Solutions of One and Two-Dimensional Compressible Navier-Stokes Equations

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Abstract

The paper deals with solving of the time dependent one and two-dimensional compressible Navier-Stokes equations using the Tanh method. The method is based on the construction called solitary wave solution. We found two family of solutions in the one dimension model, and one family of solution for the two-dimension case.

Keywords: 1d and 2d compressible Navier Stokes equation, Tanh function, solitary wave solution

1 Introduction

In general, a fluid is well described by the Navier-Stokes equations (NSEq). A further generalization is to consider a compressible fluid, which is characterized by a significant change in fluid density. It can be caused, in the most simple version, by changes in temperature or pressure. Under this considerations, density could be considered as a density field. Then, if we ask for the relation among density, pressure and temperature, it is common to appeal for the perfect gas model. If $T$ is taken constant, there is a direct relation between pressure and density $P = c\rho$, or the polytropic equation of state $P = k\rho^n$. Then, collecting all these arguments, we obtain one of the simple models for a compressible fluid given by the Navier-Stokes equations (CSEq).
Intensive research in basic mathematics and computational methods have been done in order to find particular solutions of the CNSEq. Among them, we can show viscous profiles for one-dimensional CNSEq, based on center manifold theorem their existence and a construction of a criterion are proved [1]. Also, results on uniqueness are shown in [2]. Further generalizations of this problem, include the presence of electric fields and its long behaviour global solutions [3]. Likewise, coupling heat conductions phenomena with compressible Euler and continuity equations in one dimension [4]. In the case of computational approach, important work has been done in discontinuous Galerkin methods. The method looks suitable for applications demanding computing parallelization, stability and robusticity [5]. Also, a version of this methods called goal oriented a posteriori error estimation has been used for inviscid compressible fluids [6].

On the other hand, a growing number of innovative methods have emerged in the last two decades, all those designed with the purpose to obtain solitary wave solutions from partial differential wave or hyperbolic equations. Amidst them, this paper addresses the solution of the CNSEqs using the $tanh$ method [7]. This work is organized as follows. Section (2), presents a review of the $tanh$ method. In section (3), we solve CNSEqs obtaining four family solutions in one and two dimensions. In section (4), we present results and conclude.

2 Summary of the Tanh Method

The general problem corresponds to find a solitary wave solution of a nonlinear differential equation of the fields $\Psi_j$ in variables $x, y$ and $t$, which is represented by:

$$P_i(\Psi_j, \Psi_{j,t}, \Psi_{j,x}, \Psi_{j,y}, \Psi_{j;x,x}, \Psi_{j;y,y}, \Psi_{j;x,y}, \ldots) = 0$$ (1)

Where the subindex $i$ accounts for the several differential equations describing the physical system. Then, we start doing the following coordinate transformation:

$$\xi = x + y - at + \xi_0$$ (2)

The derivatives change like:

$$\frac{\partial}{\partial t} = -a \frac{d}{d\xi}; \frac{\partial}{\partial x} = \frac{d}{d\xi}; \frac{\partial}{\partial y} = \frac{d}{d\xi}; \frac{\partial^2}{\partial x^2} = \frac{d^2}{d\xi^2}; \frac{\partial^2}{\partial y^2} = \frac{d^2}{d\xi^2}$$ (3)
The field $\Psi_j(x,t)$ is transformed into functions of $\Psi_j(x,t) = u_j(\xi)$. Then, the partial differential equations of the model become a set of ordinary differential equations.

\[ p_i(u_j, u_j; \xi, u_j; \xi, \ldots) = 0 \quad (4) \]

Where $p_i$ are several sets of polynomials of the variables $u_j$ and their ordinary derivatives of $\xi$. The tanh method is a mathematical method used to find solution of equations like (1), which are called solitary waves. In general, the solutions of the field $\Psi(x,t)$, can be expressed as finite combinations of tanh functions [7]. Now, we introduce a new independent variable:

\[ Y(x,y,t) = \tanh(\xi) \quad (5) \]

Then, the derivatives of $\xi$, are:

\[ \frac{d}{d\xi} = (1 - Y^2) \frac{d}{dY}; \quad \frac{d^2}{d\xi^2} = -2Y(1 - Y^2) \frac{d}{dY} + (1 - Y^2)^2 \frac{d^2}{dY^2} \quad (6) \]

The solutions are postulated as:

\[ u_j(\xi) = \sum_{i=1}^{m_j} a_{i,j} Y_i \quad (7) \]

3 The one-dimensional solution to the compressible Navier-Stokes equations

The one-dimensional compressible Navier-Stokes equations are:

\[ \frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + \frac{\partial \rho}{\partial x} u = 0 \quad (8) \]

\[ \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) - \nu_1 \left( \frac{\partial^2 u}{\partial x^2} \right) - \frac{\nu_2}{3} \left( \frac{\partial^2 u}{\partial x^2} \right) + \kappa n \rho^{n-1} \frac{\partial \rho}{\partial x} = 0 \quad (9) \]

We choose in one spatial dimension, the next solution:

\[ u(x,t) = u(\xi), \quad p(x,t) = p(\xi) \quad (10) \]
Where $\xi$ is given in eq. (2), but is reduced to $\xi = x - at$. Then, using eqs. (3), and (10) in eqs. (8) and (9), we obtain:

\[-a\frac{d\rho}{d\xi} + \rho \frac{du}{d\xi} + \frac{d\rho}{d\xi} u = 0 \tag{11}\]

\[\rho(-a \frac{du}{d\xi} + u \frac{du}{d\xi} - \nu_1 \frac{d^2 u}{d\xi^2} - \frac{\nu_2}{3} \frac{d^2 u}{d\xi^2} + \kappa n \rho^{n-1} \frac{d\rho}{d\xi} = 0 \tag{12}\]

Which corresponds to eqs. (4). Also, we suppose a solution for eqs. (8) and (9) are given by eq. (7):

\[u(\xi) = \sum_{i=1}^{p} a_i Y^i \tag{13}\]

\[\rho(\xi) = \sum_{i=1}^{q} c_i Y^i \tag{14}\]

Then replacing the derivatives eq. (6) in eqs. (11) and (12), we have:

\[(1 - Y^2)(-a \frac{d\rho}{d\xi} + \rho \frac{du}{d\xi} + \frac{d\rho}{d\xi} u) = 0 \tag{15}\]

\[\rho(1 - Y^2)(-a \frac{du}{dY} + u \frac{du}{dY}) - \nu_1 (-2Y(1 - Y^2) \frac{d\rho}{d\xi} + (1 - Y^2) \frac{d^2 u}{dY^2} - \frac{\nu_2}{3} (-2Y(1 - Y^2) \frac{du}{dY}) \tag{16}\]

\[+ (1 - Y^2)^2 \frac{d^2 u}{dY^2} + \kappa n \rho^{n-1} (1 - Y^2) \frac{d\rho}{dY} = 0\]

Now, we have to determine the parameters $p$ and $q$ in eqs. (13-14). Therefore, we balance the nontrivial linear term of highest-order with the highest order nonlinear terms in eqs. (17-18).

\[Y^4 \frac{d^2 u}{dY^2} \rightarrow \rho Y^2 \frac{du}{dY} \rightarrow 4 + p - 2 = q + 2 + (p - 1) \rightarrow q = 1 \tag{17}\]

\[Y^4 \frac{d^2 u}{dY^2} \rightarrow \rho Y^2 \frac{d\rho}{dY} \rightarrow 4 + p - 2 = q + 2 + (q - 1) \rightarrow p = 1 \tag{18}\]
Then, $p = 1$, $q = 1$, using $n = 2$. Therefore, $u$ and $\rho$ are:

$$u(\xi) = a_0 + a_1 Y$$  \hfill (19)

$$\rho(\xi) = c_0 + c_1 Y$$  \hfill (20)

We suppose $a_1 \neq 0$ and $c_1 \neq 0$. Replacing, eqs. (19-20) and their derivatives in eqs. (15-16), we obtain:

$$-ac_1 + c_0 a_1 + a_0 c_1 + c_1 a_1 Y + a_1 c_1 Y = 0$$  \hfill (21)

$$-aa_1 c_0 + a_0 c_0 a_1 + a_1 a_1 c_0 Y - aa_1 c_1 Y + a_0 a_1 c_1 Y + a_1 a_1 c_1 Y^2 + 2\nu_1 Y a_1 + \frac{2Y\nu_2}{3} a_1 + 2\kappa c_0 c_1 + 2\kappa c_1 c_1 Y = 0$$  \hfill (22)

Then, from equation (21)

$$Y^0 \to -ac_1 + c_0 a_1 + a_0 c_1 = 0$$  \hfill (23)

$$Y^1 \to c_1 a_1 + a_1 c_1 = 0$$  \hfill (24)

Then, from equation (22)

$$Y^0 \to -aa_1 + a_0 a_1 + 2\kappa c_1 = 0$$  \hfill (25)

$$Y^1 \to a_1 a_1 c_0 - aa_1 c_1 + a_0 a_1 c_1 + 2\nu_1 a_1 + \frac{2\nu_2}{3} a_1 + 2\kappa c_1 c_1 = 0$$  \hfill (26)

$$Y^2 \to a_1 a_1 c_1 = 0$$  \hfill (27)

Therefore,
\[
a_{0,1} = a + \frac{a^2 k \nu_2}{-ak \nu_2 - \sqrt{a^2 k^2 \nu_2^2 - 4ac_0 k^2 \nu_2^2 + 8c_0 k^3 \nu_2^2}}
\]

(28)

\[
a_{1,1} = \frac{2(2a\nu_2 - a^2 \nu_2 c_0 - 4k \nu_2 - \frac{a\sqrt{k^2(a^2-4ac_0+8c_0 k)\nu_2^2}}{c_0 k})}{3c_0 k}
\]

(29)

\[
c_{1,1} = \frac{2 \left(-ak \nu_2 - \sqrt{a^2 k^2 \nu_2^2 - 4ac_0 k^2 \nu_2^2 + 8c_0 k^3 \nu_2^2} \right)}{3c_0 k^2}
\]

(30)

\[
a_{0,2} = a + \frac{a^2 k \nu_2}{-ak \nu_2 + \sqrt{a^2 k^2 \nu_2^2 - 4ac_0 k^2 \nu_2^2 + 8c_0 k^3 \nu_2^2}}
\]

(31)

\[
a_{1,2} = \frac{2(2a\nu_2 - a^2 \nu_2 c_0 - 4k \nu_2 + \frac{a\sqrt{k^2(a^2-4ac_0+8c_0 k)\nu_2^2}}{c_0 k})}{3c_0 k}
\]

(32)

\[
c_{1,2} = \frac{2 \left(-ak \nu_2 + \sqrt{a^2 k^2 \nu_2^2 - 4ac_0 k^2 \nu_2^2 + 8c_0 k^3 \nu_2^2} \right)}{3c_0 k^2}
\]

(33)
The solutions are:

\[ u_1(\xi) = a_{0,1} + a_{1,1} \tanh(\xi) \]  
(34)

\[ \rho_1(\xi) = c_{0,1} + c_{1,1} \tanh(\xi) \]  
(35)

\[ u_2(\xi) = a_{0,2} + a_{1,2} \tanh(\xi) \]  
(36)

\[ \rho_2(\xi) = c_{0,2} + c_{1,2} \tanh(\xi) \]  
(37)

### 3.1 The two-dimensional solution to the compressible Navier-Stokes equations

The two-dimensional compressible Navier-Stokes equations are:

\[
\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} + \frac{\partial \rho}{\partial x} u + \frac{\partial \rho}{\partial y} v = 0
\]  
(38)

\[
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} \right) - \nu_1 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \nu_2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} \right) + \kappa n \rho^{n-1} \frac{\partial \rho}{\partial x} = 0
\]  
(39)

\[
\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) - \nu_1 \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \nu_2 \left( \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} \right) + \kappa n \rho^{n-1} \frac{\partial \rho}{\partial y} = 0
\]  
(40)

Where \( u \) and \( v \) represent the velocity components of the fluid and \( \rho \) its density, the parameters \( \nu_1 \) and \( \nu_2 \) are the kinematic viscosities of the fluid. The components of velocity and density using the coordinate transformation \( \xi \), eq. (2), are:

\[ u(x, y, t) = u(\xi), \quad v(x, y, t) = v(\xi), \quad p(x, y, t) = p(\xi) \]  
(41)
Then, using eqs. (3) and (40) in eqs. (37)-(39), we obtain:

\[-a \frac{d\rho}{d\xi} + \rho \frac{du}{d\xi} + \rho \frac{dv}{d\xi} + \frac{d\rho}{d\xi} u + \frac{d\rho}{d\xi} v = 0\]  (42)

\[\rho (-a \frac{du}{d\xi} + u \frac{du}{d\xi} + v \frac{dv}{d\xi}) - 2\nu_1 \frac{d^2u}{d\xi^2} - \frac{\nu_2}{3} \left( \frac{d^2u}{d\xi^2} + \frac{\partial^2 v}{\partial \xi^2} \right) + \kappa \rho^{n-1} \frac{d\rho}{d\xi} = 0\]  (43)

\[\rho (-a \frac{dv}{d\xi} + u \frac{dv}{d\xi} + v \frac{dv}{d\xi}) - 2\nu_1 \frac{d^2v}{d\xi^2} - \frac{\nu_2}{3} \left( \frac{d^2u}{d\xi^2} + \frac{\partial^2 v}{\partial \xi^2} \right) + \kappa \rho^{n-1} \frac{d\rho}{d\xi} = 0\]  (44)

Which, also corresponds to eqs. (4). Then, we suppose a solution for eqs. (42-44) are given by eq. (7):

\[u(\xi) = \sum_{i=1}^{r} a_i Y^i\]  (45)

\[v(\xi) = \sum_{i=1}^{m} b_i Y^i\]  (46)

\[\rho(\xi) = \sum_{i=1}^{q} c_i Y^i\]  (47)

Also, we assume that \(Y\) is given by eq. (5). Then, replacing the derivatives eq. (6) in eqs. (45-47), we have:

\[(1 - Y^2) (-a \frac{d\rho}{dY} + \rho \frac{du}{dY} + \rho \frac{dv}{dY} + \frac{d\rho}{dY} u + \frac{d\rho}{dY} v) = 0\]  (48)

\[\rho(1 - Y^2) (-a \frac{du}{dY} + u \frac{du}{dY} + v \frac{dv}{dY}) - 2\nu_1 (2Y(1 - Y^2) \frac{du}{dY}) + \frac{\nu_2}{3} (2Y(1 - Y^2) \frac{d^2u}{dY^2} - (1 - Y^2)^2 \frac{d^2u}{dY^2}) + \kappa \rho^{n-1} (1 - Y^2 \frac{d\rho}{dY}) = 0\]  (49)
Navier-Stokes equations

\[ \rho(1 - Y^2)(-a \frac{dv}{dY} + u \frac{dv}{dY} + v \frac{dv}{dY}) - 2\nu_1(-2Y(1 - Y^2) \frac{dv}{dY}) \]

\[ +(1 - Y^2)^2 \frac{d^2v}{dY^2} - \frac{\nu_2}{3}(-2Y(1 - Y^2) \frac{du}{dY} + (1 - Y^2)^2 \frac{d^2u}{dY^2}) \]

\[ -2Y(1 - Y^2) \frac{dv}{dY} + (1 - Y^2)^2 \frac{d^2v}{dY^2} + \kappa n \rho^{n-1}(1 - Y^2) \frac{d\rho}{dY} = 0 \] (50)

Now, we have to determine the parameters \( r, m \) and \( q \) in eqs. (45-47). Therefore, we balance the nontrivial linear term of highest-order with the highest order nonlinear term, in eqs. (48-50), we obtain.

\[ Y^4 \frac{d^2u}{dY^2} \rightarrow Y^2 u \frac{du}{dY} \rightarrow 4 + r - 2 = 2 + (r - 1) + r \] (51)

\[ Y^4 \frac{d^2v}{dY^2} \rightarrow Y^2 \frac{dv}{dY} \rightarrow 4 + m - 2 = 2 + (m - 1) + m \] (52)

\[ Y^4 \frac{d^2u}{dY^2} \rightarrow Y^2 \rho^{n-1} \frac{d\rho}{dY} \rightarrow 4 + r - 2 = 2 + (q - 1) + q(n - 1) \] (53)

Then, \( r = 1, m = 1 \), using \( n = 2 \) then \( q = 2 \). Therefore, using eqs. (45-47), \( u, v \) and \( \rho \) are:

\[ u(\xi) = a_0 + a_1 Y, \quad v(\xi) = b_0 + b_1 Y, \quad \rho(\xi) = c_0 + c_1 Y + c_2 Y^2 \] (54)

We suppose \( a_1 \neq 0, b_1 \neq 0 \) and \( c_1 \neq 0 \). Replacing, eqs. (54) and their derivatives in eqs. (48-50), we have:

\[ Y^0 \rightarrow -ac_1 + a_1c_0 + b_1c_0 + c_1a_0 + c_1b_0 = 0 \] (55)

\[ Y^1 \rightarrow 2ac_1c_2 + a_1c_1 + b_1c_1 + c_1a_1 + 2c_2a_0 + c_1b_1 + 2c_2b_0 = 0 \] (56)

\[ Y^2 \rightarrow a_1c_2 + b_1c_2 + 2c_2a_1 + 2c_2b_1 = 0 \rightarrow a_1 = -b_1 \] (57)

Then, from equation (49)
\[ Y^0 \rightarrow -aa_1c_0 + a_0c_0a_1 + b_0c_0b_1 + \kappa c_1 = 0 \]  
(58)

\[ Y^1 \rightarrow a_1^2c_0 + b_1^2c_0 - aa_1c_1 + a_0a_1c_1 + b_1b_0c_1 + 4a_1\nu_1 + \frac{2}{3}\nu_2a_1 + \frac{2}{3}\nu_2b_1 + 2\kappa c_2 = 0 \]  
(59)

\[ Y^2 \rightarrow a_1^2c_1 + b_1^2c_1 - aa_1c_2 + a_0a_1c_2 + b_1b_0c_2 = 0 \]  
(60)

\[ Y^3 \rightarrow a_1^2c_2 + b_1^2c_2 = 0 \rightarrow a_1^2 = -b_1^2 \rightarrow a_1 = \pm ib_1 \]  
(61)

Then, from equation (50)

\[ Y^0 \rightarrow -ab_1c_0 + a_0c_0b_1 + b_0c_0b_1 + \kappa c_1 = 0 \]  
(62)

\[ Y^1 \rightarrow a_1c_0b_1 + b_1^2c_0 - ab_1c_1 + a_0b_1c_1 + b_1b_0c_1 + 4b_1\nu_1 + \frac{2}{3}\nu_2a_1 + \frac{2}{3}\nu_2b_1 + 2\kappa c_2 = 0 \]  
(63)

\[ Y^2 \rightarrow a_1b_1c_1 + b_1^2c_1 - ab_1c_2 + a_0b_1c_2 + b_0b_1c_2 = 0 \]  
(64)

\[ Y^3 \rightarrow a_1b_1c_2 + b_1^2c_2 = 0 \rightarrow a_1 = -b_1 \]  
(65)

We have that:

\[ (a_0 = a; a_1 = -b_1; b_0 = 0; b_1; c_0; c_1 = 0; c_2 = \frac{2b_1\nu_1}{\kappa}) \]  
(66)

The solutions are:

\[ u(\xi) = u(x, y, t) = a - b_1 \tanh(\xi) \]  
(67)

\[ v(\xi) = v(x, y, t) = b_1 \tanh(\xi) \]  
(68)

\[ \rho(\xi) = \rho(x, y, t) = c_0 + \frac{2b_1\nu_1}{\kappa} \tanh^2(\xi) \]  
(69)
4 Conclusions and Future Work

In this work, we apply the *tanh* method finding explicit expressions for the components of the fluid velocity *u*, *v* and density *ρ*, for the compressible Navier-Stokes equations in one and two dimensions. We get two family of solutions in one dimension, and one family of solution for the two-dimension case. Extending the method to three or more dimensions to the CSEq is straightforward.

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