derivatives include the left and right Riemann-Liouville fractional derivatives [1, 2] and their properties. The left Riemann-Liouville fractional derivatives is defined as

$$aD_x^\alpha f(x) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x - \tau)^{\alpha-n-1} f(\tau) d\tau .$$

and the right Riemann-Liouville fractional derivatives has the form

$$bD_x^\alpha f(x) = \frac{1}{\Gamma(n - \alpha)} \left(-\frac{d}{dx}\right)^n \int_x^b (\tau - x)^{\alpha-n-1} f(\tau) d\tau .$$

Where $n \in N$, $n - 1 \leq \alpha < n$ and $\Gamma$ represents the Euler’s gamma function.
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Remark: If $\alpha$ is an integer, these derivatives are defined as follows
\[ \alpha D^\alpha_x f(x) = \left( \frac{d}{dx} \right)^\alpha f(x), \quad -\alpha D^\alpha_x f(x) = \left( -\frac{d}{dx} \right)^\alpha f(x). \quad \alpha = 1, 2, \ldots \] (3)

3. Fractional Hamilton-Jacobi Formulation for Holonomic Constraints

The Lagrangian formulation for holonomic constraints depending on the fractional derivatives is given by
\[ L'(q_i, \lambda_\mu, a, D^\alpha_t q_i, D^\beta_b q_i, t) = L(q_i, a, D^\alpha_t q_i, D^\beta_b q_i, t) + \lambda_\mu f_{\mu} \quad 0 < \alpha, \beta < 1, \] (4)
is subject to the constraint equation with $m$ constraints can be written as
\[ f_{\mu}(q_i, t) = 0, \quad \mu = n + 1, n + 2, \ldots, n + m \] (5)
all functions $q(t)$ have continuous LRLFD of order $\alpha$ and RRLFD of order $\beta$ for $a \leq t \leq b$, and satisfy the boundary conditions $q(a) = q_a, q(b) = q_b$.
Let $S[q]$ be a functional of the form
\[ S[q] = \int_a^b L(t, q, \lambda_\mu, a, D^\alpha_t q_i, D^\beta_b q_i) dt. \] (6)
where $\alpha, \beta \in \mathbb{R}^+$, when $\alpha = \beta = 1$, the above problem reduces to the simplest variational problem. The necessary condition for $S[q]$ to have an extremum for a given function $q(t)$ is that $q(t)$ satisfy the fractional Euler-Lagrange equation given by [7]
\[ \frac{\partial L}{\partial q} + D^\alpha_b \frac{\partial L}{\partial D^\alpha_t q} + D^\beta_b \frac{\partial L}{\partial D^\beta_b q} + \lambda_\mu \frac{\partial f_{\mu}}{\partial q} = 0. \] (7)
Eq. (7) represents the formulation of fractional Euler-Lagrange equation for holonomic constraints.

The fractional Hamiltonian for the holonomic constraints is given by
\[ H(q, \lambda_\mu, p_a, p_\beta, t) = p_a D^\alpha_t q + p_\beta D^\beta_b q + D^\alpha_t \lambda_\mu p_{\mu} - L(q_i, a, D^\alpha_t q_i, D^\beta_b q_i, t) - \lambda_\mu f_{\mu}. \] (8)
Can be calculated using the definition of the generalized momenta [18]
\[ p_a = \frac{\partial L'}{\partial D^\alpha_t q}, \quad p_\beta = \frac{\partial L'}{\partial D^\beta_b q}. \] (9a, 9b)
Following to previous references [4, 5, 18], the Hamilton’s equations of motion can be written as

\[
\frac{\partial H}{\partial p_\alpha} + \frac{\partial H'}{\partial p_\alpha} D^i_\lambda = D^i_\alpha q, \quad (10a)
\]

\[
\frac{\partial H}{\partial p_\beta} + \frac{\partial H'}{\partial p_\beta} D^i_\lambda = D^i_\beta q, \quad (10b)
\]

\[
\frac{\partial H}{\partial q} + \frac{\partial H'}{\partial q} D^i_\lambda = D^i_\beta p_\beta + i D^a_\beta p_\alpha, \quad (10c)
\]

\[
-\frac{\partial H}{\partial \lambda_\mu} - \frac{\partial H'}{\partial \lambda_\mu} D^i_\lambda = D^i_\mu. \quad (10d)
\]

It is interesting to note that Eq. (10d) leads to the holonomic constraints.

4. Hamilton-Jacobi Treatment for Holonomic Constraints

In this section, our goal to determine the Hamilton-Jacobi function. Under certain conditions it is possible to separate the variables in the Hamilton-Jacobi equations. In practice, the Hamilton-Jacobi technique becomes a useful computational tool only when such a separation can be effected [6].

Now, we will deal with a set of HJPDEs can be written in the form below

\[
H' = p_\alpha + H, \quad (11a)
\]

\[
H'_\mu = p_\mu \quad (11b)
\]

Defining the momenta \( p_\alpha \) and \( p_\mu \) as

\[
p_\alpha = \frac{\partial S}{\partial t} \quad (12a)
\]

\[
p_\mu = \frac{\partial S}{\partial \lambda_\mu} \quad (12b)
\]

where \( S \) is the Hamilton-Jacobi Function and reads as
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\[ S = S(aD_i^{\alpha-1}q_i, D_b^{\beta-1}q, E_1, E_2, \lambda_{\mu}, t) \]  

(13)

One can write Eqs. (11) in compact form as follows

\[ \frac{\partial S}{\partial t} + H_o = 0, \]  

(14a)

\[ \frac{\partial S}{\partial \lambda_{\mu}} = 0. \]  

(14b)

Eqs. (14) represent the HJPDEs and have the proposed solution

\[ S = S(aD_i^{\alpha-1}q_i, D_b^{\beta-1}q, E_1, E_2, \lambda_{\mu}, t) = f(t, E_1, E_2) + W_i(E_i, D_i^{\alpha-1}q) + W_i(E_i, D_i^{\beta-1}q) + f(\lambda_{\mu}) + A. \]  

(15)

Where \( E_1 \) and \( E_i \) are constants, \( A \) is some other constant and \( f(t, E_1, E_2) = -Et \).

The Lagrange multipliers \( \lambda_{\mu} \) are treated as independent variables, just as the time \( t \). Once we have found the Hamilton-Jacobi function \( S \), the equations of motion can be obtained by using the canonical transformations [6].

\[ \eta_1 = \frac{\partial S}{\partial E_1} = aD_i^{\alpha-1}Q, \]  

(16a)

\[ \eta_2 = \frac{\partial S}{\partial E_2} = bD_b^{\beta-1}Q, \]  

(16b)

\[ p_\alpha = \frac{\partial S}{\partial D_i^{\alpha-1}q} = \frac{\partial W_1}{\partial D_i^{\alpha-1}q}, \]  

(16c)

\[ p_\beta = \frac{\partial S}{\partial D_b^{\beta-1}q} = \frac{\partial W_2}{\partial D_b^{\beta-1}q}, \]  

(16d)

\[ p_{\mu} = \frac{\partial S}{\partial \lambda_{\mu}}. \]  

(16e)

Where \( \eta_1 \) and \( \eta_2 \) are constants and can be determined from the initial conditions.

5. Fractional WKB Approximation for Holonomic Constraints

Following to the quantization of constrained systems using WKB approximation [17, 22]. The wave function for the Holonomic system can be written as
\[ \Psi(q_i, \lambda_\mu, t) = \prod_{i=1}^{n} \psi_0(q_i) \exp \left[ \frac{i}{\hbar} S(q_i, \lambda_\mu, t) \right] \]  

(17)

Where \( \psi_0(q_i) = \frac{1}{\sqrt{p_i}} \) is the amplitude of the wave function (17).

The above wave function (17) satisfies the conditions

\[ \dot{H}_\alpha \Psi = 0 \]  

(18a)

\[ \dot{H}_\mu \Psi = 0 \]  

(18b)

In the semi-classical limit \( \hbar \rightarrow 0 \).

Thus, the wave function for the holonomic systems in the fractional form can be written as

\[ \Psi(a_{D_i}^{\alpha-1} q_i, b_{D_i}^{\beta-1} q_i, \lambda_\mu, t) = \frac{1}{\sqrt{p_\alpha p_\beta}} \exp \left[ \frac{i}{\hbar} S(a_{D_i}^{\alpha-1} q_i, b_{D_i}^{\beta-1} q_i, E_1, E_2, \lambda_\mu, t) \right] \]  

(19)

with the momenta operators

\[ \hat{p}_\alpha = \frac{\hbar}{i} \frac{\partial}{\partial a_{D_i}^{\alpha-1} q} \]  

(20a)

\[ \hat{p}_\beta = \frac{\hbar}{i} \frac{\partial}{\partial b_{D_i}^{\beta-1} q} \]  

(20b)

\[ \hat{p}_{\mu} = \frac{\hbar}{i} \frac{\partial}{\partial \lambda_\mu} \] \quad \text{where} \quad D_i^0 = 1 \]  

(20c)

Eq. (19) gives the solution of the Schrödinger equation for any given fractional system. It is interesting to notice that if \( \alpha \) and \( \beta \) are equal to unity, the usual classical solution of Schrödinger equation is satisfied and the probability is inversely proportional to the momenta

\[ |\Psi|^2 \cong \frac{1}{p(q)}. \]  

(21)

6. Example

5.1 As a first example, let us consider the motion of a disk of mass \( m \) and radius \( R \) that is rolling down an inclined plane without slipping.

The Lagrangian of our problem is given by
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\[ L = \frac{1}{2} m \dot{y}^2 + \frac{1}{4} mR^2 \dot{\theta}^2 + mg \sin \phi. \] (22)

Is subject to the holonomic constraint
\[ f = y - R\theta = 0. \] (23)

Where \( \phi \) is the angle of the incline plane.

The extended Lagrangian in fractional form can be written as
\[ L' = \frac{1}{2} m(D_0^{a} y)^2 + \frac{1}{4} mR^2 (D_1^{\beta} \theta)^2 + mg \sin \phi + \lambda(y - R\theta). \] (24)

The Euler-Lagrange equations corresponding to Eq. (7) become
\[ mg \sin \phi + m D_1^{\alpha} (D_1^{\alpha} y) + \lambda = 0, \] (25a)
\[ \frac{mR^2}{2} D_1^{\beta} (D_1^{\beta} \theta) - R\lambda = 0. \] (25b)

From Eqs. (25), one can obtain the classical results if \( \alpha \) and \( \beta \) are equal to unity. Thus, the acceleration and the angular acceleration can be calculated as
\[ \ddot{y} = \frac{2}{3} g \sin \phi, \] (26a)
\[ \ddot{\theta} = \frac{2g}{3R} \sin \phi. \] (26b)

The force of constraint (Lagrange multiplier) is given by
\[ \lambda = -\frac{1}{3} mg \sin \phi. \] (27)

One can obtain the same results using the canonical approach Eqs. (10) [16, 22, 23].

The canonical Hamiltonian of this system can be written as
\[ H_c = \frac{p_y^2}{2m} + \frac{p_{\theta}^2}{mR^2} - mg \sin \phi - \lambda(y - R\theta). \] (28)

The canonical momenta are
\[ p_y = \frac{\partial S}{\partial D_0^{a-1} y}, \] (29a)
\[ p_{\theta} = \frac{\partial S}{\partial D_1^{\beta-1} \theta}. \] (29b)
Using Eq. (14a), we can obtain
\[ H' = p_\mu + H = \frac{\partial S}{\partial t} + \frac{(\partial S/\partial D^{\alpha-1}_i y)^2}{2m} + \frac{(\partial S/\partial D^{\beta-1}_i \theta)^2}{mR^2} - mg \sin \phi - \lambda(y - R\theta) = 0, \] (30a)
\[ H'_\mu = p_\mu, \text{ where } p_\mu = \frac{\partial S}{\partial \lambda_\mu} \text{ and } \frac{\partial S}{\partial \lambda_\mu} = 0. \] (30b)

Making use of Eq. (15), the proposed action function can be written as
\[ S(\alpha D^{\alpha-1}_i y, D^{\beta-1}_i \theta, E_\gamma, E_\theta, \lambda, t) = f(t) + W_\gamma(E_\gamma, \alpha D^{\alpha-1}_i y) + W_\theta(E_\theta, D^{\beta-1}_i \theta) + f(\lambda) + A. \] (31)

Where \( A \) is a constant, \( f(t) = -Et \), and \( f(\lambda) = E_\lambda \) which is a constant.

Substituting Eq. (31) in Eq. (30), we get
\[ -E + \frac{(\partial W_\gamma / \partial \alpha D^{\alpha-1}_i y)^2}{2m} + \frac{(\partial W_\theta / \partial \beta D^{\beta-1}_i \theta)^2}{mR^2} - mg \sin \phi - \lambda(y - R\theta) = 0. \] (32)

One can write the above Eq. (32) into two separate equations as follows
\[ \frac{(\partial W_\gamma / \partial \alpha D^{\alpha-1}_i y)^2}{2m} - mg \sin \phi - \lambda y = E_\gamma, \] (33a)
\[ \frac{(\partial W_\theta / \partial \beta D^{\beta-1}_i \theta)^2}{mR^2} + \lambda R \theta = E - E_\gamma \equiv E_\theta. \] (33b)

Where \( E_\gamma \) and \( E_\theta \) are constants, we can solve Eq. (33) to get
\[ W_\gamma = \int \sqrt{2m(E_\gamma + y(\lambda + mg \sin \phi))} d\alpha D^{\alpha-1}_i y, \] (34a)
\[ W_\theta = \int \sqrt{mR^2(-E_\theta - \lambda R \theta)} d\beta D^{\beta-1}_i \theta. \] (34b)

Thus, we can write the Hamilton-Jacobi function Eq. (31) as
\[ S = -(E_\gamma + E_\theta)t + \int \sqrt{2m(E_\gamma + y(\lambda + mg \sin \phi))} d\alpha D^{\alpha-1}_i y + \int \sqrt{mR^2(-E_\theta - \lambda R \theta)} d\beta D^{\beta-1}_i \theta + E_\lambda + A \] (35)

Making use of Eqs. (16a, b), we obtain the equations of motion and they are in exact agreement with the fractional Euler-Lagrange equations and fractional Hamilton’s equations.
Using Eq. (16c, d)

\[
\begin{align*}
    p_y &= \frac{\partial S}{\partial (D_y^{\alpha-1})} = \frac{\partial W_y}{\partial (D_y^{\alpha-1})} = \sqrt{2m(E_y + y(\lambda + mg \sin \phi)}, \quad (36a) \\
    p_\theta &= \frac{\partial S}{\partial (D_\theta^{\beta-1})} = \frac{\partial W_\theta}{\partial (D_\theta^{\beta-1})} = \sqrt{mR^2(-E_\theta - \lambda R \theta)}. \quad (36b)
\end{align*}
\]

Now, we can construct the wave function (19) and has the form

\[
\Psi(y, \theta, \lambda, t) = \frac{1}{\sqrt{p_y p_\theta}} \exp \left( \frac{i}{\hbar} S(y, \theta, \lambda, E, E_\theta, \lambda, t) \right)
\]

\[
\Psi = \exp \left( \frac{i}{\hbar} \left( (E_y + E_\theta) + \sqrt{2m(E_y + y(\lambda + mg \sin \phi)D_y^{\alpha-1}y + \sqrt{mR^2(-E_\theta - \lambda R \theta)D_\theta^{\beta-1} \theta + E_\theta + A}} \right) \right.
\]

(37)

Now let us apply the HJPDE, Eq. (30a), to the wave function (37), after representing the relevant quantities as operators:

\[
\hat{H} \Psi = \left( \frac{\hbar}{i \partial_t} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial (y, D_y^{\alpha-1} y)^2} + \frac{\hbar^2}{mR^2} \frac{\partial^2}{\partial (\theta, D_\theta^{\beta-1} \theta)^2} - mgy \sin \phi - \lambda (y - R \theta) \right) \Psi = 0
\]

(38)

After some algebra and taking the semi-classical limit \( \hbar \to 0 \), we get

\[
\hat{H} \Psi = E \Psi
\]

(39)

6. Conclusion

This paper is mainly concerned with the fractional quantization of holonomic constrained systems using the WKB approximation. The fractional Hamilton-Jacobi function for these systems was obtained. The equations of motion were derived from this function. This function enables us to formulate the wave function and then to quantize these systems using the WKB approximation. Besides, we achieved that the classical results were obtained are agreement when fractional derivatives are replaced with the integer order derivatives. Also, it is proven that in the semi-classical limit \( \hbar \to 0 \), the Schrödinger equation and the equations of motion were satisfied, this means that the quantum results are in exact agreement with the classical results. One physical example was discussed to demonstrate the formalism.
References


Received: April 4, 2016; Published: May 16, 2016