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Abstract

In this paper we discuss the normal ordering procedure of the generalized Heisenberg algebra proposed by Curado and Rego-Monteiro. We also present the generalized Heisenberg algebra whose characteristic function is a Möbius transformation and study the representation for this algebra.

1 Introduction

In the last few years quantum algebras and quantum groups have been the subject of intensive research in several physics and mathematics fields. Quantum groups or q-deformed Lie algebra implies some specific deformation of classical Lie algebra. From the mathematical point of view, it is a non-commutative associative Hopf algebra. The structure and representation theory of quantum groups have been developed extensively by Jimbo [1] and Drinfeld [2].

In the study of the basic hypergeometric function Jackson [3] invented the Jackson derivative and integral, which is now called q-derivative and q-integral.
Jackson’s pioneering research enabled theoretical physicists and mathematician to study the new physics or mathematics related to the q-calculus. Much was accomplished in this direction and work is under way to find the meaning of the deformed theory.

By using the q-calculus, Arik and Coon [4] proposed the q-deformation of the Heisenberg algebra as follows;

\[ aa^\dagger - qa^\dagger a = 1 \]

where \( N = N^\dagger \) is called a number operator and \( a = (a^\dagger)^\dagger \).

Following the approach of the authors of ref [4], several deformed Heisenberg algebra has been proposed in the literature [5, 6, 7]. In most of deformed Heisenberg algebra, authors adopted the same commutation relations between the number operator and step operators and deformed the commutation relation between \( a \) and \( a^\dagger \).

In 2000, the new generalization of the Heisenberg algebra was introduced by Rego-Monteiro and Curado [8,9]. In this algebra, the commutation relations between the number operator and step operators were changed into the more general form which is characterized in terms of the function of the number operator. The authors of the ref [8,9] called this function a characteristic function and discussed the cases when the characteristic function is linear and quadratic in the number operator [9].

In this paper we discuss the normal ordering procedure of the generalized Heisenberg algebra (GHA) [11,12]. We use some operator identity to construct the generalized Stirling operator of the second kind and its generating function. We also present the generalized Heisenberg algebra whose characteristic function is a Möbius transformation. In this case, we discuss some examples giving the infinite or finite dimensional representation.

2 Brief Review of GHA

In this section we review the representation theory of the GHA [8,9]. The GHA takes the following form;

\[ Ha^\dagger = a^\dagger f(H), \quad aH = f(H)a, \quad [a, a^\dagger] = f(H) - H, \quad (2) \]

where \( a = (a^\dagger)^\dagger \), \( H \) is a hamiltonian of the physical system under consideration and \( f(H) \) is an analytic function of \( H \), called a characteristic function of the algebra. The deformation parameter is related to the the concrete form of \( f(H) \) and a large class of type Heisenberg algebra can be obtained by choosing the function \( f(H) \).
For example, if we take \( f(H) = 1 + H \), the algebra (2) reduces to the ordinary Heisenberg algebra where \( H \) is regarded as a number operator \( N = a^\dagger a \). As another example we can take \( f(H) = 1 + qH \), then we have the \( q \)-deformed Heisenberg algebra where the hamiltonian is related to the number operator \( N \) as follows

\[
H = a^\dagger a = [N]_q = \frac{1 - q^N}{1 - q}
\]

(3)

The choice of \( f(H) \) gives a lot of deformed algebra, which is the reason why \( f(H) \) is called a characteristic function of the algebra.

The Casimir operator of the GHA has the expression:

\[
C = a^\dagger a - H = aa^\dagger - f(H)
\]

(4)

Now let us construct the irreducible representation of the algebra (2) by introducing the ground state \( |0\rangle \) with the lowest eigenvalue of \( H \) obeying

\[
H|0\rangle = \epsilon_0|0\rangle
\]

(5)

Let \( |n\rangle \) be a normalized eigenstate of \( H \):

\[
H|n\rangle = \epsilon_n|n\rangle, \quad n = 0, 1, 2, \cdots
\]

(6)

Applying \( H \) on \( a^\dagger|n\rangle \) yields

\[
H(a^\dagger|n\rangle) = a^\dagger f(H)|n\rangle = f(\epsilon_n)(a^\dagger|n\rangle),
\]

(7)

which means that \( a^\dagger|n\rangle \) is an eigenstate of \( H \) with eigenvalue \( f(\epsilon_n) \). Applying \( a^\dagger \) on the ground state successively, we have

\[
H((a^\dagger)^n|0\rangle) = f^n(\epsilon_0)((a^\dagger)^n|0\rangle)
\]

(8)

where \( \epsilon_n = f^n(\epsilon_0) \) denotes the \( n \)-th iterate of \( f \) defined as \( f^n(\epsilon_0) = f(f(\cdots f(\epsilon_0))) \). If we assume that \( (a^\dagger)^n|0\rangle \) is proportional to \( |n\rangle \), we have

\[
\epsilon_n = f^n(\epsilon_0) = f(\epsilon_{n-1})
\]

(9)

So all eigenvalues of \( H \) are determined from \( \epsilon_0 \) through \( f \).

Acting \( a \) on \( |n + 1\rangle \), we get

\[
a(H|n + 1\rangle) = f(H)(a|n + 1\rangle) = \epsilon_{n+1}(a|n + 1\rangle) = f(\epsilon_n)(a|n + 1\rangle)
\]

(10)

Thus we have

\[
H(a|n + 1\rangle) = \epsilon_n(a|n + 1\rangle),
\]

(11)
which shows that $a|n+1\rangle$ is also an eigenstate of $H$ with eigenvalue $\epsilon_n$ and
$a|n+1\rangle$ is proportional to $|n\rangle$.

The representation of the GHA is then given by

$$H|n\rangle = \epsilon_n|n\rangle, \quad n = 0, 1, 2, 3, \ldots$$

$$C|n\rangle = -\epsilon_0|n\rangle$$

$$a^\dagger|n\rangle = N_n|n+1\rangle, \quad a|n\rangle = N_{n-1}|n-1\rangle,$$  \hspace{1cm} (12)

where

$$N_n^2 = f^{n+1}(\epsilon) - \epsilon_0 = \epsilon_{n+1} - \epsilon_0, \quad N_{-1} = 0$$  \hspace{1cm} (13)

The relation between step operators and hamiltonian is given by

$$aa^\dagger = f(H) - \epsilon_0,$$  \hspace{1cm} (14)

$$a^\dagger a = H - \epsilon_0$$  \hspace{1cm} (15)

Then we have

$$|n\rangle = \frac{1}{N_n^n \sqrt{|n|_f!}} (a^\dagger)^n|0\rangle,$$  \hspace{1cm} (16)

where f-number is defined as

$$[n]_f = \frac{N_n^n - N_{n-1}^{n-1}}{N_0^2} = \frac{f^n(\epsilon_0) - \epsilon_0}{f(\epsilon_0) - \epsilon_0}$$  \hspace{1cm} (17)

Indeed, we know that $[n]_f \to n$ when $f(H) = 1 + H$ and $[n]_f \to [n]_q$ when $f(H) = 1 + qH$.

In the GHA, the $(n+1)$-th eigenvalue $\epsilon_{n+1}$ of the Hamiltonian depends on the previous eigenvalue $\epsilon_n$:

$$\epsilon_{n+1} = f(\epsilon_n)$$  \hspace{1cm} (18)

so the GHA is sometimes called a one step algebra.

The representation can be rewritten in terms of the f-number as follows;

$$a^\dagger|n\rangle = N_0 \sqrt{[n+1]_f}|n+1\rangle, \quad a|n\rangle = N_0 \sqrt{[n]_f}|n-1\rangle,$$  \hspace{1cm} (19)

where

$$N_0^2 = f(\epsilon_0) - \epsilon_0$$  \hspace{1cm} (20)
3  Normal ordering process and f-Stirling operator

Now we discuss the normal ordering process for GHA. From the second relation of the eq.(2), we have

$$a \Phi(H) = \Phi(f(H))a$$

(21)

for an arbitrary function \( \Phi(H) \). For \( \Phi(x) = f(x) \), we get

$$a^k f(H) = f^{k+1}(H)a^k, \quad f(H)(a^\dagger)^k = (a^\dagger)^k f^{k+1}(H)$$

(22)

Replacing \( H \to f^{-1}(H) \) in the first relation of the eq.(2), we have

$$a^\dagger H = f^{-1}(H)a^\dagger$$

(23)

or generally

$$(a^\dagger)^k H = f^{-k}(H)a^\dagger$$

(24)

and

$$H a^k = a f^{-k}(H),$$

(25)

where

$$f^{-k} = f^{-1} \circ f^{-1} \circ \cdots \circ f^{-1}$$

(26)

Then we have the following formulas;

$$a^k (a^\dagger)^k = \prod_{j=1}^{k} [f^j(H) - \epsilon_0]$$

(27)

$$(a^\dagger)^k a^k = \prod_{j=0}^{k-1} [f^{-j}(H) - \epsilon_0],$$

(28)

where \( f^0(H) = f(H) \). The f-Stirling operator of the second kind is defined as

$$(a^\dagger a)^n = \sum_{k=1}^{n} (a^\dagger)^k S(n, k, H)a^k,$$

(29)

where \( S(n, k, H) \) is the f-Stirling operator of the second kind. Using \((a^\dagger a)^{n+1} = (a^\dagger a)(a^\dagger a)^n\), we can obtain the recurrence relation

$$S(n + 1, 1, H) = (f(H) - H)S(n, 1, H),$$

$$S(n + 1, k, H) = S(n, k - 1, f(H)) + (f^k(H) - H)S(n, k, H), \quad (1 \leq k \leq n),$$

$$S(n + 1, n + 1, H) = S(n + 1, n, f(H)),$$

(30)
where $S(i, j, H) = 0$ for $i < j$ and we used the following formulas
\[
a(a^\dagger)^k = (a^\dagger)^k a + (a^\dagger)^{k-1} (f^k(H) - H)
\]
\[
a^k a^\dagger = a^\dagger a^k + (f^k(H) - H)a^{k-1}
\]
(31)

The eq.(31) can also be written as
\[
a(a^\dagger)^k = (a^\dagger)^{k-1} (f^k(H) - \epsilon_0)
\]
\[
a^k a^\dagger = (f^k(H) - \epsilon_0)a^{k-1}
\]
(32)

The first few Stirling operator of the second kind are
\[
S(1, 1, H) = I,
\]
\[
S(2, 1, H) = f(H) - H, \quad S(2, 2, H) = I,
\]
\[
S(3, 1, H) = (f(H) - H)^2, \quad S(3, 2, H) = 2f^2(H) - f(H) - H,
\]
\[
S(3, 3, H) = I, \quad S(4, 1, H) = (f(H) - H)^3,
\]
\[
S(4, 2, H) = 3(f^2(H))^2 - 3f^2(H)f(H) + (f(H))^2 - 3Hf^2(H) + Hf(H) + H^2,
\]
\[
S(4, 3, c) = 3f^3(H) - f^2(H) - f(H) - H, \quad S(4, 4, H) = I
\]
(33)

We define the generating function of the f-Stirling operator of the second kind in the form
\[
P_k(H|x) = \sum_{n=k}^{\infty} S(n, k, H)x^n
\]
(34)

The recurrence relations are then given by
\[
P_1(H|x) = \frac{x}{1-(f(H)-H)x}
\]
(35)
\[
P_k(H|x) = \frac{x}{1-(f^k(H)-H)x} P_{k-1}(f(H)|x), \quad (k > 1)
\]
(36)

If we set $P_0(H|x) = I$, we get
\[
P_k(H|x) = \prod_{j=0}^{k-1} \frac{x}{1-(f^k(H)-f^j(H))x}
\]
(37)

The eq.(37) can be written in terms of a sum of partial fractions
\[
P_k(H|x) = a^k \sum_{r=0}^{k-1} \frac{p_r}{1-(f^k(H)-f^r(H))x},
\]
(38)

where
\[
p_r = \frac{1}{\prod_{j=0, j \neq r}^{k-1} (1-\frac{f^j(H)}{f^r(H)})}
\]
(39)

Therefore the f-Stirling operator of the second kind takes the following form;
\[
S(n, k, H) = \sum_{r=0}^{k-1} \frac{(f^k(H) - f^r(H))^{n-1}}{\prod_{j=1, j \neq r}^{k}(f^r(H) - f^j(H))}
\]
(40)
4 The deformed Heisenberg algebra related to the Möbius transformation

In this section we are going to find the representation for the algebra defined by the relation given in the eq.(2) considering

\[ f(H) = \frac{\gamma H + \delta}{\alpha H + \beta}, \]  \hspace{1cm} (41)

where \( \alpha, \beta, \gamma, \delta \) are real. The ordinary Heisenberg algebra and q-deformed Heisenberg algebra are obtained from the suitable choice of \( \alpha, \beta, \gamma, \delta \).

The inverse of the Möbius transformation is given by

\[ f^{-1}(H) = \frac{\beta H - \delta}{-\alpha H + \gamma}, \]  \hspace{1cm} (42)

where \( \beta \gamma - \alpha \delta \neq 0 \). In the choice of the characteristic function given in the eq.(41), the algebra (2) takes the following form;

\[ \beta Ha^\dagger - \gamma a^\dagger H = \delta a^\dagger - \alpha Ha^\dagger H, \]

\[ \beta aH - \gamma Ha = \delta a - \alpha HaH, \]

\[ (\alpha H + \beta)aa^\dagger - (\alpha H + \beta)a^\dagger a = \delta + (\gamma - \beta)H - \alpha H^2 \]  \hspace{1cm} (43)

A Möbius transformation is equivalent to a sequence of the following transformations;

\[ f^{(1)}(H) = H + \frac{\delta}{\beta} \]

\[ f^{(2)}(H) = H^{-1} \]

\[ f^{(3)}(H) = \frac{\alpha \delta - \beta \gamma}{\alpha^2} H \]

\[ f^{(4)}(H) = H + \frac{\gamma}{\alpha} \]

Then we have

\[ f^{(4)} \circ f^{(3)} \circ f^{(2)} \circ f^{(1)}(H) = f(H) = \frac{\gamma H + \delta}{\alpha H + \beta}, \]  \hspace{1cm} (44)

The dimension of the representation space of the algebra (43) depends on the function given in the eq.(41). If the Möbius transformation is idempotent, i.e., \( f^p(H) = I \) for some natural number \( p \), we have \( p \)-dimensional representation. If the Möbius transformation is not idempotent, we have the infinite dimensional representation.
5 Infinite dimensional representation

In this section, we discuss an example giving the infinite dimensional representation. Let us introduce the following characteristic function;

$$f(H) = \frac{\gamma H}{1 - \alpha H},$$  \hspace{1cm} (45)

where we assume that $\alpha > 0, 0 < \gamma < 1$. Then we have the following algebra

$$Ha^\dagger - \gamma a^\dagger H = \alpha Ha^\dagger H,$$

$$aH - \gamma Ha = \alpha HaH,$$

$$(1 - \alpha H)aa^\dagger - (1 - \alpha H)a^\dagger a = (\gamma - 1)H + \alpha H^2$$  \hspace{1cm} (46)

The inverse of the characteristic function is given by

$$f^{-1}(H) = \frac{\gamma H}{1 - \alpha H}$$  \hspace{1cm} (47)

The n-the iterate of $f$ is then given by

$$f^n(H) = \frac{\gamma^n H}{1 - \alpha [n]_\gamma H},$$  \hspace{1cm} (48)

where $[n]_\gamma = \frac{1 - \gamma^n}{1 - \gamma}$.

5.1 Representation

For the characteristic function given in the eq.(45), we have

$$N_0^2 = \frac{\epsilon_0(\gamma - 1 + \alpha \epsilon_0)}{1 - \alpha \epsilon_0}$$  \hspace{1cm} (49)

$$[n]_f = \frac{(1 - \alpha \epsilon_0)[n]_\gamma}{1 - \alpha [n]_\gamma \epsilon_0}$$  \hspace{1cm} (50)

From $N_0^2 > 0, [n]_f > 0$ for all $n$, we have

$$\epsilon_0 < \frac{1 - \gamma}{\alpha}$$  \hspace{1cm} (51)

The representation takes the following form;

$$H|n\rangle = \left(\frac{\gamma^n \epsilon_0}{1 - \alpha [n]_\gamma \epsilon_0}\right)|n\rangle$$

$$a|n\rangle = \sqrt{[n]_\gamma(\gamma - 1 + \alpha \epsilon_0)\epsilon_0} \left|n - 1\right\rangle$$

$$a^\dagger|n\rangle = \sqrt{\frac{[n + 1]_\gamma(\gamma - 1 + \alpha \epsilon_0)\epsilon_0}{1 - \alpha \epsilon_0 [n + 1]_\gamma}} \left|n - 1\right\rangle$$  \hspace{1cm} (52)
5.2 Coherent state

We define a coherent state as an eigenvector of the annihilation operator as follows:
\[ a |z\rangle = z |z\rangle, \]  
where \( z \) is a complex number. The coherent state can be represented by using the number state as follows:
\[ |z\rangle = \sum_{n=0}^{\infty} c_n(z) |n\rangle \]  
Inserting the eq. (54) into the eq. (53), we obtain the following relations:
\[ c_n(z) = \sqrt{\frac{1 - \alpha \epsilon \gamma}{n\gamma (\gamma - 1 + \alpha \epsilon \gamma)} z c_{n-1}(z)} \]  
Solving the eq. (55), the coherent state is given by
\[ |z\rangle = c_0(z) \sum_{n=0}^{\infty} \sqrt{\frac{\prod_{k=1}^{\infty} (1 - \alpha \epsilon_0 [k] \gamma)}{n\gamma ! (\gamma - 1 + \alpha \epsilon_0)^n \epsilon_0^n} z^n |n\rangle}, \]  
where
\[ c_0(z) = \left[ e_{\alpha, \gamma} \left( \frac{|z|^2}{(\gamma - 1 + \alpha \epsilon_0) \epsilon_0} \right) \right]^{-1/2} \]  
and
\[ e_{\alpha, \gamma}(x) = \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{\infty} (1 - \alpha \epsilon_0 [k] \gamma)}{n\gamma !} x^n \]  

6 Finite dimensional representation

In this section we discuss the examples giving the finite dimensional representation. In order to obtain the finite spectrum, the characteristic function \( f \) should be idempotent:
\[ f_p^p(H) = H, \]  
where we will write the \( f(H) \) obeying \( f_p^p(H) = H \) as \( f_p(H) \). Without loss of generality, we will set \( \beta = \delta = 1 \). The first few \( f_p(H) \)'s are then given by
\[ f_2(H) = \frac{-H + 1}{H + 1}, \]  
\[ f_3(H) = \frac{qH + 1}{-[3]_q H + 1}, \]
\[ f_4(H) = \frac{qH + 1}{-\frac{1+q^2}{2} + 1}, \]
\[ f_5(H) = \frac{qH + 1}{\frac{1}{2}((-3 \pm \sqrt{5}) + 2(-2 \pm \sqrt{5})q + (-3 \pm \sqrt{5})q^2)H + 1}, \] (60)

where \( q > 1 \) is assumed.

### 6.1 \( p = 2 \) case

In this case we have
\[ H a^\dagger a = a^\dagger - H a^\dagger H, \]
\[ a H + H a = a - H a H, \]
\[ (H + 1)a a^\dagger - (H + 1) a^\dagger a = 1 - 2H - H^2 \] (61)

From \( N_0^2 > 0 \), we have
\[ -1 - \sqrt{2} < \epsilon_0 < -1, \; \epsilon_0 > \sqrt{2} - 1 \] (62)

From \( f_2^2 = I \), we have two dimensional Fock space and the representation becomes
\[ H|0\rangle = \epsilon_0|0\rangle, \; H|1\rangle = \epsilon_1|1\rangle \]
\[ a^\dagger|0\rangle = N_0|1\rangle, \; a^\dagger|1\rangle = 0, \; a|0\rangle = 0, \; a|1\rangle = N_0|0\rangle, \] (63)

where
\[ \epsilon_1 = \frac{1 - \epsilon_0}{1 + \epsilon_0}, \; N_0 = \sqrt{\frac{1 - 2\epsilon_0 - \epsilon_0^2}{1 + \epsilon_0}} \] (64)

For \(-1 - \sqrt{2} < \epsilon_0 < -1\), we get \( \epsilon_1 > \epsilon_0 \). For \( \epsilon_0 > \sqrt{2} - 1 \), we have \( \epsilon_0 < \epsilon_1 \).

The partition function for the Hamiltonian \( H \) is defined by
\[ Z = \sum_{n=0}^{1} e^{-\beta E_n} = e^{-\beta \epsilon_0} + e^{-\beta \epsilon_1}, \] (65)

where \( \beta = \frac{1}{k_B T} \) and \( k_B \) is a Boltzmann constant and \( T \) is a temperature. For any operator \( \hat{O} \), the ensemble average is then defined by
\[ \langle \hat{O} \rangle = \frac{1}{Z} Tr(e^{-\beta H} \hat{O}) \] (66)

The Green function \( \langle a^\dagger a \rangle \) becomes
\[ \langle a^\dagger a \rangle = \frac{N_0^2}{e^{\beta(\epsilon_1 - \epsilon_0)} + 1} \] (67)
The corresponding mean energy reads

\[
\bar{E} = -\frac{\partial \ln Z}{\partial \beta} = \frac{\epsilon_0 e^{\beta(\epsilon_1 - \epsilon_0)} + \epsilon_1}{e^{\beta(\epsilon_1 - \epsilon_0)} + 1}
\]  

(68)

In the low temperature we have

\[
\bar{E} \simeq \epsilon_0
\]

(69)

and in the high temperature we get

\[
\bar{E} \simeq \epsilon_0 + \frac{\epsilon_1}{2} - \frac{\beta}{4}(\epsilon_1 - \epsilon_0)^2
\]

(70)

The specific heat is then given by

\[
C = \frac{\partial \bar{E}}{\partial T} = \beta^2 \frac{(\epsilon_1 - \epsilon_0)^2 e^{\beta(\epsilon_1 - \epsilon_0)}}{(e^{\beta(\epsilon_1 - \epsilon_0)} + 1)^2}
\]

(71)

In the low temperature we have

\[
C \simeq \beta^2 (\epsilon_1 - \epsilon_0)^2 e^{-\beta(\epsilon_1 - \epsilon_0)}
\]

(72)

and in the high temperature we get

\[
C \simeq \frac{1}{4} \beta^2 (\epsilon_1 - \epsilon_0)^2
\]

(73)

6.2 \textit{p = 3 case}

In this case we have

\[
H a^\dagger - q a^\dagger H = a^\dagger + [3]_q H a^\dagger H,
\]

\[
aH - q Ha = a + [3]_q H a H,
\]

\[
(1 - [3]_q H) a a^\dagger - (1 - [3]_q H) a^\dagger a = 1 + (q - 1)H + [3]_q H^2
\]

(74)

From \(N_0^2 > 0\), we have

\[
0 < \epsilon_0 < \frac{1}{[3]_q}
\]

(75)

From \(f_3^3 = I\), we have three dimensional Fock space and the representation becomes

\[
H|0\rangle = \epsilon_0|0\rangle, \quad H|1\rangle = \epsilon_1|1\rangle, \quad H|2\rangle = \epsilon_2|2\rangle
\]

\[
a^\dagger|0\rangle = N_0|1\rangle, \quad a^\dagger|1\rangle = N_0 \sqrt{2} f|2\rangle, \quad a^\dagger|2\rangle = 0,
\]

\[
a|0\rangle = 0, \quad a|1\rangle = N_0|0\rangle, \quad a|2\rangle = N_0 \sqrt{2} f|1\rangle,
\]

(76) (77)
where
\[ \epsilon_1 = \frac{1 + q\epsilon_0}{1 - [3]_q\epsilon_0}, \quad \epsilon_2 = \frac{1 + \epsilon_0}{q + [3]_q\epsilon_0} \]
\[ N_0 = \sqrt{\frac{1 + (q - 1)\epsilon_0 + [3]_q\epsilon_0^2}{1 - [3]_q\epsilon_0}} \]

and
\[ [0]_f = 0, \quad [1]_f = 1 \]
\[ [2]_f = \frac{(1 - [3]_q\epsilon_0)(1 - (q - 1)\epsilon_0 - [3]_q\epsilon_0^2)}{(q + [3]_q\epsilon_0)(1 + (q - 1)\epsilon_0 + [3]_q\epsilon_0)} \] (78)

It can be easily checked that \( \epsilon_1 > \epsilon_2 > \epsilon_0 \).

The partition function for the Hamiltonian \( H \) is defined by
\[ Z = \sum_{n=0}^{2} e^{-\beta E_n} = e^{-\beta \epsilon_0} + e^{-\beta \epsilon_1} + e^{-\beta \epsilon_2} \] (79)

The Green function \( \langle a^\dagger a \rangle \) becomes
\[ \langle a^\dagger a \rangle = \frac{N_0^2(e^{\beta(\epsilon_2 - \epsilon_1)} + [2]_q)}{1 + e^{\beta(\epsilon_2 - \epsilon_1)} + e^{\beta(\epsilon_2 - \epsilon_0)}} \] (80)

The corresponding mean energy reads
\[ \bar{E} = -\frac{\partial \ln Z}{\partial \beta} = \frac{\epsilon_0 e^{\beta(\epsilon_2 - \epsilon_0)} + \epsilon_1 e^{\beta(\epsilon_2 - \epsilon_1)} + \epsilon_2}{1 + e^{\beta(\epsilon_2 - \epsilon_0)} + e^{\beta(\epsilon_2 - \epsilon_1)}} \] (81)

In the low temperature we have
\[ \bar{E} \simeq \epsilon_0 \] (82)

and in the high temperature we get
\[ \bar{E} \simeq \frac{\epsilon_0 + \epsilon_1 + \epsilon_2}{3} + \frac{2\beta}{9} \{\epsilon_0(\epsilon_2 - \epsilon_0) + \epsilon_1(\epsilon_0 - \epsilon_1) + \epsilon_2(\epsilon_1 - \epsilon_2)\} \] (83)

The specific heat is then given by
\[ C = \frac{\partial \bar{E}}{\partial T} = \beta^2 \frac{(\epsilon_2 - \epsilon_0)^2 e^{\beta(\epsilon_2 - \epsilon_0)} + (\epsilon_2 - \epsilon_1)^2 e^{\beta(\epsilon_2 - \epsilon_1)}}{(1 + e^{\beta(\epsilon_2 - \epsilon_0)} + e^{\beta(\epsilon_2 - \epsilon_1)})^2} \] (84)

In the low temperature we have
\[ C \simeq \beta^2 (\epsilon_2 - \epsilon_0)^2 e^{-\beta(\epsilon_2 - \epsilon_0)} \] (85)

and in the high temperature we get
\[ C \simeq \frac{1}{9} \beta^2 \{ (\epsilon_2 - \epsilon_0)^2 + (\epsilon_2 - \epsilon_1)^2 \} \]
\[ + \frac{1}{27} \beta^3 \{ (\epsilon_2 - \epsilon_0)(\epsilon_1 - \epsilon_0)^2 + (\epsilon_0 - \epsilon_2)(\epsilon_2 - \epsilon_1)^2 + (\epsilon_1 - \epsilon_2)(\epsilon_2 - \epsilon_0)^2 \} \] (86)
Conclusion

In this paper we discussed the normal ordering procedure of the generalized Heisenberg algebra (GHA), where we introduced the generalized Stirling operator of the second kind instead of the Stirling number of the second kind and constructed its generating function. We also discussed the generalized Heisenberg algebra whose characteristic function is a Möbius transformation. We found that the dimension of the representation space of the algebra related to Möbius transformation depends on the parameters of the Möbius transformation. We showed that the dimension of the representation space is finite when the Möbius transformation is idempotent but it is infinite when the Möbius transformation is not idempotent.

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