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Causal Communication Constraint for Two Qubits in Hardy's Ladder Proof of Non-locality

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Abstract

In this paper, we consider Hardy's ladder proof of non-locality for two qubits and $K+1$ observables per qubit, and show that the maximum success probability of Hardy's ladder argument for non-locality allowed by generalized probabilistic theory reaches 50% irrespective of the value of K. This extends the known result for $K = 1$ to an arbitrary number of observables.

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1 Introduction

As is well known, quantum-mechanical probabilities can give rise to the violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality [11] up to a maximum value of $2\sqrt{2}$ (the so-called Tsirelson bound [24]). However, Popescu and Rohrlich [22] (see also [23]) showed with an explicit example that there are more general, non-quantum-mechanical probabilities which yield the maximal algebraic violation (namely, 4) of the CHSH inequality without violating the non-signaling condition which forbids faster-than-light communication

between distant observers. Following Hillery and Yurke [18], throughout this paper this condition will be referred to as the causal communication constraint.

In 1992, Hardy [15] gave a new kind of non-locality proof for two particles without using inequalities. He subsequently also showed [16] that this proof works for all entangled states of two two-level systems or qubits except for the maximally entangled state. The maximum probability of obtaining a contradiction between quantum mechanics and local realism in Hardy's setup (involving two qubits and two observables per qubit) is found to be about 9% [16]. Moreover, it was shown in [7] that the maximal probability of success of Hardy's non-locality argument can be increased up to 50% within the framework of generalized probabilistic theory (GPT) respecting the causal communication constraint. This result was later rediscovered in [10] (see also [13] for related work).

In this paper, we focus on the generalization of Hardy's non-locality proof to the case of two qubits and $K+1$ observables per qubit ($K = 1, 2, 3, \dots$), i.e., the so-called Hardy's ladder proof of non-locality [17, 2].[1] It turns out that the maximum probability of obtaining a contradiction between quantum mechanics and local realism in Hardy's ladder setup tends to 50% for a sufficient number of observables (that is, $K \to \infty$). We will show that the maximum success probability of Hardy's ladder argument for non-locality in the context of GPT can reach a maximum of 50% for *any* value of K. This finding extends the above mentioned result for $K = 1$ [7, 10] to an arbitrary number of observables.

2 Hardy's ladder proof of non-locality for two qubits

Consider two qubits A and B in the entangled state

$$|\Psi\rangle = \alpha|+\rangle_A|+\rangle_B - \beta|-\rangle_A|-\rangle_B, \tag{1}$$

where $\{|+\rangle_A, |-\rangle_A\}$ ($\{|+\rangle_B, |-\rangle_B\}$) is an arbitrary orthonormal basis in the state space of qubit A (B). Without loss of generality, it will be assumed that α and β are taken to be real and positive, with $\alpha^2+\beta^2 = 1$. The entangled qubits fly apart to two distant locations where, for each run of the experiment, one of $K+1$ available dichotomic observables is measured—A_k for qubit A and B_k for qubit B ($k = 0, 1, \dots, K$). The observables A_k and B_k have corresponding operators $\hat{A}_k = |a_k^+\rangle\langle a_k^+| - |a_k^-\rangle\langle a_k^-|$ and $\hat{B}_k = |b_k^+\rangle\langle b_k^+| - |b_k^-\rangle\langle b_k^-|$, where the eigenvectors $|a_k^\pm\rangle$ and $|b_k^\pm\rangle$ are related to the original basis vectors $|\pm\rangle_A$ and

[1] Hardy's original proof [16] corresponds to the case $K = 1$.

$|\pm\rangle_B$ by

$$|a_k^+\rangle = \cos\alpha_k|+\rangle_A + \sin\alpha_k|-\rangle_A$$
$$|a_k^-\rangle = -\sin\alpha_k|+\rangle_A + \cos\alpha_k|-\rangle_A,$$

and

$$|b_k^+\rangle = \cos\alpha_k|+\rangle_B + \sin\alpha_k|-\rangle_B$$
$$|b_k^-\rangle = -\sin\alpha_k|+\rangle_B + \cos\alpha_k|-\rangle_B.$$

For there to be a contradiction between quantum mechanics and local realism, the observables A_k and B_k must satisfy the following conditions [17, 2]:

$$P_K = P(A_K = +1, B_K = +1) \neq 0, \tag{2}$$
$$P(A_k = +1, B_{k-1} = -1) = 0 \quad \text{for } k = 1 \text{ to } K, \tag{3}$$
$$P(A_{k-1} = -1, B_k = +1) = 0 \quad \text{for } k = 1 \text{ to } K, \tag{4}$$
$$P(A_0 = +1, B_0 = +1) = 0, \tag{5}$$

where $P(A_k = i, B_{k'} = j)$ is the joint probability that, for the state (1), the measurement of A_k on qubit A gives the outcome i, and that the measurement of $B_{k'}$ on qubit B gives the outcome j ($k, k' = 0, 1, \ldots, K$ and $i, j = \pm 1$). It is easy to see that, according to a local-realistic description of Hardy's ladder experiment, fulfilment of the $2K + 1$ conditions in (3)-(5) necessarily implies that $P_K = 0$. Quantum-mechanically, however, we can have $P_K \neq 0$ while all the other conditions in (3)-(5) are satisfied. The magnitude of P_K can, therefore, be viewed as the degree of non-locality inherent in Equations (2)-(5). The quantum prediction for P_K (subject to the fulfilment of conditions (3)-(5)) depends on the free parameter α_K, the number of observables, and the coefficients α and β, through the relation [9]

$$P_K = \alpha^2 \left[1 - (\alpha/\beta)^{2K}\right]^2 \frac{\cos^2\alpha_K}{1 + (\alpha/\beta)^{4K+2}\cot^2\alpha_K}. \tag{6}$$

Note that, for $\alpha = \beta$, we have $P_K = 0$, and no contradiction with local realism arises for the maximally entangled state in Hardy's ladder setup. Moreover, it can be shown [9] that for a given quantum state (that is, for a given value of the ratio α/β), the value of $\tan^2\alpha_K$ that maximizes (6) is

$$\tan^2\alpha_K = (\alpha/\beta)^{2K+1}.$$

Using this relation into Equation (6), we obtain the optimized probability

$$P_K = \left(\frac{\alpha\beta^{2K+1} - \beta\alpha^{2K+1}}{\beta^{2K+1} + \alpha^{2K+1}}\right)^2, \tag{7}$$

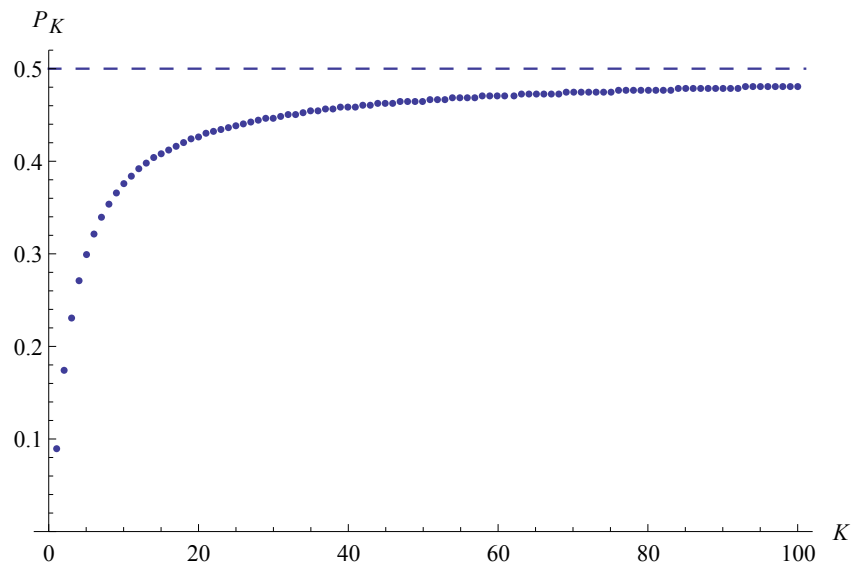

Figure 1: Maximal probability P_K of obtaining a locality contradiction in Hardy's ladder setup for $K = 1$ to 100.

which was originally derived by Hardy [17, 2]. It was also shown [17, 2] that the maximum value of P_K in Equation (7) is $(50 - \delta)\%$, which is realized for large K ($K \to \infty$) and a state that is not quite maximally entangled ($\alpha/\beta \to 1$). For a given K, the function (7) reaches the maximum value for an appropriate choice of the ratio α/β. In Figure 1, we have plotted the maximum achievable probability P_K that is obtained for $K = 1$ to 100.

3 Causal communication constraint for two qubits in Hardy's ladder setup

For the general Hardy's ladder proof of non-locality for two qubits and $K + 1$ observables for each qubit, there are a total of $4(K + 1)^2$ joint probabilities $P(A_k = i, B_{k'} = j)$. These probabilities are assumed to satisfy the following constraints:

1. Non-negativity:

$$P(A_k = i, B_{k'} = j) \geq 0, \ \forall k, k', i, j \tag{8}$$

2. Normalization:

$$\sum_{i,j=\pm 1} P(A_k = i, B_{k'} = j) = 1, \ \forall k, k' \tag{9}$$

3. Causal communication:

$$\sum_{j=\pm 1} P(A_k = i, B_0 = j) = \sum_{j=\pm 1} P(A_k = i, B_1 = j)$$
$$= \cdots = \sum_{j=\pm 1} P(A_k = i, B_K = j) \ \ \forall k, i \qquad (10)$$
$$\sum_{i=\pm 1} P(A_0 = i, B_{k'} = j) = \sum_{i=\pm 1} P(A_1 = i, B_{k'} = j)$$
$$= \cdots = \sum_{i=\pm 1} P(A_K = i, B_{k'} = j) \ \ \forall k', j \qquad (11)$$

The condition in (10) [(11)] stipulates that the marginal probability $P(A_k = i)$ $[P(B_{k'} = j)]$ of obtaining the outcome i $[j]$ in a measurement of A_k $[B_{k'}]$ on qubit A $[B]$ is independent of which measurement $B_0, B_1, \ldots, B_K$ $[A_0, A_1, \ldots, A_K]$ is performed on the distant qubit B $[A]$. The fulfilment of both (10) and (11) constitutes a physically sound requirement since the violation of either (10) or (11) would, in principle, allow two observers (one of them measuring qubit A and the other qubit B) to communicate superluminally. Both classical and quantum theories (in fact, all known physical theories) respect the causal communication constraint.

We have seen in Section 2 that, for Hardy's ladder setup, we can make the quantum probabilities satisfy all the conditions in (2)-(5) with $P_K \to 1/2$ as $K \to \infty$. One might ask, however, what is the prediction for P_K made by the less restrictive framework of GPT based solely on the assumptions (8)-(11). In what follows, it is shown that such a model gives a maximum of $P_K = 1/2$ for any value of K. We first examine the cases $K = 1, 2, 3$, and then we establish the result generally. In the rest of this section, we employ the abbreviated notation $P^{ij}_{kk'}$ to refer to the joint probability $P(A_k = i, B_{k'} = j)$.

3.1 Case $K = 1$

For $K = 1$ (i.e., two observables per qubit: A_0, A_1 for qubit A, and B_0, B_1 for qubit B), the causal communication constraint in Equations (10)-(11) reads as follows:

$$\begin{aligned}
P^{++}_{00} + P^{+-}_{00} &= P^{++}_{01} + P^{+-}_{01} \\
P^{++}_{10} + P^{+-}_{10} &= P^{++}_{11} + P^{+-}_{11} \\
P^{-+}_{00} + P^{--}_{00} &= P^{-+}_{01} + P^{--}_{01} \\
P^{-+}_{10} + P^{--}_{10} &= P^{-+}_{11} + P^{--}_{11} \\
P^{++}_{00} + P^{-+}_{00} &= P^{++}_{10} + P^{-+}_{10} \\
P^{++}_{01} + P^{-+}_{01} &= P^{++}_{11} + P^{-+}_{11} \\
P^{+-}_{00} + P^{--}_{00} &= P^{+-}_{10} + P^{--}_{10} \\
P^{+-}_{01} + P^{--}_{01} &= P^{+-}_{11} + P^{--}_{11}
\end{aligned} \qquad (12)$$

Furthermore, Hardy's conditions in (3)-(5) for $K = 1$ mean that

$$P_{00}^{++} = P_{01}^{-+} = P_{10}^{+-} = 0. \tag{13}$$

It is straightforward to check that the following joint probability distribution

$$\begin{array}{l} P_{00}^{+-} = P_{00}^{-+} = P_{01}^{++} = P_{01}^{--} = P_{10}^{++} = P_{10}^{--} = P_{11}^{++} = P_{11}^{--} = 1/2 \\ P_{00}^{++} = P_{00}^{--} = P_{01}^{+-} = P_{01}^{-+} = P_{10}^{+-} = P_{10}^{-+} = P_{11}^{+-} = P_{11}^{-+} = 0 \end{array} \tag{14}$$

satisfies all the requirements of Equations (8)-(9) and (12)-(13). Note, in particular, that $P_1 = P_{11}^{++} = 1/2$. Next we show that, in fact, 1/2 is the maximum value of P_{11}^{++} allowed by GPT in the case in which $P_{00}^{++} = P_{01}^{-+} = P_{10}^{+-} = 0$. To see this, suppose on the contrary that we have simultaneously $P_{11}^{++} > 1/2$ and $P_{00}^{++} = P_{01}^{-+} = P_{10}^{+-} = 0$. Then the following inferences hold

- Since $P_{11}^{++} > 1/2$ and $P_{01}^{-+} = 0$, from the $6th$ relation in (12) we deduce that $P_{01}^{++} > 1/2$.
- Since $P_{01}^{++} > 1/2$ and $P_{00}^{++} = 0$, from the $1st$ relation in (12) we deduce that $P_{00}^{+-} > 1/2$.
- Since $P_{11}^{++} > 1/2$ and $P_{10}^{+-} = 0$, from the $2nd$ relation in (12) we deduce that $P_{10}^{++} > 1/2$.
- Since $P_{10}^{++} > 1/2$ and $P_{00}^{++} = 0$, from the $5th$ relation in (12) we deduce that $P_{00}^{-+} > 1/2$.

This can alternately be expressed as the following two sequences of implications

$$\left.\begin{array}{l} P_{11}^{++} > \dfrac{1}{2} \Rightarrow P_{01}^{++} > \dfrac{1}{2} \Rightarrow P_{00}^{+-} > \dfrac{1}{2} \\ P_{11}^{++} > \dfrac{1}{2} \Rightarrow P_{10}^{++} > \dfrac{1}{2} \Rightarrow P_{00}^{-+} > \dfrac{1}{2} \end{array}\right\}$$

By adding the two inequalities in the last column we find that $P_{00}^{+-} + P_{00}^{-+} > 1$, which contradicts the normalization condition $P_{00}^{++} + P_{00}^{+-} + P_{00}^{-+} + P_{00}^{--} = 1$. Thus we must have that $P_{11}^{++} \leq 1/2$ when $P_{00}^{++} = P_{01}^{-+} = P_{10}^{+-} = 0$. Combining this upper bound with the non-negativity condition, we conclude that P_{11}^{++} can vary in the range $0 \leq P_{11}^{++} \leq 1/2$ without violating the causal communication constraint.

It is worth noting that the above upper bound for P_{11}^{++} can also be deduced directly from the relation [7, 8]

$$P_{11}^{++} = \frac{1}{2}\big(1 + P_{00}^{++} - P_{00}^{--} - P_{01}^{+-} + P_{01}^{-+} + P_{10}^{+-} - P_{10}^{-+} - P_{11}^{+-} - P_{11}^{-+}\big).$$

Since $P^{ij}_{kk'} \geq 0$, from this relation it quickly follows that $P^{++}_{11} \leq \frac{1}{2}(1 + P^{++}_{00} + P^{-+}_{01} + P^{+-}_{10})$. Therefore, when the probabilities P^{++}_{00}, P^{-+}_{01}, and P^{+-}_{10} vanish, we conclude that $P^{++}_{11} \leq 1/2$.

The probability distribution (14) can be written compactly as

$$P^{ij}_{kk'} = \begin{cases} \frac{1}{2}, & \text{if } \delta_{ij} = kk' \oplus k \oplus k'; \\ 0, & \text{otherwise,} \end{cases} \tag{15}$$

where $k, k' \in \{0, 1\}$, $\delta_{++} = \delta_{--} = 1, \delta_{+-} = \delta_{-+} = 0$, and where $\oplus$ indicates addition modulo 2. The set of probabilities (15) corresponds to the Popescu-Rohrlich-type correlations leading to the maximal algebraic violation of the CHSH inequality [11]

$$|E(A_0, B_1) + E(A_1, B_1) + E(A_1, B_0) - E(A_0, B_0)| \leq 2,$$

the correlation function $E(A_k, B_{k'})$ being defined by $P^{++}_{kk'} + P^{--}_{kk'} - P^{+-}_{kk'} - P^{-+}_{kk'}$.

3.2 Case $K = 2$

For $K = 2$ (i.e., three observables per qubit: A_0, A_1, A_2 for qubit A, and B_0, B_1, B_2 for qubit B), the causal communication constraint in Equations (10)-(11) implies that

$$\begin{aligned}
P^{++}_{00} + P^{+-}_{00} &= P^{++}_{01} + P^{+-}_{01} = P^{++}_{02} + P^{+-}_{02} \\
P^{++}_{10} + P^{+-}_{10} &= P^{++}_{11} + P^{+-}_{11} = P^{++}_{12} + P^{+-}_{12} \\
P^{++}_{20} + P^{+-}_{20} &= P^{++}_{21} + P^{+-}_{21} = P^{++}_{22} + P^{+-}_{22} \\
P^{-+}_{00} + P^{--}_{00} &= P^{-+}_{01} + P^{--}_{01} = P^{-+}_{02} + P^{--}_{02} \\
P^{-+}_{10} + P^{--}_{10} &= P^{-+}_{11} + P^{--}_{11} = P^{-+}_{12} + P^{--}_{12} \\
P^{-+}_{20} + P^{--}_{20} &= P^{-+}_{21} + P^{--}_{21} = P^{-+}_{22} + P^{--}_{22} \\
P^{++}_{00} + P^{-+}_{00} &= P^{++}_{10} + P^{-+}_{10} = P^{++}_{20} + P^{-+}_{20} \\
P^{++}_{01} + P^{-+}_{01} &= P^{++}_{11} + P^{-+}_{11} = P^{++}_{21} + P^{-+}_{21} \\
P^{++}_{02} + P^{-+}_{02} &= P^{++}_{12} + P^{-+}_{12} = P^{++}_{22} + P^{-+}_{22} \\
P^{+-}_{00} + P^{--}_{00} &= P^{+-}_{10} + P^{--}_{10} = P^{+-}_{20} + P^{--}_{20} \\
P^{+-}_{01} + P^{--}_{01} &= P^{+-}_{11} + P^{--}_{11} = P^{+-}_{21} + P^{--}_{21} \\
P^{+-}_{02} + P^{--}_{02} &= P^{+-}_{12} + P^{--}_{12} = P^{+-}_{22} + P^{--}_{22}
\end{aligned} \tag{16}$$

On the other hand, Hardy's conditions in (3)-(5) for $K = 2$ are

$$P^{++}_{00} = P^{-+}_{01} = P^{+-}_{10} = P^{-+}_{12} = P^{+-}_{21} = 0. \tag{17}$$

It is readily verified that the following joint probability distribution

$$P^{ij}_{kk'} = \begin{cases} \frac{1}{2}, & \text{for } i = j \text{ and } \forall k, k' \in \{0,1,2\} \text{ except for } k = k' = 0; \\ 0, & \text{for } i = j \text{ and } k = k' = 0; \\ 0, & \text{for } i \neq j \text{ and } \forall k, k' \in \{0,1,2\} \text{ except for } k = k' = 0; \\ \frac{1}{2}, & \text{for } i \neq j \text{ and } k = k' = 0, \end{cases} \tag{18}$$

satisfies all the conditions in Equations (8)-(9) and (16)-(17), with $P_2 = P^{++}_{22} = 1/2$. It can be shown that, in fact, $1/2$ is the maximum value for P^{++}_{22} allowed by GPT for the case where $P^{++}_{00} = P^{-+}_{01} = P^{+-}_{10} = P^{-+}_{12} = P^{+-}_{21} = 0$. To see this, suppose on the contrary that we have simultaneously $P^{++}_{22} > 1/2$ and $P^{++}_{00} = P^{-+}_{01} = P^{+-}_{10} = P^{-+}_{12} = P^{+-}_{21} = 0$. Then the following inferences follow

- Since $P^{++}_{22} > 1/2$ and $P^{+-}_{21} = 0$, from the $3rd$ relation in (16) we deduce that $P^{++}_{21} > 1/2$.
- Since $P^{++}_{21} > 1/2$ and $P^{-+}_{01} = 0$, from the $8th$ relation in (16) we deduce that $P^{++}_{01} > 1/2$.
- Since $P^{++}_{01} > 1/2$ and $P^{++}_{00} = 0$, from the $1st$ relation in (16) we deduce that $P^{+-}_{00} > 1/2$.
- Since $P^{++}_{22} > 1/2$ and $P^{-+}_{12} = 0$, from the $9th$ relation in (16) we deduce that $P^{++}_{12} > 1/2$.
- Since $P^{++}_{12} > 1/2$ and $P^{+-}_{10} = 0$, from the $2nd$ relation in (16) we deduce that $P^{++}_{10} > 1/2$.
- Since $P^{++}_{10} > 1/2$ and $P^{++}_{00} = 0$, from the $7th$ relation in (16) we deduce that $P^{-+}_{00} > 1/2$.

Expressing this as two sequences of implications

$$\left.\begin{array}{l} P^{++}_{22} > \dfrac{1}{2} \Rightarrow P^{++}_{21} > \dfrac{1}{2} \Rightarrow P^{++}_{01} > \dfrac{1}{2} \Rightarrow P^{+-}_{00} > \dfrac{1}{2} \\ P^{++}_{22} > \dfrac{1}{2} \Rightarrow P^{++}_{12} > \dfrac{1}{2} \Rightarrow P^{++}_{10} > \dfrac{1}{2} \Rightarrow P^{-+}_{00} > \dfrac{1}{2} \end{array}\right\}$$

we can see from the last two inequalities that $P^{+-}_{00} + P^{-+}_{00} > 1$, contradicting the normalization condition $P^{++}_{00} + P^{+-}_{00} + P^{-+}_{00} + P^{--}_{00} = 1$. We therefore conclude that, when Hardy's conditions (17) are satisfied, the probability P^{++}_{22} can vary in the range $0 \leq P^{++}_{22} \leq 1/2$ while respecting the causal communication constraint.

We further note that the probability distribution (18) gives the maximal algebraic violation (namely, 6) of the chained CHSH-type inequality [4, 25]

$$|E(A_0, B_1) + E(A_1, B_2) + E(A_2, B_2) + E(A_2, B_1) + E(A_1, B_0) - E(A_0, B_0)| \leq 4.$$

3.3 Case $K = 3$

For $K = 3$ (i.e., four observables per qubit: A_0, A_1, A_2, A_3 for qubit A, and B_0, B_1, B_2, B_3 for qubit B), the causal communication constraint in Equations (10)-(11) implies the following set of conditions:

$$\begin{aligned}
&P_{00}^{++} + P_{00}^{+-} = P_{01}^{++} + P_{01}^{+-} = P_{02}^{++} + P_{02}^{+-} = P_{03}^{++} + P_{03}^{+-}\\
&P_{10}^{++} + P_{10}^{+-} = P_{11}^{++} + P_{11}^{+-} = P_{12}^{++} + P_{12}^{+-} = P_{13}^{++} + P_{13}^{+-}\\
&P_{20}^{++} + P_{20}^{+-} = P_{21}^{++} + P_{21}^{+-} = P_{22}^{++} + P_{22}^{+-} = P_{23}^{++} + P_{23}^{+-}\\
&P_{30}^{++} + P_{30}^{+-} = P_{31}^{++} + P_{31}^{+-} = P_{32}^{++} + P_{32}^{+-} = P_{33}^{++} + P_{33}^{+-}\\
&P_{00}^{-+} + P_{00}^{--} = P_{01}^{-+} + P_{01}^{--} = P_{02}^{-+} + P_{02}^{--} = P_{03}^{-+} + P_{03}^{--}\\
&P_{10}^{-+} + P_{10}^{--} = P_{11}^{-+} + P_{11}^{--} = P_{12}^{-+} + P_{12}^{--} = P_{13}^{-+} + P_{13}^{--}\\
&P_{20}^{-+} + P_{20}^{--} = P_{21}^{-+} + P_{21}^{--} = P_{22}^{-+} + P_{22}^{--} = P_{23}^{-+} + P_{23}^{--}\\
&P_{30}^{-+} + P_{30}^{--} = P_{31}^{-+} + P_{31}^{--} = P_{32}^{-+} + P_{32}^{--} = P_{33}^{-+} + P_{33}^{--}\\
&P_{00}^{++} + P_{00}^{-+} = P_{10}^{++} + P_{10}^{-+} = P_{20}^{++} + P_{20}^{-+} = P_{30}^{++} + P_{30}^{-+}\\
&P_{01}^{++} + P_{01}^{-+} = P_{11}^{++} + P_{11}^{-+} = P_{21}^{++} + P_{21}^{-+} = P_{31}^{++} + P_{31}^{-+}\\
&P_{02}^{++} + P_{02}^{-+} = P_{12}^{++} + P_{12}^{-+} = P_{22}^{++} + P_{22}^{-+} = P_{32}^{++} + P_{32}^{-+}\\
&P_{03}^{++} + P_{03}^{-+} = P_{13}^{++} + P_{13}^{-+} = P_{23}^{++} + P_{23}^{-+} = P_{33}^{++} + P_{33}^{-+}\\
&P_{00}^{+-} + P_{00}^{--} = P_{10}^{+-} + P_{10}^{--} = P_{20}^{+-} + P_{20}^{--} = P_{30}^{+-} + P_{30}^{--}\\
&P_{01}^{+-} + P_{01}^{--} = P_{11}^{+-} + P_{11}^{--} = P_{21}^{+-} + P_{21}^{--} = P_{31}^{+-} + P_{31}^{--}\\
&P_{02}^{+-} + P_{02}^{--} = P_{12}^{+-} + P_{12}^{--} = P_{22}^{+-} + P_{22}^{--} = P_{32}^{+-} + P_{32}^{--}\\
&P_{03}^{+-} + P_{03}^{--} = P_{13}^{+-} + P_{13}^{--} = P_{23}^{+-} + P_{23}^{--} = P_{33}^{+-} + P_{33}^{--}
\end{aligned} \tag{19}$$

while Hardy's conditions in (3)-(5) for $K = 3$ are

$$P_{00}^{++} = P_{01}^{-+} = P_{10}^{+-} = P_{12}^{-+} = P_{21}^{+-} = P_{23}^{-+} = P_{32}^{+-} = 0. \tag{20}$$

It is readily checked that the following joint probability distribution

$$P_{kk'}^{ij} = \begin{cases} \frac{1}{2}, & \text{for } i = j \text{ and } \forall k, k' \in \{0,1,2,3\} \text{ except for } k = k' = 0;\\ 0, & \text{for } i = j \text{ and } k = k' = 0;\\ 0, & \text{for } i \neq j \text{ and } \forall k, k' \in \{0,1,2,3\} \text{ except for } k = k' = 0;\\ \frac{1}{2}, & \text{for } i \neq j \text{ and } k = k' = 0, \end{cases} \tag{21}$$

satisfies all the conditions in Equations (8)-(9) and (19)-(20), with $P_3 = P_{33}^{++} = 1/2$. Likewise, it turns out that $1/2$ is the maximum of P_{33}^{++} allowed by GPT when Hardy's conditions (20) are fulfilled. To see this, suppose on the contrary that we have simultaneously $P_{33}^{++} > 1/2$ and $P_{00}^{++} = P_{01}^{-+} = P_{10}^{+-} = P_{12}^{-+} = P_{21}^{+-} = P_{23}^{-+} = P_{32}^{+-} = 0$. Then the following inferences follow

- Since $P_{33}^{++} > 1/2$ and $P_{23}^{-+} = 0$, from the $12th$ relation in (19) we deduce that $P_{23}^{++} > 1/2$.

- Since $P_{23}^{++} > 1/2$ and $P_{21}^{+-} = 0$, from the $3rd$ relation in (19) we deduce that $P_{21}^{++} > 1/2$.

- Since $P_{21}^{++} > 1/2$ and $P_{01}^{-+} = 0$, from the $10th$ relation in (19) we deduce that $P_{01}^{++} > 1/2$.

- Since $P_{01}^{++} > 1/2$ and $P_{00}^{++} = 0$, from the $1st$ relation in (19) we deduce that $P_{00}^{+-} > 1/2$.

- Since $P_{33}^{++} > 1/2$ and $P_{32}^{+-} = 0$, from the $4th$ relation in (19) we deduce that $P_{32}^{++} > 1/2$.

- Since $P_{32}^{++} > 1/2$ and $P_{12}^{-+} = 0$, from the $11th$ relation in (19) we deduce that $P_{12}^{++} > 1/2$.

- Since $P_{12}^{++} > 1/2$ and $P_{10}^{+-} = 0$, from the $2nd$ relation in (19) we deduce that $P_{10}^{++} > 1/2$.

- Since $P_{10}^{++} > 1/2$ and $P_{00}^{++} = 0$, from the $9th$ relation in (19) we deduce that $P_{00}^{-+} > 1/2$.

Expressing this as two sequences of implications

$$\left.\begin{array}{l} P_{33}^{++} > \dfrac{1}{2} \;\Rightarrow P_{23}^{++} > \dfrac{1}{2} \;\Rightarrow P_{21}^{++} > \dfrac{1}{2} \;\Rightarrow P_{01}^{++} > \dfrac{1}{2} \;\Rightarrow P_{00}^{+-} > \dfrac{1}{2} \\ P_{33}^{++} > \dfrac{1}{2} \;\Rightarrow P_{32}^{++} > \dfrac{1}{2} \;\Rightarrow P_{12}^{++} > \dfrac{1}{2} \;\Rightarrow P_{10}^{++} > \dfrac{1}{2} \;\Rightarrow P_{00}^{-+} > \dfrac{1}{2} \end{array}\right\}$$

we obtain $P_{00}^{+-} + P_{00}^{-+} > 1$ from the last two inequalities, and this contradicts the normalization condition $P_{00}^{++} + P_{00}^{+-} + P_{00}^{-+} + P_{00}^{--} = 1$. Thus, we conclude that, when Hardy conditions (20) are met, the probability P_{33}^{++} can vary in the range $0 \leq P_{33}^{++} \leq 1/2$ without violating the causal communication constraint.

On the other hand, the probability distribution (21) gives the maximal algebraic violation (namely, 8) of the chained CHSH-type inequality [4, 25]

$$\begin{aligned} &|E(A_0, B_1) + E(A_1, B_2) + E(A_2, B_3) + E(A_3, B_3) \\ &\quad + E(A_3, B_2) + E(A_2, B_1) + E(A_1, B_0) - E(A_0, B_0)| \leq 6. \end{aligned}$$

3.4 The general case

The above results for $K = 1, 2, 3$ generalize in a straightforward way to an arbitrary number $K + 1$ of observables per qubit (i.e., $A_0, A_1, \ldots A_K$ for qubit

A, and $B_0, B_1, \ldots B_K$ for qubit B). To show this, we write the causal communication constraint in Equations (10)-(11) in the form

$$\begin{cases} P_{00}^{++} + P_{00}^{+-} = P_{01}^{++} + P_{01}^{+-} = P_{02}^{++} + P_{02}^{+-} = \ldots = P_{0K}^{++} + P_{0K}^{+-} \\ P_{10}^{++} + P_{10}^{+-} = P_{11}^{++} + P_{11}^{+-} = P_{12}^{++} + P_{12}^{+-} = \ldots = P_{1K}^{++} + P_{1K}^{+-} \\ \quad \vdots \\ P_{K0}^{++} + P_{K0}^{+-} = P_{K1}^{++} + P_{K1}^{+-} = P_{K2}^{++} + P_{K2}^{+-} = \ldots = P_{KK}^{++} + P_{KK}^{+-} \end{cases}$$

$$\begin{cases} P_{00}^{-+} + P_{00}^{--} = P_{01}^{-+} + P_{01}^{--} = P_{02}^{-+} + P_{02}^{--} = \ldots = P_{0K}^{-+} + P_{0K}^{--} \\ P_{10}^{-+} + P_{10}^{--} = P_{11}^{-+} + P_{11}^{--} = P_{12}^{-+} + P_{12}^{--} = \ldots = P_{1K}^{-+} + P_{1K}^{--} \\ \quad \vdots \\ P_{K0}^{-+} + P_{K0}^{--} = P_{K1}^{-+} + P_{K1}^{--} = P_{K2}^{-+} + P_{K2}^{--} = \ldots = P_{KK}^{-+} + P_{KK}^{--} \end{cases}$$

$$\begin{cases} P_{00}^{++} + P_{00}^{-+} = P_{10}^{++} + P_{10}^{-+} = P_{20}^{++} + P_{20}^{-+} = \ldots = P_{K0}^{++} + P_{K0}^{-+} \\ P_{01}^{++} + P_{01}^{-+} = P_{11}^{++} + P_{11}^{-+} = P_{21}^{++} + P_{21}^{-+} = \ldots = P_{K1}^{++} + P_{K1}^{-+} \\ \quad \vdots \\ P_{0K}^{++} + P_{0K}^{-+} = P_{1K}^{++} + P_{1K}^{-+} = P_{2K}^{++} + P_{2K}^{-+} = \ldots = P_{KK}^{++} + P_{KK}^{-+} \end{cases} \tag{22}$$

$$\begin{cases} P_{00}^{+-} + P_{00}^{--} = P_{10}^{+-} + P_{10}^{--} = P_{20}^{+-} + P_{20}^{--} = \ldots = P_{K0}^{+-} + P_{K0}^{--} \\ P_{01}^{+-} + P_{01}^{--} = P_{11}^{+-} + P_{11}^{--} = P_{21}^{+-} + P_{21}^{--} = \ldots = P_{K1}^{+-} + P_{K1}^{--} \\ \quad \vdots \\ P_{0K}^{+-} + P_{0K}^{--} = P_{1K}^{+-} + P_{1K}^{--} = P_{2K}^{+-} + P_{2K}^{--} = \ldots = P_{KK}^{+-} + P_{KK}^{--} \end{cases}$$

with a total of $4(K+1)$ relations, and K equals signs in each relation. Furthermore, Hardy's conditions in (3)-(5) read as

$$P_{00}^{++} = P_{01}^{-+} = P_{10}^{+-} = P_{12}^{-+} = P_{21}^{+-} = \ldots = P_{K-1,K}^{-+} = P_{K,K-1}^{+-} = 0. \tag{23}$$

The following joint probability distribution (which is a direct generalization of the previous particular distributions (14), (18), and (21))

$$P_{kk'}^{ij} = \begin{cases} \frac{1}{2}, & \text{for } i = j \text{ and } \forall k, k' \in \{0, 1, \ldots, K\} \text{ except for } k = k' = 0; \\ 0, & \text{for } i = j \text{ and } k = k' = 0; \\ 0, & \text{for } i \neq j \text{ and } \forall k, k' \in \{0, 1, \ldots, K\} \text{ except for } k = k' = 0; \\ \frac{1}{2}, & \text{for } i \neq j \text{ and } k = k' = 0, \end{cases} \tag{24}$$

then satisfies all the conditions in Equations (8)-(9) and (22)-(23), with $P_K = P_{KK}^{++} = 1/2$. Similarly, it can be shown that $1/2$ is the maximum value of P_{KK}^{++} allowed by GPT when Hardy's conditions (23) are fulfilled. Indeed, assuming that $P_{KK}^{++} > 1/2$ and $P_{00}^{++} = P_{01}^{-+} = P_{10}^{+-} = \ldots = P_{K-1,K}^{-+} = P_{K,K-1}^{+-} = 0$, from the relations in the first and third blocks of Equation (22) one can derive the following two sequences of implications:

- For $K = 1, 3, 5, \ldots$

$$P_{KK}^{++} > \frac{1}{2} \Rightarrow P_{K-1,K}^{++} > \frac{1}{2} \Rightarrow P_{K-1,K-2}^{++} > \frac{1}{2} \Rightarrow P_{K-3,K-2}^{++} > \frac{1}{2}$$
$$\Rightarrow P_{K-3,K-4}^{++} > \frac{1}{2} \Rightarrow \ldots \Rightarrow P_{21}^{++} > \frac{1}{2} \Rightarrow P_{01}^{++} > \frac{1}{2} \Rightarrow P_{00}^{+-} > \frac{1}{2}$$
$$P_{KK}^{++} > \frac{1}{2} \Rightarrow P_{K,K-1}^{++} > \frac{1}{2} \Rightarrow P_{K-2,K-1}^{++} > \frac{1}{2} \Rightarrow P_{K-2,K-3}^{++} > \frac{1}{2}$$
$$\Rightarrow P_{K-4,K-3}^{++} > \frac{1}{2} \Rightarrow \ldots \Rightarrow P_{12}^{++} > \frac{1}{2} \Rightarrow P_{10}^{++} > \frac{1}{2} \Rightarrow P_{00}^{-+} > \frac{1}{2}$$

- For $K = 2, 4, 6, \ldots$

$$P_{KK}^{++} > \frac{1}{2} \Rightarrow P_{K,K-1}^{++} > \frac{1}{2} \Rightarrow P_{K-2,K-1}^{++} > \frac{1}{2} \Rightarrow P_{K-2,K-3}^{++} > \frac{1}{2}$$
$$\Rightarrow P_{K-4,K-3}^{++} > \frac{1}{2} \Rightarrow \ldots \Rightarrow P_{21}^{++} > \frac{1}{2} \Rightarrow P_{01}^{++} > \frac{1}{2} \Rightarrow P_{00}^{+-} > \frac{1}{2}$$
$$P_{KK}^{++} > \frac{1}{2} \Rightarrow P_{K-1,K}^{++} > \frac{1}{2} \Rightarrow P_{K-1,K-2}^{++} > \frac{1}{2} \Rightarrow P_{K-2,K-3}^{++} > \frac{1}{2}$$
$$\Rightarrow P_{K-3,K-4}^{++} > \frac{1}{2} \Rightarrow \ldots \Rightarrow P_{12}^{++} > \frac{1}{2} \Rightarrow P_{10}^{++} > \frac{1}{2} \Rightarrow P_{00}^{-+} > \frac{1}{2}$$

where each constituent sequence involves exactly $K+1$ implication signs. For either odd or even K, we end up with $P_{00}^{+-} + P_{00}^{-+} > 1$ from the last two inequalities, contradicting the normalization condition $P_{00}^{++} + P_{00}^{+-} + P_{00}^{-+} + P_{00}^{--} = 1$. Hence, it follows that, when Hardy's conditions (23) are fulfilled, the probability P_{KK}^{++} can vary in the range $0 \leq P_{KK}^{++} \leq 1/2$ without violating the causal communication constraint.

We note that the probability distribution (24) gives the maximal algebraic violation (namely, $2K+2$) of the chained CHSH inequality [4, 25]

$$\left| \sum_{k=1}^{K} E(A_{k-1}, B_k) + \sum_{k=1}^{K} E(A_k, B_{k-1}) + E(A_K, B_K) - E(A_0, B_0) \right| \leq 2K, \quad (25)$$

where the $2K+2$ pairs of observables $(A_0, B_0), (A_K, B_K), (A_{k-1}, B_k), (A_k, B_{k-1})$, $k = 1, 2, \ldots, K$, appearing in the left-hand side of inequality (25) are precisely those in Equations (2)-(5). Moreover, we point out that the maximum value of the left-hand side of (25) predicted by quantum mechanics is given by [25] $2(K+1)\cos\frac{\pi}{2(K+1)}$. For sufficiently large K, we have that $\cos\frac{\pi}{2(K+1)} \approx 1$, and then the left-hand side of (25) approaches the algebraic limit $2K+2$. This corresponds to the case in which the quantum-mechanical probabilities satisfy all Hardy's non-locality conditions (2)-(5) with $P_K \to 1/2$ as $K \to \infty$. It is easily seen that, in this limit, a direct ("all or nothing") contradiction between quantum mechanics and local realism arises for Hardy's ladder setup [9].

Finally, we mention that a complete characterization of the extremal non-signaling bipartite probability distributions P_{xy}^{ab} for $a, b \in \{0, 1\}$, $x \in \{0, 1, \ldots, d_x - 1\}$ and $y \in \{0, 1, \ldots, d_y - 1\}$, has been given in [19]. The probability distribution (24) constitutes a representative element of a fully non-local, non-deterministic extremal distribution corresponding to the case in which $d_x = d_y = K + 1$.

4 Conclusion

In conclusion, we have shown that the success probability of Hardy's ladder argument for non-locality for two qubits and $K + 1$ observables per qubit in the framework of GPT reaches a maximum of 50% for any value of K, thereby generalizing the known result for $K = 1$ [7, 10] to an arbitrary number of observables. Incidentally, we observe that, as shown in [10], the maximum success probability of Hardy's non-locality argument for three qubits and two observables per qubit also reaches a value of 50% in the context of GPT.

In view of our results, we conclude that the causal communication constraint by itself cannot account (unless in the limit of a truly infinite number of observables; see Figure 1) for the upper bound of P_K predicted by quantum mechanics for Hardy's ladder setup. It is therefore worthwhile to investigate whether such quantum bound for P_K could naturally emerge as a necessary consequence of some additional constraints, other than causal communication, such as those ensuing from the physical principles of non-trivial communication complexity [3], information causality [21], macroscopic locality [20], and local orthogonality [14] (also known as the exclusivity principle [5, 27, 6]). In this respect, Ahanj et al. [1] (see also [26]) showed that, for Hardy's setup for two qubits and two observables per qubit, the 50% bound prescribed by the causal communication constraint is lowered to $\frac{\sqrt{2}-1}{2} \approx 0.207$ under the condition of information causality. Subsequently, Das et al. [12] applied the local orthogonality principle to two copies of Hardy's set of correlations and found that the maximum success probability of Hardy's argument is reduced to ≈ 0.177 which is relatively closer to the corresponding quantum value ≈ 0.09. It will be interesting to look for the bounds that are imposed by the above mentioned physical principles on Hardy's ladder setup.

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