A Novel Chaotic Attractors in Piecewise Version of the 3D Hénon Map

M. Mammeri

Department of Mathematics
University Mentouri Constantine (1), Algeria

Abstract

In this paper, we have proposed a 3D piecewise chaotic map with six terms. Our proposed map can display a new type of chaotic attractor which has not been studied before in the literature. The chaotic attractor results from a reverse border-collision bifurcation from a stable 1-periodic orbit route to chaos.

Keywords: Fixed point, piecewise map, chaotic attractors, three-dimensional Hénon map

1 Introduction

In recent years, many papers have described 3D chaotic map with a quadratic inverse and constant Jacobian, one of the most famous being a three-dimensional Hénon map (1) suggested by [7] and studied in detail by others [8, 9, 10, 11] and given by:

\[ H(x, y, z) = \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} y \\ z \\ a + bx + cy - z^2 \end{pmatrix} \] (1)

where \((x, y, z) \in \mathbb{R}^3\) is the state space and \((a, b, c) \in \mathbb{R}^3\) are bifurcations parameters. The Hénon map (1) analogous of the famous two-dimensional Hénon map [5, 6]. Furthermore, the map (1) is the simplest example of a
dissipative 3D map with chaotic orbits, is quadratic with a quadratic inverse and it has a constant Jacobian matrix determinant equal to $b$ and it has a single quadratic nonlinearity. The chaotic attractor exhibited by the Hénon map (1) called wild-hyperbolic Lorenz attractor (see Fig.1) is similar to Lorenz chaotic attractor, but not equivalent in the topological structure, and it is possible to change the form of this map to obtain other types of chaotic attractors. For our map (1) a classical period-doubling route to chaos is observed as shown in Fig.2. Fig.3 shows the spectrum of the largest Lyapunov exponent of map (1) with respect to the parameter $c$, $c \in [0, 0.9]$.

2 The new piecewise map

The study of piecewise maps is an interesting contribution to the development of the theory of dynamical systems with many possible applications in science and engineering [1, 2, 3, 4]. A large number of physical and engineering systems have been found that are represented by piecewise map where the discrete time state is divided in two or more comportments with different functional forms separated by borderlines.

The new 3D piecewise map (2) is a simplification of 3D the Hénon map (1), the quadratic nonlinear term in $z^2 (-z^2)$ is replaced by $-|z|$. The form of the new 3D piecewise map (2) is given by:

Figure 1: Chaotic attractor of the 3D Hénon (1) with $a = 0.002$, $b = 0.7$, and $c = 0.87$. 
Chaotic attractors

Figure 2: Bifurcation diagram of the 3D Hénon map (1) of the variable $x$ plotted versus the parameter $c \in [0, 0.9]$ with $a = 0$ and $b = 0.7$.

$$f(x, y, z) = \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} y \\ z \\ a + bx + cy - |z| \end{pmatrix}$$

(2)

where $(x, y, z) \in \mathbb{R}^3$ and $(a, b, c) \in \mathbb{R}^3$ are bifurcations parameters, our map (2) displays chaotic attractors, for certain values of its bifurcation parameters. On the other hand the can be by converting the map (2) into a third order difference equation. Let $(x_t, y_t, z_t), t = 0, 1, ...$ be a trajectory of the map (2) and we suppose $x = z_{t-2}, y = z_{t-1}$ and $z = z_t$, then the map (2) can be written as:

$$z_{t+1} = a + bz_{t-2} + cz_{t-1} - |z_t|$$

(3)

Due to the shape of the vector field $f$ of map (2), the space can be divided into two linear regions defined by:

\[
\begin{align*}
\Sigma_1 &= \{(x, y; z) \in \mathbb{R}^3 : z \geq 0\}, \\
\Sigma_2 &= \{(x, y, z) \in \mathbb{R}^3 : z < 0\}.
\end{align*}
\]
Figure 3: Variation of the largest Lyapunov exponent of the 3D Hénon map (1) versus the parameter \( c \in [0, 0.9] \) with \( a = 0 \) and \( b = 0.7 \).

In the two regions \( \Sigma_1 \) and \( \Sigma_2 \), map (2) can be rewritten as follows:

\[
\begin{align*}
    f(x, y, z) &= \begin{cases} 
        \begin{pmatrix} y \\ z \\ a + bx + cy \end{pmatrix} & \text{if } z \in \Sigma_1, \\
        \begin{pmatrix} y \\ z \\ a + bx + cy + z \end{pmatrix} & \text{if } z \in \Sigma_2.
    \end{cases}
\end{align*}
\]

(5)

3 Analytical study

3.1 Fixed points

In this section, we begin by studying the existence of the fixed point of the map (2) and determine their stability type.

**Theorem 1** For all values of the bifurcations parameters \( b + c - 2 \neq 0 \) and \( b + c \neq 0 \) the map (2) has two fixed points, and they are given by:

\[
S_1 = \frac{-a}{b + c - 2}(1, 1, 1), \quad S_2 = \frac{-a}{b + c}(1, 1, 1)
\]

(6)

**Proof.** The fixed point of the map (2) are the real solutions of the system:

\[
y = x, \quad z = y, \quad a + bx + cy - |z| = z
\]

(7)
Hence, one may easily obtain the equation:

\[ a + bz + cz - |z| = z \]  \hspace{1cm} (8)

For \( a = 0 \), there is one fixed point \((0, 0, 0)\) for the map (2). If \( S_1 \in \Sigma_1 \), from (8) one has that \((b + c - 2)z = -a\), which implies that \( z = \frac{-a}{b + c - 2}\) when \( a > 0 \) and \( b < 2 - c \), or when \( a < 0 \) and \( b > 2 - c \). If \( S_2 \in \Sigma_2 \) from (8) one has that \((b + c)z = -a\), which implies that \( z = \frac{-a}{b + c} \) when \( a > 0 \) and \( b > -c \) or when \( a < 0 \) and \( b < -c \).

**Conclusion 2** If \( b + c = 2 \) and \( b + c = 0 \), all orbits of the map (2) are unbounded, and if \( b + c \neq 2 \) or \( b + c \neq 0 \), then the map (2) has possible bounded orbits.

### 3.2 Stability of fixed points

We give necessary and sufficient condition for the local stability of the two fixed points \( S_1 \) and \( S_2 \). Note that in this subsection we use the simple result available in [12], equivalent conditions exist in [13,14], then we have the following theorem:

**Theorem 3** If \( a > 0 \) and \( b < 2 - c \), the fixed point \( S_1 \) of the map (2) is stable if and only if the following conditions hold:

\[
\begin{cases}
-1 < b < 1 \\
c < b < 2 - c < 3 - b - b^2
\end{cases}
\]  \hspace{1cm} (9)

**Proof.** The characteristic polynomial of the Jacobian matrix of the map (2) calculated at a fixed point \( S_1 \), which takes the form: \( P_{S_1}(\lambda) = \lambda^3 + \lambda^2 - c\lambda - b \), according to the result available in [12], we conclude that the fixed point \( S_1 \) of the map (2) is locally stable if the following conditions hold: (i) \( |1 - b| < 1 - c \), (ii) \( |b| < 1 \), and (iii) \( b - c < 1 - b^2 \). From (i) and (ii) we get \( c < b < 2 - c \) and \( -1 < b < 1 \) and from (iii) we get \( 2 - c < 3 - b - b^2 \). Finally, the conditions of stability margin for the fixed point \( S_1 \) are: \( -1 < b < 1 \), \( c < b < 2 - c \) and \( 2 - c < 3 - b - b^2 \). ■

For example we fix the parameters \( a = 0.1 \), \( b = 0.3 \) and \( c = 0.1 \). With this values the fixed point \( S_1(-0.0625, -0.0625, -0.0625) \) is stable, and we have the following three eigenvalues: \( \lambda_1 = -0.7422 - 0.2613i \), \( \lambda_2 = -0.7422 + 0.2613i \) and \( \lambda_3 = 0.4845 \), thus \( |\lambda_{1,2,3}| < 1 \).

**Theorem 4** If \( a < 0 \) and \( b < -c \), the fixed point \( S_2 \) of the map (2) is stable if and only if the following conditions hold:
\[
\begin{aligned}
-1 < b < 1 \\
c - 2 < b < -c < 1 + b - b^2
\end{aligned}
\] (10)

**Proof.** The characteristic polynomial of the Jacobian matrix of the map (2) calculated at a fixed point \( S_2 \), which takes the form: \( P_{S_2}(\lambda) = \lambda^3 - \lambda^2 - c\lambda - b \), according to the result available in [12], we conclude that the fixed point \( S_1 \) of the map (2) is locally stable if the following conditions hold: 

(i) \( |1 + b| < 1 - c \),

(ii) \( |b| < 1 \), and

(iii) \( -b < c < 1 - b^2 \).

From (i) and (ii) we get \( c - 2 < b < -c \) and \( -1 < b < 1 \) and From (iii) we get \( -c < 1 + b - b^2 \). Finally, the conditions of stability margin for fixed point \( S_2 \) are: 

\(-1 < b < 1 \), \( c - 2 < b < -c \) and \( -c < 1 + b - b^2 \). ■

For example we fix the parameters \( a = -0.1, b = -0.3 \) and \( c = -0.1 \). With this values the fixed point \( S_2(-0.25, -0.25, -0.25) \) is stable, and we have the following three eigenvalues: \( \lambda_1 = 0.7125 - 0.4450i \), \( \lambda_2 = 0.7125 + 0.4450i \) and \( \lambda_3 = -0.4251 \), thus \( |\lambda_{1,2,3}| < 1 \).

4 Numerical study

From the above analysis, it is visible that the stability of the fixed point of the map (2) will be changed along with the change of the bifurcation parameters of the map (2), and the map (2) will also be in different state. The dynamical behaviors of the map (2) are investigated numerically, then to determine the long-time behavior and chaotic regions, we numerically computed the bifurcation diagram and largest Lyapunov exponent that are obtained at different values of bifurcation parameter \( a \).

We fix \( b = -0.29, c = -0.9 \), and the initial condition \( x = y = z = 2.01 \), and let \( a \) vary. Then for \( a \in [0, 2] \), the map (2) exhibits the following dynamical behaviors as shown in Fig.5: For \( 0 \leq a < 1.56 \) the map (2) is chaotic with a positive Lyapunov exponent, for example, with \( a = 1, b = -0.29 \) and \( c = -0.9 \) the map (2) has two fixed points \( S_1(0.31, 0.31, 0.31) \) and \( S_2(0.84, 0.84, 0.84) \). The Jacobian matrix of the map (2) evaluated at \( S_1 \) has the following three eigenvalues: \( \lambda_1 = 0.6208 + 0.9026i \), \( \lambda_2 = 0.6208 - 0.9026i \), and \( \lambda_3 = -0.2417 \), thus \( |\lambda_{1,2}| > 1 \) and \( |\lambda_3| < 1 \) in this case \( S_1 \) is a saddle, i.e., the map (2) is unstable at the fixed point. The corresponding Lyapunov exponents are \( L_1 = L_2 = 0.0912 \) and \( L_2 = -1.4201 \). Therefore, the Lyapunov dimension of the map (2) is 2.1302, the chaotic attractor is shown in Fig.8 (e). For \( 1.56 \leq a < 1.66 \) there is stable 1-periodic orbit zone (a chaotic attractor is suddenly destroyed, that produces the periodic windows as the single bifurcation parameter \( a \) is varied observed in Fig.4), for \( 1.66 \leq a < 1.82 \) there is chaotic zone (a chaotic attractor is suddenly created, as the single bifurcation parameter \( a \) is varied observed in Fig.4), for \( 1.82 \leq a \leq 2 \) there is stable 1-periodic orbit zone. Fig.5
Chaotic attractors shows the spectrum of the largest Lyapunov exponent of map (2) with respect to the bifurcation parameter $a$, $a \in [0, 2]$. The phase portraits of the map (2) are shown in Figs.6-7-8-9-10-11.

5 Conclusion

In this paper we investigate the results of a analytical and numerical study of the piecewise version of the 3D Hénon map (2) capable of generating chaotic attractors via border-collision bifurcation route to chaos.

References


Received: March 6, 2015; Published: May 29, 2015
Figure 4: Bifurcation diagram of the map (2) of the variable $x$ plotted versus the parameter $a \in [0, 2]$ with $b = -0.29$ and $c = -0.9$. 
Figure 5: Variation of the largest Lyapunov exponent of the map (2) versus the parameter $a \in [0, 2]$ with $b = -0.29$ and $c = -0.9$.

Figure 6: Projection onto the $(x, y)$ plane of the attractor of the map (2) with $a = 0$, $b = -0.277$ and $c = -0.9$. 
Figure 7: Projection onto the \((x, y)\) plane of the chaotic attractor of the map (2) with \(a = 0.001\), \(b = -0.29\) and \(c = -0.9\).

Figure 8: Projection onto the \((x, y)\) plane of the chaotic attractor of the map (2) with \(a = 0.001\), and \(b = c = -0.9\).
Figure 9: Projection onto the \((x, y)\) plane of the chaotic attractor of the map (2) with \(a = 0.006\), \(b = -0.29\) and \(c = -0.9\).

Figure 10: Projection onto the \((x, y)\) plane of the chaotic attractor of the map (2) with \(a = 0\), \(b = -0.29\) and \(c = -0.9\).
Figure 11: Projection onto the \((x, y)\) plane of the chaotic attractors of the map (2) with \(a = 0.005\), \(b = -0.29\) and \(c = -0.9\).