Causal Communication Constraint for Two Qubits
in Hardy’s Ladder Proof of Non-locality

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Abstract

In this paper, we consider Hardy’s ladder proof of non-locality for two qubits and $K+1$ observables per qubit, and show that the maximum success probability of Hardy’s ladder argument for non-locality allowed by generalized probabilistic theory reaches 50% irrespective of the value of $K$. This extends the known result for $K = 1$ to an arbitrary number of observables.

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1 Introduction

As is well known, quantum-mechanical probabilities can give rise to the violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality [11] up to a maximum value of $2\sqrt{2}$ (the so-called Tsirelson bound [24]). However, Popescu and Rohrlich [22] (see also [23]) showed with an explicit example that there are more general, non-quantum-mechanical probabilities which yield the maximal algebraic violation (namely, 4) of the CHSH inequality without violating the non-signaling condition which forbids faster-than-light communication
between distant observers. Following Hillery and Yurke [18], throughout this paper this condition will be referred to as the causal communication constraint.

In 1992, Hardy [15] gave a new kind of non-locality proof for two particles without using inequalities. He subsequently also showed [16] that this proof works for all entangled states of two two-level systems or qubits except for the maximally entangled state. The maximum probability of obtaining a contradiction between quantum mechanics and local realism in Hardy’s setup (involving two qubits and two observables per qubit) is found to be about 9% [16]. Moreover, it was shown in [7] that the maximal probability of success of Hardy’s non-locality argument can be increased up to 50% within the framework of generalized probabilistic theory (GPT) respecting the causal communication constraint. This result was later rediscovered in [10] (see also [13] for related work).

In this paper, we focus on the generalization of Hardy’s non-locality proof to the case of two qubits and $K+1$ observables per qubit ($K = 1, 2, 3, \ldots$), i.e., the so-called Hardy’s ladder proof of non-locality [17, 2].

It turns out that the maximum probability of obtaining a contradiction between quantum mechanics and local realism in Hardy’s ladder setup tends to 50% for a sufficient number of observables (that is, $K \to \infty$). We will show that the maximum success probability of Hardy’s ladder argument for non-locality in the context of GPT can reach a maximum of 50% for any value of $K$. This finding extends the above mentioned result for $K = 1$ [7, 10] to an arbitrary number of observables.

\section{Hardy’s ladder proof of non-locality for two qubits}

Consider two qubits $A$ and $B$ in the entangled state

$$|\Psi\rangle = \alpha|+\rangle_A|+\rangle_B - \beta|--\rangle_A|--\rangle_B,$$

where $\{|+\rangle_A, |--\rangle_A\} \{|--\rangle_B, |--\rangle_B\}$ is an arbitrary orthonormal basis in the state space of qubit $A$ ($B$). Without loss of generality, it will be assumed that $\alpha$ and $\beta$ are taken to be real and positive, with $\alpha^2 + \beta^2 = 1$. The entangled qubits fly apart to two distant locations where, for each run of the experiment, one of $K + 1$ available dichotomic observables is measured—$A_k$ for qubit $A$ and $B_k$ for qubit $B$ ($k = 0, 1, \ldots, K$). The observables $A_k$ and $B_k$ have corresponding operators $A_k = |a_k^+\rangle\langle a_k^+| - |a_k^-\rangle\langle a_k^-|$ and $B_k = |b_k^+\rangle\langle b_k^+| - |b_k^-\rangle\langle b_k^-|$, where the eigenvectors $|a_k^\pm\rangle$ and $|b_k^\pm\rangle$ are related to the original basis vectors $|\pm\rangle_A$ and

\footnote{Hardy’s original proof [16] corresponds to the case $K = 1$.}
Causal communication constraint for two qubits

\[ |\pm\rangle_B \text{ by} \]

\[ |a^+_k\rangle = \cos \alpha_k |+\rangle_A + \sin \alpha_k |-\rangle_A \]
\[ |a^-_k\rangle = -\sin \alpha_k |+\rangle_A + \cos \alpha_k |-\rangle_A, \]

and

\[ |b^+_k\rangle = \cos \alpha_k |+\rangle_B + \sin \alpha_k |-\rangle_B \]
\[ |b^-_k\rangle = -\sin \alpha_k |+\rangle_B + \cos \alpha_k |-\rangle_B. \]

For there to be a contradiction between quantum mechanics and local realism, the observables \( A_k \) and \( B_k \) must satisfy the following conditions [17, 2]:

\[ P_K = P(A_K = +1, B_K = +1) \neq 0, \]  
\[ P(A_k = +1, B_{k-1} = -1) = 0 \quad \text{for } k = 1 \text{ to } K, \]  
\[ P(A_{k-1} = -1, B_k = +1) = 0 \quad \text{for } k = 1 \text{ to } K, \]  
\[ P(A_0 = +1, B_0 = +1) = 0, \]

where \( P(A_k = i, B_{k'} = j) \) is the joint probability that, for the state (1), the measurement of \( A_k \) on qubit \( A \) gives the outcome \( i \), and that the measurement of \( B_{k'} \) on qubit \( B \) gives the outcome \( j \) \((k, k' = 0, 1, \ldots, K \text{ and } i, j = \pm 1)\). It is easy to see that, according to a local-realistic description of Hardy’s ladder experiment, fulfilment of the \( 2K + 1 \) conditions in (3)-(5) necessarily implies that \( P_K = 0 \). Quantum-mechanically, however, we can have \( P_K \neq 0 \) while all the other conditions in (3)-(5) are satisfied. The magnitude of \( P_K \) can, therefore, be viewed as the degree of non-locality inherent in Equations (2)-(5). The quantum prediction for \( P_K \) (subject to the fulfilment of conditions (3)-(5)) depends on the free parameter \( \alpha_K \), the number of observables, and the coefficients \( \alpha \) and \( \beta \), through the relation [9]

\[ P_K = \alpha^2 \left[ 1 - (\alpha/\beta)^{2K} \right]^2 \frac{\cos^2 \alpha_K}{1 + (\alpha/\beta)^{4K+2} \cot^2 \alpha_K}. \]  

(6)

Note that, for \( \alpha = \beta \), we have \( P_K = 0 \), and no contradiction with local realism arises for the maximally entangled state in Hardy’s ladder setup. Moreover, it can be shown [9] that for a given quantum state (that is, for a given value of the ratio \( \alpha/\beta \)), the value of \( \tan^2 \alpha_K \) that maximizes (6) is

\[ \tan^2 \alpha_K = (\alpha/\beta)^{2K+1}. \]

Using this relation into Equation (6), we obtain the optimized probability

\[ P_K = \left( \frac{\alpha \beta^{2K+1} - \beta \alpha^{2K+1}}{\beta^{2K+1} + \alpha^{2K+1}} \right)^2, \]  

(7)
which was originally derived by Hardy [17, 2]. It was also shown [17, 2] that the maximum value of $P_K$ in Equation (7) is $(50 - \delta)\%$, which is realized for large $K$ ($K \to \infty$) and a state that is not quite maximally entangled ($\alpha/\beta \to 1$).

For a given $K$, the function (7) reaches the maximum value for an appropriate choice of the ratio $\alpha/\beta$. In Figure 1, we have plotted the maximum achievable probability $P_K$ that is obtained for $K = 1$ to 100.

3 Causal communication constraint for two qubits in Hardy’s ladder setup

For the general Hardy’s ladder proof of non-locality for two qubits and $K + 1$ observables for each qubit, there are a total of $4(K + 1)^2$ joint probabilities $P(A_k = i, B_{k'} = j)$. These probabilities are assumed to satisfy the following constraints:

1. Non-negativity:

\[ P(A_k = i, B_{k'} = j) \geq 0, \ \forall k, k', i, j \]  

2. Normalization:

\[ \sum_{i,j=\pm1} P(A_k = i, B_{k'} = j) = 1, \ \forall k, k' \]
3. Causal communication:

\[
\sum_{j=\pm 1} P(A_k = i, B_0 = j) = \sum_{j=\pm 1} P(A_k = i, B_1 = j) = \cdots = \sum_{j=\pm 1} P(A_k = i, B_K = j) \quad \forall k, i \quad (10)
\]

\[
\sum_{i=\pm 1} P(A_0 = i, B_{k'} = j) = \sum_{i=\pm 1} P(A_1 = i, B_{k'} = j) = \cdots = \sum_{i=\pm 1} P(A_K = i, B_{k'} = j) \quad \forall k', j \quad (11)
\]

The condition in (10) [(11)] stipulates that the marginal probability \( P(A_k = i) \) [\( P(B_{k'} = j) \)] of obtaining the outcome \( i \) [\( j \)] in a measurement of \( A_k \) \([B_{k'}] \) on qubit \( A \) \([B] \) is independent of which measurement \( B_0, B_1, \ldots, B_K \) \([A_0, A_1, \ldots, A_K] \) is performed on the distant qubit \( B \) \([A] \). The fulfilment of both (10) and (11) constitutes a physically sound requirement since the violation of either (10) or (11) would, in principle, allow two observers (one of them measuring qubit \( A \) and the other qubit \( B \)) to communicate superluminally. Both classical and quantum theories (in fact, all known physical theories) respect the causal communication constraint.

We have seen in Section 2 that, for Hardy’s ladder setup, we can make the quantum probabilities satisfy all the conditions in (2)-(5) with \( P_K \to 1/2 \) as \( K \to \infty \). One might ask, however, what is the prediction for \( P_K \) made by the less restrictive framework of GPT based solely on the assumptions (8)-(11). In what follows, it is shown that such a model gives a maximum of \( P_K = 1/2 \) for any value of \( K \). We first examine the cases \( K = 1, 2, 3 \), and then we establish the result generally. In the rest of this section, we employ the abbreviated notation \( P_{ij}^{k_{k'}} \) to refer to the joint probability \( P(A_k = i, B_{k'} = j) \).

### 3.1 Case \( K = 1 \)

For \( K = 1 \) (i.e., two observables per qubit: \( A_0, A_1 \) for qubit \( A \), and \( B_0, B_1 \) for qubit \( B \)), the causal communication constraint in Equations (10)-(11) reads as follows:

\[
\begin{align*}
P_{00}^{++} + P_{00}^{+-} &= P_{01}^{++} + P_{01}^{+-} \\
P_{10}^{++} + P_{10}^{+-} &= P_{11}^{++} + P_{11}^{+-} \\
P_{00}^{--} + P_{00}^{-+} &= P_{01}^{--} + P_{01}^{-+} \\
P_{10}^{--} + P_{10}^{-+} &= P_{11}^{--} + P_{11}^{-+} \\
P_{00}^{++} + P_{00}^{+-} &= P_{10}^{++} + P_{10}^{+-} \\
P_{01}^{++} + P_{01}^{+-} &= P_{11}^{++} + P_{11}^{+-} \\
P_{00}^{--} + P_{00}^{-+} &= P_{10}^{--} + P_{10}^{-+} \\
P_{01}^{--} + P_{01}^{-+} &= P_{11}^{--} + P_{11}^{-+} \\
\end{align*}
\] (12)
Furthermore, Hardy’s conditions in (3)-(5) for \( K = 1 \) mean that

\[
P_{00}^{++} = P_{01}^{+} = P_{10}^{+} = 0. \tag{13}
\]

It is straightforward to check that the following joint probability distribution

\[
\begin{align*}
P_{00}^{+} &= P_{00}^{+} = P_{01}^{+} = 1/2, \\
P_{00}^{0} &= P_{00}^{0} = P_{01}^{0} = P_{10}^{0} = P_{11}^{0} = 1/2
\end{align*}
\]

satisfies all the requirements of Equations (8)-(9) and (12)-(13). Note, in particular, that \( P_{1} = P_{11}^{++} = 1/2 \). Next we show that, in fact, \( 1/2 \) is the maximum value of \( P_{11}^{++} \) allowed by GPT in the case in which \( P_{00}^{++} = P_{01}^{+} = P_{10}^{+} = 0 \). To see this, suppose on the contrary that we have simultaneously \( P_{11}^{++} > 1/2 \) and \( P_{00}^{++} = P_{01}^{+} = P_{10}^{+} = 0 \). Then the following inferences hold

- Since \( P_{11}^{++} > 1/2 \) and \( P_{00}^{+} = 0 \), from the 6th relation in (12) we deduce that \( P_{01}^{+} > 1/2 \).
- Since \( P_{01}^{+} > 1/2 \) and \( P_{00}^{+} = 0 \), from the 1st relation in (12) we deduce that \( P_{00}^{+} > 1/2 \).
- Since \( P_{11}^{++} > 1/2 \) and \( P_{10}^{+} = 0 \), from the 2nd relation in (12) we deduce that \( P_{10}^{+} > 1/2 \).
- Since \( P_{10}^{+} > 1/2 \) and \( P_{11}^{++} = 0 \), from the 5th relation in (12) we deduce that \( P_{00}^{+} > 1/2 \).

This can alternately be expressed as the following two sequences of implications

\[
\begin{align*}
P_{11}^{++} > \frac{1}{2} &\Rightarrow P_{01}^{+} > \frac{1}{2} \Rightarrow P_{00}^{+} > \frac{1}{2} \\
P_{11}^{++} > \frac{1}{2} &\Rightarrow P_{10}^{+} > \frac{1}{2} \Rightarrow P_{00}^{+} > \frac{1}{2}
\end{align*}
\]

By adding the two inequalities in the last column we find that \( P_{00}^{+} + P_{01}^{+} > 1 \), which contradicts the normalization condition \( P_{00}^{+} + P_{01}^{+} + P_{10}^{+} + P_{11}^{+} = 1 \). Thus we must have that \( P_{11}^{++} \leq 1/2 \) when \( P_{00}^{++} = P_{01}^{+} = P_{10}^{+} = 0 \). Combining this upper bound with the non-negativity condition, we conclude that \( P_{11}^{++} \) can vary in the range \( 0 \leq P_{11}^{++} \leq 1/2 \) without violating the causal communication constraint.

It is worth noting that the above upper bound for \( P_{11}^{++} \) can also be deduced directly from the relation [7, 8]

\[
P_{11}^{++} = \frac{1}{2} (1 + P_{00}^{++} - P_{00}^{+} - P_{01}^{+} + P_{01}^{+} + P_{10}^{+} - P_{10}^{+} - P_{11}^{+} - P_{11}^{+}).
\]
Since \( P_{kk}^{ij} \geq 0 \), from this relation it quickly follows that \( P_{11}^{++} \leq \frac{1}{2}(1 + P_{00}^{++} + P_{01}^{--} + P_{10}^{+--}) \). Therefore, when the probabilities \( P_{00}^{++} \), \( P_{01}^{--} \), and \( P_{10}^{+--} \) vanish, we conclude that \( P_{11}^{++} \leq 1/2 \).

The probability distribution (14) can be written compactly as

\[
P_{kk}^{ij} = \begin{cases} 
\frac{1}{2}, & \text{if } \delta_{ij} = kk' \oplus k \oplus k'; \\
0, & \text{otherwise},
\end{cases}
\]

where \( k, k' \in \{0, 1\}, \delta_{++} = \delta_{--} = 1, \delta_{+-} = \delta_{-+} = 0 \), and where \( \oplus \) indicates addition modulo 2. The set of probabilities (15) corresponds to the Popescu-Rohrlich-type correlations leading to the maximal algebraic violation of the CHSH inequality [11]

\[
|E(A_0, B_1) + E(A_1, B_1) + E(A_1, B_0) - E(A_0, B_0)| \leq 2,
\]

the correlation function \( E(A_k, B_{k'}) \) being defined by \( P_{kk'}^{++} + P_{kk'}^{--} = P_{kk'}^{+-} - P_{kk'}^{-+} \).

### 3.2 Case \( K = 2 \)

For \( K = 2 \) (i.e., three observables per qubit: \( A_0, A_1, A_2 \) for qubit \( A \), and \( B_0, B_1, B_2 \) for qubit \( B \)), the causal communication constraint in Equations (10)-(11) implies that

\[
\begin{align*}
P_{00}^{++} + P_{00}^{+-} &= P_{00}^{++} + P_{00}^{+-} = P_{00}^{++} + P_{00}^{++} \\
P_{01}^{++} + P_{01}^{+-} &= P_{01}^{++} + P_{01}^{+-} = P_{01}^{++} + P_{10}^{++} \\
P_{10}^{++} + P_{10}^{+-} &= P_{10}^{++} + P_{10}^{+-} = P_{10}^{++} + P_{10}^{++} \\
P_{20}^{++} + P_{20}^{+-} &= P_{20}^{++} + P_{20}^{+-} = P_{20}^{++} + P_{20}^{++} \\
P_{00}^{+--} + P_{00}^{-+-} &= P_{00}^{+--} + P_{00}^{-+-} = P_{00}^{+--} + P_{00}^{+--} \\
P_{01}^{+--} + P_{01}^{-+-} &= P_{01}^{+--} + P_{01}^{-+-} = P_{01}^{+--} + P_{01}^{+--} \\
P_{10}^{+--} + P_{10}^{-+-} &= P_{10}^{+--} + P_{10}^{-+-} = P_{10}^{+--} + P_{10}^{+--} \\
P_{20}^{+--} + P_{20}^{-+-} &= P_{20}^{+--} + P_{20}^{-+-} = P_{20}^{+--} + P_{20}^{+--}
\end{align*}
\]

(16)

On the other hand, Hardy’s conditions in (3)-(5) for \( K = 2 \) are

\[
P_{00}^{++} = P_{01}^{--} = P_{10}^{+--} = P_{21}^{+--} = 0.
\]

(17)
It is readily verified that the following joint probability distribution

\[
P_{kk'}^{ij} = \begin{cases} 
\frac{1}{2}, & \text{for } i = j \text{ and } \forall k, k' \in \{0, 1, 2\} \text{ except for } k = k' = 0; \\
0, & \text{for } i = j \text{ and } k = k' = 0; \\
0, & \text{for } i \neq j \text{ and } \forall k, k' \in \{0, 1, 2\} \text{ except for } k = k' = 0; \\
\frac{1}{2}, & \text{for } i \neq j \text{ and } k = k' = 0,
\end{cases}
\]  

(18)

satisfies all the conditions in Equations (8)-(9) and (16)-(17), with \( P_2 = P_{22}^{++} = 1/2 \). It can be shown that, in fact, 1/2 is the maximum value for \( P_{22}^{++} \) allowed by GPT for the case where \( P_{00}^{++} = P_{01}^{++} = P_{10}^{++} = P_{12}^{++} = P_{21}^{++} = 0 \). To see this, suppose on the contrary that we have simultaneously \( P_{22}^{++} > 1/2 \) and \( P_{00}^{++} = P_{01}^{++} = P_{10}^{++} = P_{12}^{++} = P_{21}^{++} = 0 \). Then the following inferences follow

- Since \( P_{22}^{++} > 1/2 \) and \( P_{21}^{++} = 0 \), from the 3rd relation in (16) we deduce that \( P_{21}^{++} > 1/2 \).
- Since \( P_{21}^{++} > 1/2 \) and \( P_{01}^{++} = 0 \), from the 8th relation in (16) we deduce that \( P_{01}^{++} > 1/2 \).
- Since \( P_{01}^{++} > 1/2 \) and \( P_{00}^{++} = 0 \), from the 1st relation in (16) we deduce that \( P_{00}^{++} > 1/2 \).
- Since \( P_{22}^{++} > 1/2 \) and \( P_{12}^{++} = 0 \), from the 9th relation in (16) we deduce that \( P_{12}^{++} > 1/2 \).
- Since \( P_{12}^{++} > 1/2 \) and \( P_{10}^{++} = 0 \), from the 2nd relation in (16) we deduce that \( P_{10}^{++} > 1/2 \).
- Since \( P_{10}^{++} > 1/2 \) and \( P_{00}^{++} = 0 \), from the 7th relation in (16) we deduce that \( P_{00}^{++} > 1/2 \).

Expressing this as two sequences of implications

\[
\left\{ \begin{align*}
P_{22}^{++} &> \frac{1}{2} \Rightarrow P_{21}^{++} > \frac{1}{2} \Rightarrow P_{01}^{++} > \frac{1}{2} \Rightarrow P_{00}^{++} > \frac{1}{2} \\
P_{22}^{++} &> \frac{1}{2} \Rightarrow P_{12}^{++} > \frac{1}{2} \Rightarrow P_{10}^{++} > \frac{1}{2} \Rightarrow P_{00}^{++} > \frac{1}{2}
\end{align*} \right\}
\]

we can see from the last two inequalities that \( P_{00}^{++} + P_{00}^{-+} > 1 \), contradicting the normalization condition \( P_{00}^{++} + P_{00}^{-+} + P_{00}^{+-} + P_{00}^{-} = 1 \). We therefore conclude that, when Hardy’s conditions (17) are satisfied, the probability \( P_{22}^{++} \) can vary in the range \( 0 \leq P_{22}^{++} \leq 1/2 \) while respecting the causal communication constraint.

We further note that the probability distribution (18) gives the maximal algebraic violation (namely, 6) of the chained CHSH-type inequality [4, 25]

\[ |E(A_0, B_1) + E(A_1, B_2) + E(A_2, B_2) + E(A_2, B_1) + E(A_1, B_0) - E(A_0, B_0)| \leq 4. \]
3.3 Case $K = 3$

For $K = 3$ (i.e., four observables per qubit: $A_0, A_1, A_2, A_3$ for qubit $A$, and $B_0, B_1, B_2, B_3$ for qubit $B$), the causal communication constraint in Equations (10)-(11) implies the following set of conditions:

$$
\begin{align*}
P^{++}_{10} + P_{10}^{+-} &= P_{01}^{++} + P_{01}^{+-} = P_{02}^{++} + P_{02}^{+-} = P_{03}^{++} + P_{03}^{+-} \\
&= P_{11}^{++} + P_{11}^{+-} = P_{12}^{++} + P_{12}^{+-} = P_{13}^{++} + P_{13}^{+-} \\
&= P_{21}^{++} + P_{21}^{+-} = P_{22}^{++} + P_{22}^{+-} = P_{23}^{++} + P_{23}^{+-} \\
&= P_{31}^{++} + P_{31}^{+-} = P_{32}^{++} + P_{32}^{+-} = P_{33}^{++} + P_{33}^{+-} \\
&= P_{00}^{++} + P_{00}^{+-} = P_{01}^{++} + P_{01}^{+-} = P_{02}^{++} + P_{02}^{+-} = P_{03}^{++} + P_{03}^{+-} \\
&= P_{10}^{++} + P_{10}^{+-} = P_{11}^{++} + P_{11}^{+-} = P_{12}^{++} + P_{12}^{+-} = P_{13}^{++} + P_{13}^{+-} \\
&= P_{20}^{++} + P_{20}^{+-} = P_{21}^{++} + P_{21}^{+-} = P_{22}^{++} + P_{22}^{+-} = P_{23}^{++} + P_{23}^{+-} \\
&= P_{30}^{++} + P_{30}^{+-} = P_{31}^{++} + P_{31}^{+-} = P_{32}^{++} + P_{32}^{+-} = P_{33}^{++} + P_{33}^{+-} \\
&= P_{00}^{++} + P_{00}^{+-} = P_{10}^{++} + P_{10}^{+-} = P_{20}^{++} + P_{20}^{+-} = P_{30}^{++} + P_{30}^{+-} \\
&= P_{01}^{++} + P_{01}^{+-} = P_{11}^{++} + P_{11}^{+-} = P_{21}^{++} + P_{21}^{+-} = P_{31}^{++} + P_{31}^{+-} \\
&= P_{02}^{++} + P_{02}^{+-} = P_{12}^{++} + P_{12}^{+-} = P_{22}^{++} + P_{22}^{+-} = P_{32}^{++} + P_{32}^{+-} \\
&= P_{03}^{++} + P_{03}^{+-} = P_{13}^{++} + P_{13}^{+-} = P_{23}^{++} + P_{23}^{+-} = P_{33}^{++} + P_{33}^{+-} \\
\end{align*}
$$

while Hardy’s conditions in (3)-(5) for $K = 3$ are

$$
P_{00}^{++} = P_{01}^{+-} = P_{10}^{--} = P_{12}^{--} = P_{21}^{--} = P_{23}^{++} = P_{32}^{++} = 0. \tag{20}
$$

It is readily checked that the following joint probability distribution

$$
P_{k,k'}^{ij} = \begin{cases} 
\frac{1}{2}, & \text{for } i = j \text{ and } \forall k, k' \in \{0, 1, 2, 3\} \text{ except for } k = k' = 0; \\
0, & \text{for } i = j \text{ and } k = k' = 0; \\
0, & \text{for } i \neq j \text{ and } \forall k, k' \in \{0, 1, 2, 3\} \text{ except for } k = k' = 0; \\
\frac{1}{2}, & \text{for } i \neq j \text{ and } k = k' = 0,
\end{cases} \tag{21}
$$

satisfies all the conditions in Equations (8)-(9) and (19)-(20), with $P_3 = P_{33}^{++} = 1/2$. Likewise, it turns out that $1/2$ is the maximum of $P_{33}^{++}$ allowed by GPT when Hardy’s conditions (20) are fulfilled. To see this, suppose on the contrary that we have simultaneously $P_{33}^{++} > 1/2$ and $P_{00}^{++} = P_{01}^{-+} = P_{10}^{--} = P_{21}^{--} = P_{23}^{++} = P_{32}^{++} = P_{33}^{++} = 0$. Then the following inferences follow

- Since $P_{33}^{++} > 1/2$ and $P_{33}^{++} = 0$, from the 12th relation in (19) we deduce that $P_{33}^{++} > 1/2$. 

• Since $P_{23}^{++} > 1/2$ and $P_{21}^{+-} = 0$, from the 3rd relation in (19) we deduce that $P_{21}^{++} > 1/2$.

• Since $P_{21}^{++} > 1/2$ and $P_{01}^{++} = 0$, from the 10th relation in (19) we deduce that $P_{01}^{++} > 1/2$.

• Since $P_{01}^{++} > 1/2$ and $P_{00}^{++} = 0$, from the 1st relation in (19) we deduce that $P_{00}^{++} > 1/2$.

• Since $P_{33}^{++} > 1/2$ and $P_{32}^{+-} = 0$, from the 4th relation in (19) we deduce that $P_{32}^{++} > 1/2$.

• Since $P_{32}^{++} > 1/2$ and $P_{12}^{+-} = 0$, from the 11th relation in (19) we deduce that $P_{12}^{++} > 1/2$.

• Since $P_{12}^{++} > 1/2$ and $P_{10}^{+-} = 0$, from the 2nd relation in (19) we deduce that $P_{10}^{++} > 1/2$.

• Since $P_{10}^{++} > 1/2$ and $P_{00}^{++} = 0$, from the 9th relation in (19) we deduce that $P_{00}^{++} > 1/2$.

Expressing this as two sequences of implications

\[
\begin{align*}
P_{33}^{++} > \frac{1}{2} & \Rightarrow P_{23}^{++} > \frac{1}{2} \Rightarrow P_{21}^{++} > \frac{1}{2} \Rightarrow P_{01}^{++} > \frac{1}{2} \Rightarrow P_{00}^{++} > \frac{1}{2} \\
P_{33}^{++} > \frac{1}{2} & \Rightarrow P_{32}^{++} > \frac{1}{2} \Rightarrow P_{12}^{++} > \frac{1}{2} \Rightarrow P_{10}^{++} > \frac{1}{2} \Rightarrow P_{00}^{++} > \frac{1}{2}
\end{align*}
\]

we obtain $P_{00}^{++} + P_{00}^{+-} > 1$ from the last two inequalities, and this contradicts the normalization condition $P_{00}^{++} + P_{00}^{+-} + P_{00}^{--} + P_{00}^{--} = 1$. Thus, we conclude that, when Hardy conditions (20) are met, the probability $P_{33}^{++}$ can vary in the range $0 \leq P_{33}^{++} \leq 1/2$ without violating the causal communication constraint.

On the other hand, the probability distribution (21) gives the maximal algebraic violation (namely, 8) of the chained CHSH-type inequality [4, 25]

\[
|E(A_0, B_1) + E(A_1, B_2) + E(A_2, B_3) + E(A_3, B_3) \\
+ E(A_3, B_2) + E(A_2, B_1) + E(A_1, B_0) - E(A_0, B_0)| \leq 6.
\]

### 3.4 The general case

The above results for $K = 1, 2, 3$ generalize in a straightforward way to an arbitrary number $K + 1$ of observables per qubit (i.e., $A_0, A_1, \ldots A_K$ for qubit
A, and \(B_0, B_1, \ldots, B_K\) for qubit \(B\). To show this, we write the causal communication constraint in Equations (10)-(11) in the form

\[
\begin{align*}
&\begin{aligned}
P_{00}^{++} + P_{00}^{+-} + P_{01}^{++} + P_{01}^{+\cdot} = \ldots = P_{0K}^{++} + P_{0K}^{+\cdot} \\
P_{10}^{++} + P_{10}^{+-} + P_{11}^{++} + P_{11}^{+\cdot} = \ldots = P_{1K}^{++} + P_{1K}^{+\cdot} \\
\quad \vdots
\end{aligned} \\
&\begin{aligned}
P_{K0}^{++} + P_{K0}^{+-} + P_{K1}^{++} + P_{K1}^{+\cdot} = \ldots = P_{KK}^{++} + P_{KK}^{+\cdot}
\end{aligned}
\end{align*}
\]

(22)

with a total of \(4(K+1)\) relations, and \(K\) equals signs in each relation. Furthermore, Hardy’s conditions in (3)-(5) read as

\[
P_{00}^{++} = P_{01}^{+-} = P_{10}^{++} = P_{11}^{+\cdot} = \ldots = P_{K-1,K}^{++} = P_{K,K-1}^{+\cdot} = 0.
\]

(23)

The following joint probability distribution (which is a direct generalization of the previous particular distributions (14), (18), and (21))

\[
P_{kk'}^{ij} = \begin{cases} 
\frac{1}{2}, & \text{for } i = j \text{ and } \forall k, k' \in \{0, 1, \ldots, K\} \text{ except for } k = k' = 0; \\
0, & \text{for } i = j \text{ and } k = k' = 0; \\
0, & \text{for } i \neq j \text{ and } k, k' \in \{0, 1, \ldots, K\} \text{ except for } k = k' = 0; \\
\frac{1}{2}, & \text{for } i \neq j \text{ and } k = k' = 0,
\end{cases}
\]

(24)

then satisfies all the conditions in Equations (8)-(9) and (22)-(23), with \(P_K = P_{KK}^{++} = 1/2\). Similarly, it can be shown that \(1/2\) is the maximum value of \(P_{KK}^{++}\) allowed by GPT when Hardy’s conditions (23) are fulfilled. Indeed, assuming that \(P_{KK}^{++} > 1/2\) and \(P_{00}^{++} = P_{01}^{+-} = P_{10}^{++} = \ldots = P_{K-1,K}^{++} = P_{K,K-1}^{+\cdot} = 0\), from the relations in the first and third blocks of Equation (22) one can derive the following two sequences of implications:
For $K = 1, 3, 5, \ldots$

$$P_{KK}^{++} > \frac{1}{2} \Rightarrow P_{K-1,K}^{++} > \frac{1}{2} \Rightarrow P_{K-2,K-2}^{++} > \frac{1}{2} \Rightarrow P_{K-3,K-3}^{++} > \frac{1}{2} \Rightarrow \ldots$$

$$P_{K,K}^{++} > \frac{1}{2} \Rightarrow P_{K,K-1}^{++} > \frac{1}{2} \Rightarrow P_{K-1,K}^{++} > \frac{1}{2} \Rightarrow P_{K-2,K-2}^{++} > \frac{1}{2} \Rightarrow \ldots$$

where each constituent sequence involves exactly $K + 1$ implication signs. For either odd or even $K$, we end up with $P_{00}^{++} + P_{00}^{--} > 1$ from the last two inequalities, contradicting the normalization condition $P_{00}^{++} + P_{00}^{--} + P_{00}^{+-} + P_{00}^{-+} = 1$. Hence, it follows that, when Hardy’s conditions (23) are fulfilled, the probability $P_{KK}^{++}$ can vary in the range $0 \leq P_{KK}^{++} \leq 1/2$ without violating the causal communication constraint.

We note that the probability distribution (24) gives the maximal algebraic violation (namely, $2K + 2$) of the chained CHSH inequality [4, 25]

$$\left| \sum_{k=1}^{K} E(A_{k-1}, B_k) + \sum_{k=1}^{K} E(A_{k}, B_{k-1}) + E(A_{K}, B_K) - E(A_0, B_0) \right| \leq 2K, \quad (25)$$

where the $2K + 2$ pairs of observables $(A_0, B_0), (A_K, B_K), (A_{k-1}, B_k), (A_k, B_{k-1}), k = 1, 2, \ldots, K$, appearing in the left-hand side of inequality (25) are precisely those in Equations (2)-(5). Moreover, we point out that the maximum value of the left-hand side of (25) predicted by quantum mechanics is given by [25] $2(K + 1) \cos \frac{\pi}{2(K+1)}$. For sufficiently large $K$, we have that $\cos \frac{\pi}{2(K+1)} \approx 1$, and then the left-hand side of (25) approaches the algebraic limit $2K + 2$. This corresponds to the case in which the quantum-mechanical probabilities satisfy all Hardy’s non-locality conditions (2)-(5) with $P_K \rightarrow 1/2$ as $K \rightarrow \infty$. It is easily seen that, in this limit, a direct (“all or nothing”) contradiction between quantum mechanics and local realism arises for Hardy’s ladder setup [9].
Finally, we mention that a complete characterization of the extremal non-signaling bipartite probability distributions $P_{ab}^{xy}$ for $a, b \in \{0, 1\}$, $x \in \{0, 1, \ldots, d_x - 1\}$ and $y \in \{0, 1, \ldots, d_y - 1\}$, has been given in [19]. The probability distribution (24) constitutes a representative element of a fully non-local, non-deterministic extremal distribution corresponding to the case in which $d_x = d_y = K + 1$.

4 Conclusion

In conclusion, we have shown that the success probability of Hardy’s ladder argument for non-locality for two qubits and $K + 1$ observables per qubit in the framework of GPT reaches a maximum of 50% for any value of $K$, thereby generalizing the known result for $K = 1$ [7, 10] to an arbitrary number of observables. Incidentally, we observe that, as shown in [10], the maximum success probability of Hardy’s non-locality argument for three qubits and two observables per qubit also reaches a value of 50% in the context of GPT.

In view of our results, we conclude that the causal communication constraint by itself cannot account (unless in the limit of a truly infinite number of observables; see Figure 1) for the upper bound of $P_K$ predicted by quantum mechanics for Hardy’s ladder setup. It is therefore worthwhile to investigate whether such quantum bound for $P_K$ could naturally emerge as a necessary consequence of some additional constraints, other than causal communication, such as those ensuing from the physical principles of non-trivial communication complexity [3], information causality [21], macroscopic locality [20], and local orthogonality [14] (also known as the exclusivity principle [5, 27, 6]). In this respect, Ahanj et al. [1] (see also [26]) showed that, for Hardy’s setup for two qubits and two observables per qubit, the 50% bound prescribed by the causal communication constraint is lowered to $\frac{\sqrt{2} - 1}{2} \approx 0.207$ under the condition of information causality. Subsequently, Das et al. [12] applied the local orthogonality principle to two copies of Hardy’s set of correlations and found that the maximum success probability of Hardy’s argument is reduced to $\approx 0.177$ which is relatively closer to the corresponding quantum value $\approx 0.09$. It will be interesting to look for the bounds that are imposed by the above mentioned physical principles on Hardy’s ladder setup.

References


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