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Cosmological Exact Solutions Set of a Perfect Fluid in an Anisotropic Space-Time in Petrov Type D

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Abstract

In this document a set of exact solutions is obtained by using the model of a perfect fluid on an homogeneous and anisotropic space-time that belongs to (Local Rotational Symmetry) Petrov's type D. This solutions represent the possible scenarios of a Universe for which the pressure P and the energetic density μ of a fluid are proportional ($P = \lambda\mu$), where λ can have any value. Hence, the aforementioned solution has importance for the fluid models with standard matter ($\lambda \in [0, 1/3]$), and also for other models, like quintessence, dark energy, phantom, ekpyrotic etc. It is established that for any state of the fluid, there are two possible solutions that depend on the matter expanding faster on a perpendicular coordinate on a plane, rather tan the plane itself, or if the situation is the opposite. Possible singularities are studied, and cases where the solutions are not singular in $t = 0$ are established; moreover, to the possibility for the solutions, depending of the t values, to become isotropic, pointing out that going from a model that was anisotropic at the beginning to an isotropic one with the augmentation of t, depends on whether the pressure P of the fluid is lower or not than the density of the energy μ.

Keywords: cosmology, exact, solution, Einstein

1 Introduction

Cosmology has become more dynamic since the data related to the microwave background radiation obtained with the satellites COBE,WMAP and PLANCK. By using the data of WMAP, it has been discovered [1] a spatial preference on the part of radiation to the microwave background radiation (CMB), referred to as the axis of evil when showing that some hot and cold spots on the CMB do not distribute randomly, but instead they do aligned along the axis. This discovery is confirmed on other works related to the direction of the polarization of Quasars [2] and the rotation of spiral galaxies [3].
Other important discoveries, though exposed to criticisms and refutation, are the ones obtained with the data of the satellites WMAP and Planck: the non-Gaussianity in the CMB (see examples [4],[5]), vast empty spaces [6],large-scale peculiar velocities larger than the expected on the standard cosmological model (Dark Flow anomaly) [7], [8], adverse data analysis of gravitational waves for some cosmological models of inflation and string theory [9] and inside the LCDM cosmological model, discrepancies with a few observations; such as the measurement of the current rate of expansion H_0, the galaxies shear power spectrum and the number of galaxy clusters, as a result of the Plank satellite's rates measurements [10], and to a lesser extent a deficit of $5-10\%$ at the low power spectrum mode $[l<40]$ vs. the standard model LCDM [11] , the lack of evidence of dynamic dark energy [12], the reduction of the possible standard cosmological model, such as exponential potential model, simple hybrid inflation models, and monomial potential models of degree $n>2$ [13]. Moreover, and this being another restrictions to models with variable physical constants, both temporal and special, it was found that the time variable independent of the fine-structure constant α and the mass of the electron m_e are limited by the data of the Plank satellite for $\Delta\alpha/\alpha=(3.6\pm3.7)\cdot10^{-3}$ and $\Delta m_e/m_e=(4\pm11)\cdot10^{-3}$ at a 68% level of confidence and for a spatial variation of the fine-structure constant α, at $\Delta\alpha/\alpha=(-2.4\pm3.7)\cdot10^{-2}$ [14]; related to the same topic and base on the data of VLT/UVES and the Keck telescope, it has been observed a dipole of the $\Delta\alpha/\alpha$ variation with a amplitude of $0.97^{+0.22}_{-0.20}\cdot10^{-5}$ [15] this implying an anisotropic variation of the α constant.
The aforementioned motivates to look for new alternatives, both inside and outside the standard cosmology. An alternative is to consider that our position in the Universe is special [16] this opposing to the Copernican principle, which would mean that the Universe does not have to isotropic and homogeneous or at least have both characteristics at once. Possible heterogeneous isotropic models are achieved by using the symmetric Lemaitre-Tolman-Bondi model [17] this giving satisfactory results related to some astronomical data. Other suggested model is the homogeneous anisotropic, this usually trans-

forms into isotropic models (see example [18], [19]), with the augmentation of the temporal coordinate. This not only finds support from the theoretical point of view, but also within the experimental. According to the satellites PLANK and WMAP it has surfaced evidence for a dipolar power modulation in the temperature's fluctuations of the CMB signal at low multipoles l; the prior implies certain deviation in the isotropic statistical, favoring anisotropic models. [20].

Anisotropic and isotropic models have been obtained and analyzed vastly using, in much cases, the Bianchi-I, V y IX symmetric models. For example, Bianchi-IX symmetric model has not only generated models such as the FRWL closed Universe and the Misner's Mixmaster Universe[21], but also models that serve to the space-time study and its oscillating deformations neighboring singularity. [22, 23]

It has been obtained models for a vast range of equations of fluid states, using for this the perfect fluid, see example [24], [25] where several sceneries of the Universe are described, that depend on the type of equation and allowing exact analytical solutions for some limited cases of types of equations of state or with metrics that contain complex elements ($g_{\mu\nu} \in \mathbb{C}$).

Also, solutions considering a binary mixture of dark energy, or quintessence, and a perfect fluid in conjoint movement [25] and dark energy and dark mass with different velocities [26] have been obtained; for example, the investigation on gravitational waves in a Bianchi-I space-time has been carried out fundamentally, but not restricted, for the Kasner solution and important differences have been determined between a symmetry of the anisotropic Bianchi-I type and isotropic of the Robertson and Walker type, an important influence was established, generated by the anisotropic expansion of space, on the gravitational waves and the influence of the gravitational waves energy on the anisotropic space in expansion [27], [28]; in addition to the anisotropy role, in the coupling between gravitational waves and density modes and the density perturbations as a possible cause in the formation of galaxies [29].

The equations of state of $\lambda\mu = P$, usually analyzed in literature in relation to cosmology are:

1. $\lambda < -1$ ("phantom" model [30]), for which there are no restrictions for values lower thant -1 of λ [31]
2. $\lambda = -1$ (model with dark energy or quintessence), which is part of the current cosmological standard model)
3. $-1 < \lambda < 0$ (quintessence [32]),
4. $\lambda = 0$ (dust type model with zero pressure [33]),
5. $\lambda = 1/3$ (ultrarelativistic model or of radiation [33]),
6. $1/3 < \lambda < 1$ (hard Universe [24], [34], [35]),
7. $\lambda = 1$ (Zeldovich Universe or stiff matter [24], [34], [35]) and
8. $\lambda > 1$ (ekpyrotic matter) equation of state considered for some cosmologists

as impossible to do [34], [35]. Of the above cases it is noticed a missing of 9. $0 < \lambda < 1/3$ (ordinary relativistic fluid under pressure) that has been of less interest in literature but with a greater mathematical difficulty in the process of obtaining the cosmological models with this fluid.

In the present work all the possible scenarios will be analyzed (the previous nine), using a symmetry close to the classic Bianchi-I, but with the component $g_{00} = g_{00}(t)$.

2 The Interval Metric

The interval metric to be used is given in its most general form, under the following structure

$$ds^2 = d^2dt^2 - a^2dx^2 - b^2dy^2 - c^2dz^2, \tag{1}$$

where d, a, b,and c are any functions of t and the metrics represent and anisotropic and homogeneous space-time (in the case when $d = 1$, Bianchi-I's symmetry is obtained).
Using the Newman-Penrouse formalism (NP formalism) [36], it is possible to determine the the Petrov's type of space-time to which may belong a metric. The null tetras of the metric (1) at the NP formalism are
$l_\mu = \left(d/\sqrt{2}, a/\sqrt{2}, 0, 0\right)$, $n_\mu = \left(d/\sqrt{2}, -a/\sqrt{2}, 0, 0\right)$,
$m_\mu = \left(0, 0, b/\sqrt{2}, -ic/\sqrt{2}\right)$, $\bar{m}_\mu = \left(0, 0, b/\sqrt{2}, ic/\sqrt{2}\right)$.
The 5 scalars of Weyl-NP, are defined as follows:
$\Psi_0 = C_{\alpha\beta\gamma\delta}l^\alpha m^\beta l^\gamma m^\delta$, $\Psi_1 = C_{\alpha\beta\gamma\delta}l^\alpha n^\beta l^\gamma m^\delta$, $\Psi_2 = C_{\alpha\beta\gamma\delta}l^\alpha m^\beta \bar{m}^\gamma n^\delta$,
$\Psi_3 = C_{\alpha\beta\gamma\delta}l^\alpha n^\beta \bar{m}^\gamma n^\delta$, $\Psi_4 = C_{\alpha\beta\gamma\delta}n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta$,
where $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor. For the metric (1), the previous scalars have the following form

$$\Psi_0 = \Psi_4 = \frac{-1}{4d^2cb}\left(\ddot{c}b - \ddot{b}c\right) - \frac{1}{4d^3acb}(ad)^{\cdot}\left(c\dot{b} - b\dot{c}\right), \tag{2}$$

$$\Psi_2 = -\frac{1}{3}\Psi_0 - \frac{1}{6d^2ca}(\ddot{c}a - \ddot{a}c) - \frac{1}{6d^3acb}(bd)^{\cdot}(\dot{a}c - \dot{c}a), \tag{3}$$

$$\Psi_1 = \Psi_3 = 0, \tag{4}$$

And the components non zero and independent of the Weyl tensor are:

$$C_{0101} = 2a^2d^2\Psi_2, \quad C_{0202} = b^2d^2(\Psi_0 - \Psi_2), \quad C_{0303} = -c^2d^2(\Psi_0 + \Psi_2), \tag{5}$$

$$C_{1212} = b^2a^2(\Psi_0 - \Psi_2), \quad C_{1313} = -a^2c^2(\Psi_0 - \Psi_2), \quad C_{2323} = -2b^2c^2\Psi_2 \tag{6}$$

It is of interest in the present work when the metric is of Petrov's O or D type; it is simple to note that when considering the roots of a polynomial [37]

$$Q = \Psi_0z^4 + 4\Psi_1z^3 + 6\Psi_2z^2 + 4\Psi_3z + \Psi_4 = \Psi_0z^4 + 6\Psi_2z^2 + \Psi_0 \tag{7}$$

if $a = b$ it has $\Psi_2 = \Psi_0/3$, and the roots of Q are $i, -i, i, -i$, it's analogous if $a = c$ it has $\Psi_2 = -\Psi_0/3$, and the roots of Q are $1, 1, -1, -1$ or if $c = b$ it obtains that $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$, $\Psi_2 \neq 0$; so if two of the three functions a, b, c are equal, any metric with this form is of Petrov type D (the same happens for similar cases in the metric structure), if $b = a = c$, then $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0$, in all the space-time, so a metric will then be isotropic-homogeneous and of the type Petrov O.

Another important thing to notice, is that the symmetry (1), under the Szekeres criterion [38], is absent from solutions with terms of radiation of longitudinal waves ($\Psi_1 = \Psi_3 = 0$) and of transversal waves, as null tetrads can be selected such that $\Psi_0 = \Psi_4 = 0$ ($b = c$); therefore, it is conclude as well that the left term on the equation in Ψ_2

$$-\frac{1}{6ad^3b^2}\left(\dot{a}b - \dot{b}a\right)\left(\dot{d}b + \dot{b}d\right) + \frac{1}{6ad^2b}\left(\ddot{a}b - \ddot{b}a\right) \tag{8}$$

it must represent, to some extent, a type of monopole of "Coulomb" (for further details see [38]).

It should be noted that the symmetry (1) allows several solutions both type O as well as D of Petrov, for example, similarly to the symmetry of Bianchi-I with $g_{00} = 1$. The article [39] says: "...it can be shown that there exist only two families of conformally flat Bianchi I metrics (we ignore the case of the Friedmann-Robertson-Walker space-time) given by:..".This is no true. First of all, the general solution, of Petrov O type, to which is referred to at [39] is

$$ds^2 = R(\tau)^2\left(d\tau^2 - \left(\left(c_1 f + \frac{c_2}{f}\right)^2 dx^2 - \left(\frac{c_1 f}{c_3} - \frac{c_2}{f^{\frac{1}{c_3}}}\right)^2 dy^2 - dz^2\right)\right), \tag{9}$$

where $f = exp(c_3\tau)$. In [39] conformal families are considered: 1) $c_1 = c_2 = 1/2$, $c_3 = i/a$ and 2) $c_1 = c_2 = 1/2$, $c_3 = 1/a$, where $i = \sqrt{-1}$ y $y = y'(c_3^4)^{1/4}$, as the only two solutions.

A conformably flat solution of the type Petrov O, different to the previous one is

$$ds^2 = F\left(dt^2 - t^{2/3}\frac{(dy^2 + dx^2)}{(C_1 t^{2/3} + C_2)} - t^{2/3}(C_1 t^2/3 + C_2)^2 dz^2\right), \tag{10}$$

where $F = F(t)$.

The above metric is flat ($R_{\alpha\beta\gamma\tau} = 0$), si $F = C_3(C_1 + C_2 t^{-2/3})$, i.e

$$ds^2 = C_3\left(C_1 + \frac{C_2}{t^{2/3}}\right)dt^2 - C_3\left(dx^2 + dy^2\right) - C_3\left(C_1 t^{2/3} + C_2\right)^3 dz^2, \tag{11}$$

where C_1, C_2 y C_3 are arbitrary constants.

The previous symmetry (10), serves as a base for the following work. In another publication where it is considered that the components of the tetra-dimensional velocity u_0, u_3 are not constant or null, it will be use a symmetry in the form (10), to obtain anisotropic, homogeneous, cosmological solutions, but conformably flat. Will proceed as follows: it will be deemed that in (1) $a = b$, but $c \neq b$, thus keeping the spatial anisotropy and the type of Petrov to which it belongs (tipo D) in the following form

$$ds^2 = Fdt^2 - t^{2/3}K(dx^2 + dy^2) - \frac{t^{2/3}}{K^2}dz^2, \tag{12}$$

Where F y K, are functions of t.

3 Solutions of the Einstein's Equations for the Perfect Fluid Model

The model of the perfect fluid in cosmology represents a fluid without viscosities, isentropic ($P = P(\mu)$) and without shear stress; this can be written as follows: [33]

$$T_{\alpha\beta} = (\mu + P)\, u_\alpha u_\beta - g_{\alpha\beta}P, \tag{13}$$

where $T_{\alpha\beta}$ is the tensor of the impulse-energy of the perfect fluid, u_α the tetra-dimensional velocity, $g_{\alpha\beta}$ the metric tensor, μ y P the energy density and the pressure of the fluid respectively.

The equation of the state for the analyzed fluid will be take from $P = \lambda\mu$, where λ is constant.

The components of the Einstein's tensor $G_\alpha^\beta = R_\alpha^\beta - 1/2\delta_\alpha^\beta R$ different from zero are

$$G_0^0 = \frac{4\,K^2 - 9\,t^2\dot{K}^2}{12t^2K^2F}, \tag{14}$$

$$G_1^1 = -\frac{3Kt\dot{K}\left(2F - \dot{F}t\right) + 3Ft^2\left(2K\ddot{K} - 5\dot{K}^2\right) + 4K^2\left(\dot{F}t + F\right)}{12t^2K^2F^2}, \tag{15}$$

$$G_2^2 = G_1^1, \tag{16}$$

$$G_3^3 = -\frac{-6Kt\dot{K}\left(2F - \dot{F}t\right) - 3Ft^2\left(4K\ddot{K} - \dot{K}^2\right) + 4K^2\left(\dot{F}t + F\right)}{12t^2K^2F^2}, \tag{17}$$

A fluid with tetra-dimensional velocity will be considered $u_\alpha = (u_0, 0, 0, 0)$, hence the components of the energy impulse tensor (13) different from zero are

$$T_0^0 = \mu, \tag{18}$$

$$T_1^1 = T_2^2 = T_3^3 = P, \tag{19}$$

Implying that from Einstein's equations $G_\alpha^\beta = \kappa T_\alpha^\beta$, it must be complied that $G_1^1 = G_3^3$, so be that from (15) y (17) is obtained

$$\dot{K} K \left(2F - \dot{F}t\right) - 2Ft\left(-K\ddot{K} + \dot{K}^2\right) = 0 \tag{20}$$

therefore

$$K = K_0 e^{C_1 \int \frac{F^{1/2}}{t} dt}, \tag{21}$$

without loss of generality, the constant K_0 en (21) will be considered equal to 1.

From the equation of the state $P = \lambda\mu$, it has that $G_1^1 = \lambda G_0^0$, and from (15),(17) and (21) the following equations is obtained

$$F\left(-1+\lambda\right)\left(9\,FC_1^2 - 4\right) + 4\dot{F}t = 0, \tag{22}$$

its solution is

$$F = \begin{cases} \frac{4}{9C_1^2 + 4t^{1-\lambda}C_2} & \text{if } \lambda \neq 1 \\ const & \text{if } \lambda = 1 \end{cases} \tag{23}$$

or in an equivalent way and without loss of generality

$$F = \frac{1}{1+\alpha t^{1-\lambda}},$$
$$K_\pm = \left(\frac{4(\sqrt{1+\alpha\, t^{1-\lambda}}-1)}{\alpha(\sqrt{1+\alpha\, t^{1-\lambda}}+1)}\right)^{\pm\frac{2}{3(1-\lambda)}} = \left(\frac{4t^{1-\lambda}}{\left(\sqrt{1+\alpha\, t^{1-\lambda}}+1\right)^2}\right)^{\pm\frac{2}{3(1-\lambda)}} \tag{24}$$

for $\lambda \neq 1$ and

$$F = 1, \quad K_{Z\pm} = t^{\pm 2/3\sqrt{1-3M}} \tag{25}$$

when $\lambda = 1$. In (24) and (25) α (different from each λ) and M are considered positive constants (by requiring that $\mu \geq 0$), and $0 \leq M \leq 1/3$ (by requiring that $g_{\mu\nu} \in \mathbb{R}$). The density and the pressure, independently of the sign chosen for K, are

$$\mu = \frac{\alpha}{3t^{1+\lambda}}, \quad P = \frac{\lambda\,\alpha}{3t^{1+\lambda}}, \tag{26}$$

for the case when $\lambda \neq 1$, and when $\lambda = 1$, it should be change to $\alpha/3$ por M. The prior solutions (26) are multi-fluids and determine the possible cosmological scenarios when using the model of the perfect fluid for the used symmetry. All the solutions transform into the metric (11), of Kasner E_{D_0} of the Petrov type D (see below for the definition of E_{D_0}); if $\alpha = 0$ or $M = 0$ and it is considered the negative sign in (24) or (25),but if the positive sign is considered in (24) or (25), a Kasner vacuum (external) solution E_{D_1} of Petrov type D, where [40] $ds^2 = dt^2 - t^{2a_1}dx^2 - t^{2a_2}dy^2 - t^{2a_3}dz^2$, $a_1 + a_2 + a_3 = 1$ and $a_1^2 + a_2^2 + a_3^2 = 1$, being in this case $a_1 = a_2 = -2a_3 = 2/3$, ($E_{D_1} = (2/3, 2/3, -1/3)$), and the Flat world solution $a_1 = a_2 = 0$ $a_3 = 1$, ($E_{D_0} = (0, 0, 1)$).

For future analysis of the solutions, in proximity to $t = 0$ or $t \longrightarrow \infty$, it would be use the Kretschmann invariant. The condition $Krets = R_{\alpha\beta\gamma\tau}R^{\alpha\beta\gamma\tau} < \infty$ is necessary and sufficient [41] for the finitude of all the algebraic invariants curve, and for that solution it has the form

$$Krets = \frac{32\, t^{2\,\lambda} \left(1 \pm \sqrt{1 + \alpha\, t^{1-\lambda}}\right) + t^2\,(5 + 6\,\lambda + 9\,\lambda^2)\,\alpha^2 + 16\,\alpha\, t^{\lambda} t}{27 t^{2\,\lambda+4}}, \quad (27)$$

which for the values of $\alpha t^{1-\lambda} \longrightarrow 0$ tends to

$$Krets \simeq -1/27\, \frac{-32(1 \pm 1) - 16\,(1 \pm 1)\,\alpha\, t^{1-\lambda} + t^{-2\,\lambda+2}\alpha^2\,(\pm 4 - 5 - 6\,\lambda - 9\,\lambda^2)}{t^4}$$

and for values of $\alpha t^{1-\lambda} \gg 1$

$$Krets \simeq \pm\frac{32}{27}\, t^{-1/2\,\lambda - 7/2}\sqrt{\alpha} + 1/27\, t^{-2-2\,\lambda}\left(5 + 6\,\lambda + 9\,\lambda^2\right)\alpha^2 + \frac{16}{27}\,\alpha\, t^{-3-\lambda} \approx$$

$$\approx 1/27\, t^{-2-2\,\lambda}\left(5 + 6\,\lambda + 9\,\lambda^2\right)\alpha^2.$$

4 Analysis of the Solutions

Solution (24) can be rewritten in another way, so as to allow to be compared to other known solutions. One of the preferred scenarios for the analysis of solutions in cosmology is the dust type when $\lambda \longrightarrow 0$, since it represents, vastly, the state of the Universe currently observable. The observations point out that the Universe can be represented through the flat model of Friedman Robertson Walker and Lemaitre (FRWL); although there are discussions about it (see introduction), there is consent that the model allows to elaborate astronomic data with a quite acceptable consistency. Because of this, and other theoretical aspects, it desirable to compare the solution obtained with the FRWL, and similar for which, $g_{00} = 1$, so that it can determined whether or not these solutions provide information that has been somehow missing in studies related

to cosmology using the FRWL model and other like it.
When changing the temporal coordinate t, in (12) and considering (24) so that $dt/\sqrt{F} = d\eta$, one has, for the values of $\lambda \neq 1$ that

$$\eta = t\,{}_2F_1\left(1/2, 1/(1-\lambda); 1 + 1/(1-\lambda), -\alpha t^{1-\lambda}\right) + const \tag{28}$$

where ${}_2F_1\left(1/2, 1/(1-\lambda); 1 + 1/(1-\lambda), -\alpha t^{1-\lambda}\right)$ is a is a hypergeometric function, allowing that system to be synchronic $g_{00} = 1, g_{0i} = 0$ for this new coordinate η.

4.1 Phantom Model

The Phantom model is the one whose state equation is $\lambda < -1$ whose solution (24) it can be written in the following way:

$$F = \frac{1}{1 + \alpha t^{1+|\lambda|}}, \quad K = \left(\frac{4(\sqrt{1 + \alpha\, t^{1+|\lambda|}} - 1)}{\alpha(\sqrt{1 + \alpha\, t^{1+|\lambda|}} + 1)}\right)^{\pm 2/3\,(1+|\lambda|)^{-1}}. \tag{29}$$

The solution (29), in proximity to $t \longrightarrow 0$, tends to $F \longrightarrow 1$ and $K \longrightarrow t^{\pm 2/3}$, therefore the metric (12) tends to

$$ds^2 \longrightarrow dt^2 - t^{2/3 \pm 2/3}(dx^2 + dy^2) - t^{2/3 \mp 4/3} dz^2, \tag{30}$$

which because of its form, are the Kasner vacuum or the flat world depending on whether the positive or negative sign is taken in (29), but though the metric tends to look alike the prior ones, it does not necessarily tends to be, as it will be seen in other models (for a deeper approach see subsection E). In the case of Phantom model, when using a positive sign in (29), the metric tends to be the Kasner vacuum E_{D_1} for the λ values of said model. Moreover, both the metric and the Kretschmann invariant of (29) ($Krest \longrightarrow 64/(27t^4)$, when $t \longrightarrow 0$) tends to be the Kasner solution E_{D_1} and the Kretschann invariant of it ($Krets = 64/(27t^4)$), an analog situation presents when considering the negative sign in (29), this situation opens the possibilities to analyze the singularities based on the behavior of the Kretschmann invariant and the solutions obtained, with its Kasner analogues of type D. Related to the solution, whether or not the positive sign is taken in (29), this one tends to the flat world for all the values in the Phantom model, i.e. the Kretschann invariant tends to zero when $t \longrightarrow 0$, so the solution is not singular. For all of these, the solutions might tend to not been singular and flat (E_{D_0}) if the negative sign is taken in (29) and singularities of the Kasner type E_{D_1}, if the positive sign is taken. Although one would want to evaluate the singularity when considering whether the density or the pressure for this models tends, or not, to infinite when $t \longrightarrow 0$, it is not appropriate for all the options, since the solutions tend

to be external, of Kasner vacuum type E_{D_1}, presents a singularity in $t = 0$.
From the tetra-dimensional point of view, the solutions provide a picture of a World in its beginnings in the form of an infinitely long tube on the z axis and an infinitely small radius, if the solution is considered with the positive sign in K, or the form of thin plane (disk) in case the negative sign is used in K.
Another important limit to analyze is when t has the large values, being that they are related to our current observations. In such case, assuming that $\alpha t^{1+|\lambda|} \gg 1$, $|\lambda| \nrightarrow 1$ and making a change of temporal coordinate $t = \left(-1/2\,\eta\,\sqrt{\alpha}\,\left(|\lambda|-1\right)\right)^{-2\,(|\lambda|-1)^{-1}}$ ($t \longrightarrow \infty$, $\eta \longrightarrow 0$, $\eta < 0$), results that

$$ds^2 \longrightarrow d\eta^2 - \left(-1/2\,\eta\,\sqrt{\alpha}\,\left(|\lambda|-1\right)\right)^{-4/3\,(|\lambda|-1)^{-1}} \left(dx'^2 + dy'^2 + dz'^2\right), \quad (31)$$

where $x' = (4/\alpha)^{\pm 1/(3(1+|\lambda|))}x$, analogue y, and $z' = (4/\alpha)^{\mp 2/(3(1+|\lambda|))}z$, where it can be seen that the space time tends to be isotropic with the augmentation of t, creating an increase in the volume of the space, disproportionate with the change presented in the time for all the cases. The previous is known in literature as the Big Rip. Whether the positive or negative sign is taken, the Kretschmann invariant is singular when $t \longrightarrow \infty$, so a double singularity can be presented for the cases where the positive sign is taken, both when it is $t \longrightarrow 0$, as when it is $t \longrightarrow \infty$.
A direct calculation of the solution (when $\Psi_0 = \Psi_4 = 0$) results for the case of $\lambda < -1$

$$\Psi_2 = \frac{1 \pm \sqrt{1 + \alpha\, t^{1+|\lambda|}}}{9t^2} \longrightarrow \pm \frac{\sqrt{\alpha}}{9t^{3/2-|\lambda|/2}} \quad (32)$$

of the above, for example, if $|\lambda| = 3$, when $t \longrightarrow \infty$ it is obtained $\Psi_2 \longrightarrow \pm\sqrt{\alpha}/9$, meaning there is no decay with time towards zero, the terms "Coulomb" [38] which arise from the model. Because of this, the Phantom model is divided into two types: the ones that DO NOT allow that the Weyl scalar los $\Psi_2 \longrightarrow 0$ when $t \longrightarrow \infty$ ($\lambda \leq -3$) and the solutions that do allow it $-3 < \lambda$, to the latter joins the Kasner vacuum solution and similarly the solutions of Schwarzschild, Kerr, Kerr-Newman and others for which $\Psi_2 \longrightarrow 0$ when $r \longrightarrow \infty$.

4.2 Dark Energy Model

The model of dark energy has as a state equation the relation $\mu = -P \sim \Lambda$ (where Λ is the cosmological constant) or what is the same $\lambda = -1$. The solution obtained results, if $\lambda = -1$, that $\mu = -P = -\alpha/3$ or if it is considered the usual relation that is given to the cosmological constant Λ, with the dark energy $\alpha = 3c^2\Lambda/\left(8\pi G\right)$.
Analogous to what done in the prior section, the following change of a temporal

coordinate will be done $t = \frac{\sinh\left(\eta\sqrt{\alpha}\right)}{\sqrt{\alpha}}$. The metric in this case can be written in the form of

$$\begin{aligned} ds^2 = d\eta^2 - \left(\sinh\left(\sqrt{\alpha}\eta\right)\right)^{2/3}\left(4\,\frac{\cosh\left(\sqrt{\alpha}\eta\right)-1}{\cosh\left(\sqrt{\alpha}\eta\right)+1}\right)^{\pm 1/3}\alpha^{\frac{-1\mp 1}{3}}\times \\ \times\left(dx^2+dy^2+\left(4\,\frac{\cosh\left(\sqrt{\alpha}\eta\right)-1}{\alpha\,\left(\cosh\left(\sqrt{\alpha}\eta\right)+1\right)}\right)^{\mp 1}dz^2\right). \end{aligned} \tag{33}$$

The metric (33) can be singular or not, near to $t \longrightarrow \eta \longrightarrow 0$ and can be written, for such proximities, as

$$ds^2 = d\eta^2 - \eta^{2/3\pm 2/3}\left(dx^2+dy^2\right) - \eta^{2/3\mp 4/3}dz^2, \tag{34}$$

where it can be noticed that the space elongates over the z axis and shrinks over the plane x, y as $\eta \longrightarrow 0$, if the sign (+) is tanking in (33), or flattens suppressing the z axis, if the solution with the sign (-), in a similar way to the case of the Phantom model. The solution, as in the case of the Phantom model, tends to the Kasner vacuum solution E_{D_1}, when $t \longrightarrow 0$ and when the positive sign is taken in (33) or Kasner E_{D_0} if the negative sign is taken, for the latter solutions with analogous behaviors are obtained in the closeness with $t = 0$. For example, a possible solution for a magnetic field in Bianchi I, tends to Kasner's E_{D_0} in the proximities of $t = 0$ and the energy of the magnetic field tends to a constant, for a value of a parameter $p = 1$(see [42] pag.529), and the density $\mu \nrightarrow 0$ when $t \longrightarrow 0$ since it is constant.

Therefore, a difference start to stand out from the dark energy model and the successors models, in regard to the Phantom model where $\mu \longrightarrow 0$ and $P \longrightarrow 0$ when $t \longrightarrow 0$. In the case of the model of the dark energy of a perfect fluid, and the successors, it is wise to ask for the Kretschmann invariant and to what tends those proximities, to determine whether the metric becomes or not the Kasner's, together with such invariant. For the energy dark model with the (+) sign solution results that $Krest \longrightarrow 64/(27t^4)$, thus in this case the solution also tends to the Krasner's together with Kretschmann.

For the solution with the sign (-) in (33) results a space-time, that though in appearance tends to be flat it is not; however, it is not singular either, that is, the singularity in this case is suppressed, for the Kretschmann invariant $Krets \longrightarrow 2^2/3^2\alpha^2$.
For the η values that do not tend to zero rapidly, the metric tends to enter to an isotropic regime of form

$$ds^2 \approx {ds_{isot}}^2 + {ds_{pert}}^2 \tag{35}$$

$${ds_{isot}}^2 = d\eta^2 - e^{2/3\sqrt{\alpha}\eta}\left(dx'^2 + dy'^2 + dz'^2\right) \tag{36}$$

$${ds_{pert}}^2 = \pm 4/3\,\frac{\left(dx'^2 + dy'^2 - 2\,dz'^2\right)}{e^{1/3\sqrt{\alpha}\eta}}, \tag{37}$$

where $x' = 2^{-1/3\pm 1/3}\alpha^{-1/6\mp 1/6}x$, analogous y, and $z' = 2^{\mp 2/3-1/3}\alpha^{-1/6\pm 1/3}z$. From the previous approach it can be noticed that ${ds_{pert}}^2$ decays rapidly being this solution soon isotropic and close to

$$ds^2 \approx d\eta^2 - e^{2/3\sqrt{\alpha}\eta}\left(dx'^2 + dy'^2 + dz'^2\right) \tag{38}$$

and it presents the characteristics of a De Sitter inflationary spacetime in flat slicing coordinates.
The dark energy model do not present singularity when $t \longrightarrow \infty$, for such case the invariant $Krets \longrightarrow 2^3/3^3\alpha^2$.. Because of the latter, the dark energy model cannot be singular in $t \longrightarrow 0$, but it is not either, in any other case, when $t \longrightarrow \infty$.
In regard to Weyl's scalars, the model does present problems as to the decay of the scalars with time. For this models the scalar $\Psi_2 \neq 0$ has the following for:

$$\Psi_2 = \frac{1 \pm \sqrt{1 + \alpha\,t^2}}{9t^2} \tag{39}$$

and decays with time in the form of $\sim t^{-1}$.

4.3 Quintessence Model

For this model an equation of stay is used, where $-1 < \lambda < 0$. Mathematically, this model is similar to the Phantom mode; for example, for the proximities when $t \longrightarrow 0$, the solution of the Quintessence model tends to be (30). If $t \longrightarrow \infty$, and making the change of the temporal coordinate $\xi = 1/2\,\eta\,\sqrt{\alpha}\,(1 - |\lambda|)$ where

$$t = \xi^{2\,(1-|\lambda|)^{-1}}\left(1 - \xi^{-2\,\frac{1+|\lambda|}{1-|\lambda|}}\alpha^{-1}\left(1 + 3\,|\lambda|\right)^{-1}\right), \tag{40}$$

($\alpha \neq 0$ y $\lambda \neq -1$), results that the metric tends to be

$$ds^2 \approx ds^2_{isot_Q} + ds^2_{pert1_Q} + ds^2_{pert2_Q}, \tag{41}$$

$${ds_{isot_Q}}^2 = d\eta^2 - \xi^{4/3\,(1-|\lambda|)^{-1}}\left(dx'^2 + dy'^2 + dz'^2\right) \tag{42}$$

$$ds^2_{pert1_Q} = \pm 4/3\,\xi^{\frac{-3\,|\lambda|+1}{3(1-|\lambda|)}}\left(dx'^2 + dy'^2 - 2\,dz'^2\right)(1+|\lambda|)^{-1}\,\frac{1}{\sqrt{\alpha}}, \qquad (43)$$

$$ds^2_{pert2_Q} = -\frac{8}{9}\,\xi^{\frac{-2(1+3\,|\lambda|)}{3(1-|\lambda|)}}\left(dx'^2 + dy'^2 + 4\,dz'^2\right)(1+|\lambda|)^{-2}\,\alpha^{-1}, \qquad (44)$$

where $x' = \left(4\,\alpha^{-1}\right)^{\pm 1/3(1+|\lambda|)^{-1}} x$, analogous y and $z' = (1/4\,\alpha)^{\pm 2/3(1+|\lambda|)^{-1}} z$. From the solution it is noticed that this tends to be (42), when $t \longrightarrow \infty$, thus it is clear that the metric tends to enter to an isotropic regime regardless the λ value; however, there are values for which $ds^2_{pert1_Q}$, decays to zero or not depending on λ. The above is highly important, though this term behaves as a perturbation over an isotropic bottom for large values of t, it may represent a perturbation that could be rapidly lost or not. For values of $-1 < \lambda < -1/3$, and for the values of the Phantom model, the perturbation is rapidly lost, but for values between $-1/3 \leq \lambda < 0$, this may be significant at least for some values of ξ. Another thing related to the previous values for which the perturbation rapidly decays is that for $-1 < \lambda < -1/3$ and all the latter models, one of the conditions in the Hawking Penrose theorem, in relation to the singularities in cosmology, $\mu + 3P \geq 0$ (strong energy condition) [42], does not comply.
A difference of this model in relation to the Phantom model is that the solutions are from the flat world, when $t \longrightarrow 0$, for the case when the positive sign is taken in (24), since the invariant

$$Krest \longrightarrow 1/27\,\alpha^2 t^{2\,|\lambda|-2}\left(5 - 6\,|\lambda| + 9\,\lambda^2\right)$$

when $t \longrightarrow 0$ and all the solutions in this case are singular. The solutions when the positive sign is taken in en (24), present the same tendency to transform into Kasner's space-time E_{D_1} and with the same tendency of the value of the Kretschmann invariant when $t \longrightarrow 0$ presented above. The only difference is that the tendency is less accentuated than for the previous cases; this is to say that it is necessary to get closer to $t = 0$ to obtain a Kasner vacuum E_{D_1}. The Weyl scalar has the form given in (32), but it decays faster than in the previous models.

4.4 Dust Model

For this model, when considering in (28) a *const* $= 1$, it is obtained that $\eta = 2\,\frac{\sqrt{1+t\alpha}}{\alpha}$, so $t = \frac{-4+\eta^2\alpha^2}{4\alpha}$. The solution can be written in the form of

$$ds^2 = d\eta^2 - (\eta\,\alpha/2 - 1)^{\frac{2\pm 2}{3}}\,(\eta\,\alpha/2 + 1)^{\frac{2\mp 2}{3}}\left(dx'^2 + dy'^2 + \left(\frac{\eta\,\alpha + 2}{\eta\,\alpha - 2}\right)^{\pm 2} dz'^2\right) \qquad (45)$$

where $x' = 4^{\pm 1/3}\alpha^{-1/3\mp 1/3}x$, analogous y, $z' = 4^{\mp 2/3}\alpha^{\pm 2/3-1/3}z$. From the solution (45) it can be noticed that when $t \longrightarrow 0$, $\eta \longrightarrow 2/\alpha$, so assuming $\eta > 0$ and noticing that $\eta \longrightarrow (\alpha t+2)/\alpha$, it has that the metric can be written returning to the temporal coordinate t, of the form

$$ds^2 \longrightarrow dt^2 - t^{2/3\pm 2/3}(dx^2+dy^2) - t^{2/3\mp 4/3}dz^2, \tag{46}$$

The solution in the proximities with con $t=0$, is singular for all the cases. Although in appearance the solution tends to the one of a flat world when the negative sign is chosen in (46), the Krestchmann invariant, when $t \longrightarrow 0$, tends to $Krets \longrightarrow 1/3\alpha^2 t^{-2} \longrightarrow \infty$, so it is singular a does not represent the flat world. In regards to whether the positive sign is taken in K, the invariant tends to $Krets \longrightarrow 64/27t^{-4}$, that though less rapidly than other models, tends to the Kasner vacuum solution E_{D_1}, so it is also singular in $t=0$.
For large values of η, the metric can be written as

$$ds^2 \simeq ds^2_{isot} + ds^2_{pert} \approx ds^2_{isot}, \tag{47}$$

$$ds^2_{isot} = d\eta^2 - 2^{-1/3}\,(\eta\,\alpha)^{4/3}\left(dx'^2+dy'^2+dz'^2\right), \tag{48}$$

$$ds^2_{pert} = \pm 2^{8/3}/3\sqrt[3]{\eta\,\alpha}\left(dx'^2+dy'^2-2\,dz'^2\right). \tag{49}$$

where ds^2_{isot} is the flat FRWL metric [33], and ds^2_{pert} is a perturbation in regards to the flat FRWL metric, that results from the anisotropy considered in the original model. The metric's perturbation generates anisotropies in a special axis, specifically in z axis. This situation finds its reflection in investigations related to some anisotropies experimentally found (see introduction).
For the Dust model, the Weyl scalars decay toward zero as follow:

$$\Psi_2 = \frac{1 \pm \sqrt{1+\alpha\, t}}{9t^2} \longrightarrow \frac{\pm\sqrt{\alpha}}{9t^{3/2}} \tag{50}$$

so, its fall to zero is more pronounced with the augmentation of t.

4.5 Ordinary Relativistic Fluid Model Under Pressure

This model represents a perfect fluid which has pressure and is relativistic. Its stated equation is determined in ways that $0 < \lambda < 1/3$. This model has importance if considered that the Universe has gone through several scenarios; for example, if it has its beginnings with a state equation where $\lambda = 1/3$, until having an existence regimen described by the state equation of the type dust model. A more general idea is that $\lambda = \lambda(t)$, and that in a continuous way it transforms from one scenario to another; therefore, one more term would present for such case in the approximation when $t \longrightarrow \infty$, with the intention of, in case of needing for example in a visualization, being able to access this

resource.
When $t \longrightarrow 0$, one can change the following temporal coordinate $t \longrightarrow \eta\left(1+1/2\,\frac{\alpha\,\eta^{1-\lambda}}{2-\lambda}\right)$ in which case the solution can be written as

$$ds^2 \simeq ds_0^2 + ds_1^2, \tag{51}$$

$$ds_0{}^2 = d\eta^2 - \eta^{2/3\pm 2/3}\left(dx^2+dy^2\right) - \eta^{2/3\mp 4/3}dz^2 \tag{52}$$

$$ds_1{}^2 = 1/3\,\frac{\eta^{5/3-\lambda\pm 2/3}\left(\left(\pm 1-1+\lambda\right)\left(dx^2+dy^2\right)+\left(\mp 2-1+\lambda\right)\eta^{\mp 2}dz^2\right)\alpha}{\left(-1+\lambda\right)\left(-2+\lambda\right)} \tag{53}$$

where it can be noticed that the solution is singular at point $t = 0$. When the sign $+$ is taken in K, once again is obtained that the solution tends to Kasner's previously mentioned and if the negative sign is taken, the invariant is singular in $t = 0$; thus, the metric does not tend to be flat though it looks like it in appearance. The rhythm in which the Kretschmann invariant gets closer to being singular is bigger than in previous cases. In this solution, as in the previous, the space can have, at the beginning, the shape of a infinitely long tube of negligible radius or tend to be flat with $z = 0$. In the case when t has large values and making the change of the temporal coordinate so that

$$\xi = 1/2\,\eta\,\sqrt{\alpha}\left(1+\lambda\right)$$

and

$$t = \xi^{2\,(1+\lambda)^{-1}}\left(1+\xi^{2\,\frac{-1+\lambda}{1+\lambda}}\alpha^{-1}\left(-1+3\,\lambda\right)^{-1}\right), \tag{54}$$

results that the metric approximates to

$$ds^2 \approx ds^2_{isot} + ds^2_{pert1} + ds^2_{pert2}, \tag{55}$$

$$ds_{isot}{}^2 = d\eta^2 - \xi^{4/3\,(1+\lambda)^{-1}}\left(dx'^2+dy'^2+dz'^2\right) \tag{56}$$

$$ds^2_{pert1} = \pm 4/3\,\xi^{1/3\,\frac{3\,\lambda+1}{1+\lambda}}\left(dx'^2+dy'^2-2\,dz'^2\right)\left(1-\lambda\right)^{-1}\frac{1}{\sqrt{\alpha}}, \tag{57}$$

$$ds^2_{pert2} = -\frac{8}{9}\,\xi^{2/3\,\frac{-1+3\,\lambda}{1+\lambda}}\left(dx'^2+dy'^2+4\,dz'^2\right)\left(1-\lambda\right)^{-2}\alpha^{-1}, \tag{58}$$

which, when $t \longrightarrow \infty$ tends to

$$ds_{isot}{}^2 = d\eta^2 - \xi^{4/3\,(1+\lambda)^{-1}}\left(dx'^2+dy'^2+dz'^2\right) \tag{59}$$

and where $x' = \left(4\,\alpha^{-1}\right)^{\pm 1/3(1-\lambda)^{-1}} x$, analogous y and $z' = \left(1/4\,\alpha\right)^{\pm 2/3(1-\lambda)^{-1}} z$ thus it is clear that the metric tends to enter at an isotropic regimen.
For this models the Weyl scalar decays towards zero as follows:

$$\Psi_2 = \frac{1\pm\sqrt{1+\alpha\,t^{1-\lambda}}}{9t^2} \longrightarrow \frac{\pm\sqrt{\alpha}}{9t^{3/2+\lambda/2}} \tag{60}$$

Therefore, they decay at a faster rate in relation to the above cases.

4.6 Ultrarelativistic model or model of radiation

The ultrarelativistc model has an specific value of $\lambda = 1/3$. This model could have been the Universe's scenario in the proximities with its formation; therefore, it is important to analyze it both in the proximities with $t = 0$ as with large values of t.
In the proximities of $t = 0$, the metric can be written as

$$ds^2 \longrightarrow dt^2 - t^{2/3 \pm 2/3}\left(dx^2 + dy^2\right) - t^{-4/3 \pm 2/3} dz^2, \tag{61}$$

where it can be noticed that a similar situation to the analyzed in the previous models is presented .
For large values of t the following change in the variable can be done

$$t = \left(2/3\, \eta\, \alpha^{3/2} + 1/2 \ln\left(\eta \alpha^{3/2}\right)\right)^{3/2} \alpha^{-3/2},$$

where approximately can be obtained that

$$ds^2 \approx ds^2_{isot} + ds^2_{pert}, \tag{62}$$

$$ds_{isot}{}^2 = d\eta^2 - (2/3\, \xi + 1/2\, \ln(\xi))\left(dx'^2 + dy'^2 + dz'^2\right) \tag{63}$$

$$ds_{pert}{}^2 = \pm 2/3\sqrt{6}\sqrt{\xi}\left(dx'^2 + dy'^2 - 2\, dz'^2\right), \tag{64}$$

where $\xi = \eta \alpha^{3/2}$, $x' = 2^{\pm 1} \alpha^{-1/2 \mp 1/2} x$, analogous y and $z' = 2^{\mp 2} \alpha^{\pm 1 - 1/2} z$.

From the previous solution, it is noticed that when t has large values, the solution keeps an isotropic level when conserving the logarithmic term, which tough increases in lower reason that $\xi^{1/2}$, it does generates a larger scale factor in the considered isotropic term of the metric.
The Weyl scalar different than zero for this case, has the following form

$$\Psi_2 = \frac{1 \pm \sqrt{1 + \alpha\, t^{2/3}}}{9t^2} \longrightarrow \frac{\pm\sqrt{\alpha}}{9t^{5/3}}. \tag{65}$$

Thus, they decay in a faster rate in relation to the previous cases.

4.7 Hard Universe Model

For the model Hard Universe, the state equation where $1/3 < \lambda < 1$ is used. Mathematically, this model is similar to the Ordinary Fluid relativistic to pressure model, the asymptotes deductions, singularities obtained, are the same for most of the cases when $\lambda \neq 1$. The Weyl scalars have the form given in (50), but since λ is larger than in this case, in this model those scalars decay with a faster rated than in the previous models .
It is important to notice that there are solutions of the Hard Universe model for which (57) and (58) diverge when the values of $\lambda = 1 - \epsilon$, where $\epsilon > 0$ and

$\epsilon \longrightarrow 0$; this, because those approximations lose validity in those cases. For the aforementioned, the equation (22), considering $C_1 = 2/3$ it can be written as

$$-F\epsilon\,(4\,F-4)+4\dot{F}t=0, \tag{66}$$

where, if $\epsilon \longrightarrow 0$, the equation tends to be the one that has to be resolve for the Zeldovich model. This is understood as well because of the fact that physicality, for this type of model $P \approx \mu$; therefore, intuitively it must be expected that the solution behave in a similar way. To prove the aforementioned it must be noticed that the general solution of (22), is $F = \frac{1}{1+\alpha t^{1-\lambda}}$, for which if $\lambda = 1$, $F = \frac{1}{1+\alpha} = const.$ The integral in the exponent of (21), results for the general solution of F, that

$$INT = \pm 2/3\int \frac{F^{1/2}}{t}dt = \pm 2/3\,\ln\left(\frac{\sqrt{1+t^{1-\lambda}\alpha}-1}{\sqrt{1+t^{1-\lambda}\alpha}+1}\right)(1-\lambda)^{-1}+const. \tag{67}$$

The integral (67), when $\lambda \longrightarrow 1$, can approximate through a series a

$$INT = C_3 + 2/3\,\frac{\ln(t)}{\sqrt{1+\alpha}} + 1/6\,\frac{(\ln(t))^2\,\alpha\,(\lambda-1)}{(1+\alpha)^{3/2}} + O\left((1-\lambda)^2\right), \tag{68}$$

where

$$C_3 = -2/3\,\ln\left(\frac{\sqrt{1+\alpha}-1}{\sqrt{1+\alpha}+1}\right)(\lambda-1)^{-1}+const.$$

If $K_0 e^{C_3} \longrightarrow K_0'$ is substituted in (21), without loss of generality it can be considered that $K_0' = 1$ and obtain that

$$K \approx t^{\frac{\pm 2}{3\sqrt{1+\alpha}}}\left(1 \pm \frac{(\ln(t))^2\,\alpha\,(\lambda-1)}{6\,(1+\alpha)^{3/2}}\right) + O\left((1-\lambda)^2\right), \tag{69}$$

$$K^{-2} \approx t^{\frac{\mp 4}{3\sqrt{1+\alpha}}}\left(1 \mp \frac{(\ln(t))^2\,\alpha\,(\lambda-1)}{3\,(1+\alpha)^{3/2}}\right) + O\left((1-\lambda)^2\right). \tag{70}$$

Moreover, when $\lambda \longrightarrow 1$, F approximates to

$$F \approx (1+\alpha)^{-1} + \frac{\ln(t)\,\alpha\,(\lambda-1)}{(1+\alpha)^2}. \tag{71}$$

Of (69), (70), and (71), the solution tends to be

$$ds^2 \approx ds_Z^2 + ds_1^2 \approx ds_Z^2, \tag{72}$$

$$ {ds_Z}^2 = dT^2 - T^{\frac{\pm 2}{3\sqrt{1+\alpha}}+2/3}\left(dX^2+dY^2\right) - T^{\mp\frac{4}{3\sqrt{1+\alpha}}+2/3}dZ^2, \tag{73} $$

where $T = \frac{t}{\sqrt{1+\alpha}}$, $X = x\left((1+\alpha)^{\frac{\pm 1}{3\sqrt{1+\alpha}}+1/3}\right)^{-1}$, analogous y and $Z = z\left((1+\alpha)^{\frac{\mp 2}{3\sqrt{1+\alpha}}+1/3}\right)^{-1}$ and

$$ \begin{gathered} {ds_1}^2 = \ln(t)\,\alpha\,(\lambda-1)\times \\ \times\left(dt^2 \mp 1/6\ln(t)\sqrt{1+\alpha}t^{2/3\,\frac{\pm 1+\sqrt{1+\alpha}}{\sqrt{1+\alpha}}}\left(dx^2+dy^2-2\,t^{\frac{\mp 2}{\sqrt{1+\alpha}}}dz^2\right)\right)(1+\alpha)^{-2}. \end{gathered} \tag{74} $$

The solution (73) is the one obtained in (25) if one changes $1/\sqrt{1+\alpha} = \sqrt{1-3M}$, $T=t$, $X=x$, $Y=y$ and $Z=z$.
Due to the above, the solutions tend to behave like the ones of the Zedovich model, when $\lambda \longrightarrow 1$ and approximate like $ds^2 \approx ds_Z^2 + ds_1^2$. For the values of the Hard Universe model, ds_1^2 is negative, but for the ekpyrotic models is positive if $\lambda \longrightarrow 1+\epsilon$. The analysis of ds_Z^2, will be done in the following subsection.

4.8 Model Universe Zeldovich or stiff matter)

This type of models have as state equation $P = \mu$, i.e. $\lambda = 1$. The solution for such model, of (25), has the form

$$ ds^2 = dt^2 - t^{2/3(1\pm\sqrt{1-3M})}\left(dx^2+dy^2\right) - t^{2/3(1\mp 2\sqrt{1-3M})}dz^2. \tag{75} $$

From the previous solution, it is noticed that the constant $M \leq 1/3$, requires the metric to be real ($g_{\mu\nu} \in \mathbb{R}$) and ($M \geq 0$), requires the density μ, to be positive.
The solution of this model does not have asymptote to an isotropic regime for any value of M except $M = 1/3$, for which

$$ ds^2 = dt^2 - t^{2/3}\left(dx^2+dy^2+dz^2\right) $$

in such case is isotropic in every moment of the type Petrov O. Furthermore, the solution, if $M=0$ with the positive sign in K (25), represent a vacuum (external solution) of the Kasner type E_{D_1}, and in case that the sign in K (25) is negative the solution represent the flat world E_{D_0}, for other values of M there is not tendency for the solution, o at least in appearance, of being the flat world when $t \longrightarrow 0$.
An important aspect to noticed is that the solution of the density μ, relates to M in the way $\mu = M/t^2$, where the constant $\kappa = \frac{8\pi G}{c^4}$, in Einsten's the equations had taken equal to one for convenience, but in the solution can be

written when changing $\mu \longrightarrow \kappa\mu'$, with the intention of investigating some values of times for which was able to accomplish such scenerio. The bulk density of ϱ matter for the case of our interest, according to Zeldovich [35], can be considered within the limits of $\varrho_{zeld} =\sim 10^{52} gr/cm^3$, with the assistance of the above, a formula to determine the time when the scenario of the Zeldovich fluid type approaches, depending on M, this is $t(M) = \sqrt{M/(\kappa \varrho_{zeld} c^2)}$. For example, in the case of $M = 1/3$, the time at which such scenario occurs is $t(1/3) \approx 0.25 \cdot 10^{-23} s$. For a minimum time of $t_{min} = t_{planck} \approx 5.4 \cdot 10^{-44} s$, it is obtained that the value $M \approx 0.48 \cdot 10^{-40}$; from the foregoing, it can be expected that a scenerio of the Zeldovich fluid occurs in times $t \in [5.4 \cdot 10^{-44}, 0.25 \cdot 10^{-23}] s$ and for the possible values of M are $M \in [0.48 \cdot 10^{-40}, 1/3]$.
The Kretschmann invariant in said solution is

$$Krets_{Z\pm} = \frac{4}{27} \frac{-36\,M + 8 \pm 8\,(1 - 3\,M)^{3/2} + 81\,M^2}{t^4}, \tag{76}$$

with this it can be noticed the solution tending to singularity, proportionally, in the same way as in the case of the Kasner's vacuum $\sim t^{-4}$, but not so the in the speed in which it gets closer to such singularity if $M \neq 0$ (if $M = 0$ the two possible solutions of Kasner's Petrov type D are present, one of those is the flat world). In the solution of the model type Zeldovich, exists a value of M, for which the structure of space in the proximities of $t = 0$ (the singularity) corresponds to the type of solution of the Kasner's vacuum if the sign $(+)$ is taken in K; it is not the Petrov type D, thus results contradictory with the initial hypothesis of symmetry. However, it is known that the Kasner's solutions, in the proximities with $t = 0$, ranging from a time of Kasner to another due to gravitational perturbations [22, 23], so it can be expected that although the final solution is of the type Petrov D, it has not necessarily always been accompanied by a space-time of the type Petrov D in the proximities with $t = 0$. In the previous models, the situation presented in the solution of the Zeldovich type does not have an equivalent, both the solutions as well as the Kretschmann invariant used to tend to the two only Kasner vacuums of the type Petrov D when $t \longrightarrow 0$. Assuming that the solution passed through a Kasner's time when $t \longrightarrow 0$ then, what kind of solution could this have been? This question cannot be respond with the tools at hand; for example, if one tries to couple somehow the solution obtained with anyone of Kasner's for which the constants a_i can be parameterized in the manner of Lifshitz-Khalatnikov [43],

$$a_1 = -\frac{p}{p^2 + p + 1},\ a_2 = \frac{p^2 + p}{p^2 + p + 1},\ a_3 = \frac{p + 1}{p^2 + p + 1},$$

where $0 \leq p \leq 1$ (the D types of Petrov are given if $p = 0$ y $p = 1$), and since $0 \leq M \leq 1/3$, M can be parameterized as $M = p/3$. It can be considered that in an analogous way to other solutions, the Kretschmann invariant, both for the solution of Kasner

$$Krets_K = \frac{16\,p^2\,(p+1)^2}{t^4\,(p^2+p+1)^3}$$

As for the Zedovich type $Krets_{Z\pm}$, should match when $t \longrightarrow 0$, which accomplishes only for the case of $Krets_+$ (the sign + en K), when $p = 0.3862683158$; for the rest of the values of M, equality cannot be accomplished. The Kasner solution, evidently, is not of the Petrov type D, but it neither is about to be; therefore, of being the hypothesis, related to the correct Kretschmann invariants, results that for a fluid of the Zeldovich type, it does not present a time of the type Kasner when $t \longrightarrow 0$, so the type of the fluid and the time $t \in [5.4 \cdot 10^{-44}, 0.25 \cdot 10^{-23}]s$ in which can emerge such scenario, limits the possibilities for the times of gravitational perturbations of the Kasner type [22] that in other models where able to be present.

The Weyl scalar different from zero for this case, has the shape of

$$\Psi_2 = \frac{\left(\pm 1 + \sqrt{1-3\,M}\right)\sqrt{1-3\,M}}{9t^2} \tag{77}$$

From the prior relation it can be noticed that if $M = 1/3$,the solution is conformably flat and, as it was noticed, isotropic; moreover; if the negative sign is taken in (25) and $M = 0$, results that the scalars are null, since the space-time is flat in said case.
For he values of $M \in [0.48 \cdot 10^{-40}, 1/3[$ the Weyl invariants decline at a rapid pace, in relation to the previous cases.

4.9 Conflagration Modelo or Ekpyrotic (ekpyrotic matter)

This type of model has a state equation with values of $\lambda > 1$. This situation marks a fundamental difference in the space-time properties, both in the proximities with $t = 0$, as when $t \longrightarrow \infty$.
In the proximities when $\longrightarrow 0$, a temporal change of variable can be made ($\alpha \neq 0$) de la forma $t = \xi^{2\,(\lambda+1)^{-1}} + \xi^{2\,\frac{\lambda}{\lambda+1}}\alpha^{-1}\,(3\,\lambda-1)^{-1}$, where $\xi = 1/2\,\sqrt{\alpha}\,(\lambda+1)\,\eta$ and obtain that

$$ds^2 \approx ds^2_{isot} + ds^2_{pert} \tag{78}$$

$$ds^2_{isot} = d\eta^2 - \xi^{4/3\,(\lambda+1)^{-1}}\left(dx'^2 + dy'^2 + dz'^2\right) \tag{79}$$

$$ds^2_{pert} = \pm 4/3\xi^{1/3\,\frac{3\,\lambda+1}{\lambda+1}}\left(dx'^2 + dy'^2 - 2\,dz'^2\right)(\lambda-1)^{-1}\frac{1}{\sqrt{\alpha}}, \tag{80}$$

where $x' = x\ (\alpha/4)^{\pm 2(3(\lambda-1))^{-1}}$, analogous y and $z' = z(\alpha/4)^{\mp 4(3(\lambda-1))^{-1}}$. From the above solution can it is noticed that of existing an ekpyrotic scenario, this had to be, at a large extent, isotropic in its beginnings.
The solution, if $t \longrightarrow \infty$ y $\lambda \neq 2$, can be written doing a change in the time variable, valid for this case, in the form of $t = \frac{\alpha\,\eta^{2-\lambda}+2\,\eta\,(2-\lambda)}{2(2-\lambda)}$, with which

$$ds^2 \approx ds_0^2 + ds_1^2 \quad (81)$$

$$ds_0{}^2 = d\eta^2 - \eta^{\pm 2/3+2/3}\left(dx^2+dy^2\right) - \eta^{\mp 4/3+2/3}dz^2 \quad (82)$$

$$ds_1{}^2 = 1/3\,\frac{\eta^{\pm 2/3+5/3-\lambda}\alpha\,\left(\left(\pm 1-1+\lambda\right)\left(dx^2+dy^2\right)+\eta^{\mp 2}\left(\mp 2-1+\lambda\right)dz^2\right)}{\left(-1+\lambda\right)\left(-2+\lambda\right)}, \quad (83)$$

when $\lambda = 2$, and making the temporal coordinate change $t = \eta + 1/2\,\alpha\,\ln(\eta)$, it is obtained that when $t \longrightarrow \infty$, the solution tends to

$$ds^2 \approx ds_0^2 + ds_1^2 \quad (84)$$

$$ds_0{}^2 = d\eta^2 - \eta^{\pm 2/3+2/3}\left(dx^2+dy^2\right) - \eta^{\mp 4/3+2/3}dz^2 \quad (85)$$

$$ds_1{}^2 = \mp\eta^{\pm 2/3-1/3}\left(dx^2+dy^2-2\,\eta^{\mp 2}dz^2\right) \quad (86)$$

From the prior solutions the following is noticed: if $t \longrightarrow \infty$, none of this enters in an isotropic regimen in the same way as in the previous models. If the positive sign is considered in K, this tends to the Kasner's vacuum space-time and the energy density μ, declines strongly towards zero with the augmentation of t, diluting the Universe matter to a large extent, and if the solution is considered with the negative sign in K, the solution tend to become flat with the augmentation of t, though with a perturbation on the z axis, this because the metric $ds^2 = d\eta^2 - (dx^2 + dy^2 + \eta^2 dz^2)$ is the same metric as that of Minskowski (if a coordinates change is made $\eta = \sqrt{t'^2 + z'^2}$ and $z = arctanh(z'/t')$ the above is determined). From the results for the Zeldovich model and for this model, it can be concluded that the passage of an anisotropic space-time to an isotropic one with the augmentation of t does not occur in these solutions; this points out that said passage depends on whether or not the pressure P is lower than the density μ.
The Kretschmann invariant tends, in proximity to $t = \xi^{2\,(\lambda+1)^{-1}} = 0$, where $\xi = \sqrt{\alpha}\,(1/2 + 1/2\,\lambda)\,\eta$, to

$$Krets \approx 1/27\,t^{-2-2\,\lambda}\left(5+6\,\lambda+9\,\lambda^2\right)\alpha^2 = \frac{16}{27}\,\frac{5+6\,\lambda+9\,\lambda^2}{\left(1+\lambda\right)^4\eta^4}.$$

therefore, the speed with which deepens into the singularity is proportional to the deepening of the Kasner vacuum solution, similarly to all the models seen. The question in this case is whether there can be found values of λ for which the Krestchmann invariant for the solution tend towards singularity in an equivalent way to the Kretschmann, of some Kasner time (solutions), so that it is of the type Petro D. If this were possible, the solution should comply with the following:

$$Krets_K = 16\,\frac{p^2\,(p+1)^2}{(p^2+p+1)^3\,\eta^4} \approx \frac{16}{27}\,\frac{5+6\,\lambda+9\,\lambda^2}{(1+\lambda)^4\,\eta^4} \approx Krets_{\pm}, \tag{87}$$

this makes clear that for values of $\lambda \gg 1$ one has that $a_1 \approx a_2 \approx 0$ and $a_3 \approx 1$, so it is interpreted that the gravitational perturbations rapidly become small [22]; when λ is large, resulting in a space-time structure, without being one, but it is close to the flat world of the Kasner's solution E_{D_0}, and conserving singularity in $\eta = 0$ (the prior is valid for the flat world solution of FRWL). There are no values for which the Kretschmann invariant can tend to the Kasner's vacuum E_{D_1}. The solutions with the sign (+) or (-) in the function K (24), behave in an analogous manner in the proximities to $t = 0$.
With the above, it is deducted that for the Ecpirotic model, the space-time in its beginnings should have been very isotropic; however, of arising some type of anisotropy, this would degenerate into an anisotropic space-time similar in structure to Kasner's vacuum E_{D_1}, and deepening over time in such anisotropy or resurging in its isotropic form and flattening.
The Weyl scalar, different from zero in this case, has the following form

$$\Psi_2 = \frac{1 \pm \sqrt{1+\alpha\,t^{1-\lambda}}}{9t^2} \longrightarrow \begin{cases} \frac{2}{9t^2}\, if\, the\, sign\, is\, +\, in\, K \\ \\ \frac{-\alpha}{18t^{\lambda+1}}\, if\, the\, sign\, is\, -\, in\, K \end{cases} \tag{88}$$

Therefore they decay at a faster pace, compared to the previous cases, if the negative sign is taken; they would decay in the same pace if the positive sign is take; the prior is unique for this type of models.

5 Conclusion

The solutions for an anisotropic space-time and a homogeneous non static of the Petrov type D, under de model of a perfect fluid, with one equation of state where the pressure is proportional to the energetic density $P = \lambda\mu$, give as a result, after determining certain symmetry in the metric, that exists two kinds of possible solutions that differ from each other for any picked model; i.e., every model can generate two possible scenarios: the first one when the z axis expands more intensely than the flat x, y and when the contrary occurs. At the same time, there are five groups of models that have common characteristics related to whether the values of λ are lower than -1, equal to -1, larger than -1 but lower tan 1, equal to 1 or larger than1. When studying some points of interest, for example at the beginning when $t = 0$ or when $t \longrightarrow \infty$, results that for the models with $\lambda < -1$, Phantom type models, when using one of the two possible solutions (when K takes the positive sign), the metric tends to be the Kasner's vacuum (external solution with $-2a_3 = a_1 = a_2 = 2/3$) when $t \longrightarrow 0$ for all the $\lambda < -1$ values of such model therefore being singular at $t = 0$, and related to the other solution, if the positive sign is taken in K, this tends to the flat world for all the values of the Phantom model; as a result, that solutions is not singular at$t = 0$. It is observed that the Kretschmann invariants of the solutions tend to be the same Kretschmann invariants for the solutions of the Kasner's vacuum of the type Petrov D. Therefore, intuitively, presents the opportunity of analyzing the singularity of the form for the Kasner's solutions. Both kinds of solutions enter very quickly into an isotropic space regimen augmenting the volume at a high speed, until a new singularity that begins to manifest for large t values (Big Rip). The Weyl scalars $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$, $\Psi_2 \longrightarrow \infty$, for values of $\lambda < -3$, whereby said models can be divided into the ones that decay with the augmentation of time to zero ($-3 < \lambda < -1$) and the ones that augment their value or keep it constant with the augmentation of time ($\lambda \leq -3$).

A similar analysis to the one done for Phantom models, for the case when $\lambda = -1$ (dark energy), point out that in proximity to $t = 0$, the situation does not change largely related to the observed for the Phantom model, though in such proximity one of the solutions, the one with the negative sign in K, stops being flat, in the sense that the value of Kretschmann does not tend to zero when $t \longrightarrow 0$, but does not present singularity at $t = 0$. Both solutions become isotropic, although with some anisotropic perturbation, over time, tending to become into a space-time of De Sitter. The Weyl scalars for this case, decay without problem towards zero, more pronounced than for those that decay in the Phantom model, when cuando $t \longrightarrow \infty$.

For the models with values of $-1 < \lambda < 1$ (Quintessence, Dust, ordinary relativistic at pressure, of radiation and Hard Universe) the situation is more or less similar. An analysis equivalent to the one commented previously, shows that the two possible solutions are singular at $t = 0$, one of these keeps tending, structurally, to the solutions of the Kasner's vacuum in the same way that for the preceding models and the other tends structurally to being the flat world, but singular. All solutions tend to be isotropic with some perturbations that decreases as t augments and the Weyl scalars decay with the augmentation of t.

For the case of the Ordinary Relativistic Fluid at Pressure model, it was augmented the perturbation over the base of a change of coordinates where $g_{00} = 1$, in a term when $t \longrightarrow 0$, with the intention of functioning, for example, as a help of some possible visualization job, making clear with this that in the proximity to $t = 0$ or when $t \longrightarrow \infty$, the solutions remains without important

changes if the values of $\lambda \neq 1, 2$ (in fact, for $\lambda \geq 1$ the approximation is not valid).
When the value of $\lambda = 1$ (Zeldovich model), the situation changes greatly. In first place, the values of the energetic density, assuming they are positive, are limited trough an M constant with restrained values of this $M \in [0.48 \cdot 10^{-40}, 1/3[$ required to ensure that the metric is real ($g_{\mu\nu} \in \mathbb{R}$) besides a few general requirements of the quantum theory. From the solution it was determined that an scenario of the Zeldovich fluid can occur at the times of $t \in [5.4 \cdot 10^{-44}, 0.25 \cdot 10^{-23}]s$. The solution is singular in $t = 0$ on a rate proportional to the spacetime singularity of Kasner, based on the Kretschmann invariants, though there is no value of M; considering the previous interval; for which decays to the singularity in an equivalent way to the case of the Kasner type D solution with $-2a_3 = a_1 = a_2 = 2/3$. A calculation outside the presented in this paper, gives a value of $M \simeq 0.1287561053$ for which the Kretschmann invariant of the Zeldovich model, approaches the singularity with the same speed with which the Krestchmann invariant of a Kasner solution, with a value of the parameter $p = 0.3862683158$, approaches as well to the singularity; evidently said Kasner solution is not of the type Petrov D, therefore for a Zeldovich type fluid no Kasner time is presented when $t \longrightarrow 0$. The prior is possibly because the type of fluid and the time in which may appear such scenario limit the necessary possibilities to ensure that the gravitational perturbations decide the future behavior of the solution, Kasner type, that are present in other models. Although the Zeldovich scenario is limited to times close to $t = 0$, it was determined that there is no asymptote to an isotropic regimen, assuming an initial anisotropic symmetry; that if the solution can be isotropic, if $M = 1/3$, in which case it is for any value of t and corresponds to the FRWL flat model for the same type of fluid. The Weyl invariants decay to zero at a same pace $\sim t^{-2}$ for any value of t. This is another difference from previous models where those scalars decay in a more undetermined way in relation to t.
For values of $\lambda > 1$ (ekpirotic model) the solutions present substantial changes related to the previous models. First, the solutions present a high level of isotropy when $\longrightarrow 0$, thay is to say that the model only allows anisotropies in the range of small perturbations in the beginning that vary according to which of the two kinds of solutions is taken. However, when $t \longrightarrow \infty$, the small perturbations become very significant and may into a Kasner space-time, in which case the expansion would be greater on a perpendicular planer to one of its z axis; on the contrary, it would degenerate into a space time with a flat appearance (11), further expanding on the z axis, in both cases diluting the Universe matter rapidly. As for the Zeldovich model and for this model, it is concluded that the passage of an anisotropic space-time to another isotropy, with the augmentation of t, is not presented, which indicates that said step

depends on whether the P pressure is lower than the μ density or not. As a result, one can conclude that if an ekpirotic scenario took place, then it should had been highly isotropic since a small anisotropic perturbation over this would have degenerate into an anisotropic space-time greatly pronounced, at least for one of the solutions. For the ekpirotic model, the solutions are singular when $t = 0$ and the Weyl scalars decay with the augmentation of time in a different way depending on the selected solution; the prior does not have an equal with the previous models and represent a new difference that is marked in the two possible solutions in relation to the other Weyl scalars and in such way that for a type of solutions (when the $+$ sign is taken in K), the scalar $\Psi_2 \sim 1/t^2$ regardless of the λ value (similarly to the Zeldovich model) and for the other type $\Psi_2 \sim 1/t^{\lambda+1}$, so that changes according to the value of λ.

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