New Fundamental Dynamical Equation for Higher Derivative Quantum Field Theories

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Abstract

In space-time with the Minkowski metric, the group of the metric is the inhomogeneous Lorentz group, which is also known as the Poincaré group. A dynamical equation is called fundamental if it is invariant with respect to the group of the metric, which consists of all transformations that leave the metric invariant. A method based on this group is used to derive two infinite sets of Poincaré invariant linear dynamical equations for scalar and analytical functions that represent free, spin-zero, massive elementary particles. The sets contain two different types of fundamental higher derivative dynamical equations, which are used to construct higher derivative quantum field theories. One of these sets contains the original Klein-Gordon equation and it is shown that all physically acceptable solutions to the higher derivative equations in this set are the same as the solutions to the Klein-Gordon equation. This means that none of the higher order equation in this set can be considered as new and that the Klein-Gordon is the only fundamental dynamical equation available to construct local quantum field theories. However, for the other set, it is demonstrated that all physically acceptable solutions to the higher derivative equations are the same as the solutions to the lowest order fundamental equation in this set. As a result, this lowest order equation is the only new fundamental equation in the set, and it is used to construct higher derivative (non-local) quantum field

theories. It is suggested that such higher derivative theories may apply to string theories, modeling of dark energy and to non-localities in quantum mechanics.

Keywords: Minkowski space-time, Poincaré group, scalar wave functions, higher derivative Klein-Gordon equations

1 Introduction

The measure of space-time introduced by Minkowski plays an important role in formulating relativistic field theories of elementary particles. The basic properties of space-time are determined by the Minkowski metric, which shows how space and time are connected, and defines a measure of separation between events. The group of transformations leaving the Minkowski metric invariant is the inhomogeneous Lorentz group, which is also called the Poincaré group. A set of coordinate transformations that leaves this metric invariant forms a representation of the group. Wigner [1] was the first who classified all irreducible representations (irreps) of the Poincaré group and identified an elementary particle with an irreducible unitary representation of the group. By using this definition, Wigner was able to determine classes of elementary particles that may exist in Minkowski space-time [2,3].

In this paper, we consider free, massive, spin-zero elementary particles and represent them by a scalar and analytical wave function. As it is well-known, relativistic field theories of such particles have already been formulated [3,4] and the basic dynamical equation of these theories is the original (second-order) Klein-Gordon (KG) equation [5,6], which has also been used to describe the Higgs field in the standard model of electroweak interaction [3,7]. The KG equation describes the evolution of the wave function in space-time with the Minkowski metric, and it has been shown that the equation is Poincaré invariant [8,9]. This simply means that the KG equation is invariant with respect to all the transformations that leave the metric invariant, and we shall refer to such an equation as fundamental. Observers who use the Minkowski metric and whose coordinate systems are related by the transformations that leave the metric invariant are in inertial frames of reference in Minkowski space-time. Thus to guarantee that all inertial observers identify the same elementary particle, it is sufficient to require that the wave function describing this particle transforms like one of the irreps of the group of the metric. The latter requirement is the basis for a method that uses the group of the metric to formally derive fundamental dynamical equations of non-relativistic [10-12] and relativistic [13] wave theories, without employing the usual procedure of 'promoting' certain classical quantities to operator status.

In our previous work [10], we used the method to derive a Schrödinger-like

equation [14] for a scalar wave function, and we also demonstrated that this equation was the only Galilean invariant equation among an infinite number of dynamical equations that were formally obtained by the method [11]. In addition, we derived the basic dynamical equations of classical mechanics [12] and recently extended the method to relativistic theories of waves and particles [13], including generalized Lévy-Leblond and Schrödinger equations [15,16], and generalized Dirac and Klein-Gordon equations for spinor wave functions [17]. The method is based on two principles, namely, the Principle of Relativity and the Principle of Analyticity. Our version of the Principle of Relativity requires that all inertial observers formulate the same physical law and that they identify the same elementary particle or field. According to this Principle, all dynamical equations describing evolution of the wave function in time and space must be invariant with respect to all transformations that leave the metric unchanged. The Principle of Analyticity demands that the wave function describing elementary particles be analytic. In our previous papers as well as in this paper, it is implicitly assumed that the metric is unchanged by the presence of the objects being described, i.e. all objects are test particles. Changes in the metric of space-time must be treated with methods of General Theory of Relativity, which is beyond the scope of this paper.

The main purpose of this paper is to use our method to derive additional possible dynamical equations describing free, massive, spin-zero elementary particles in flat space-time with the Minkowski metric, and investigate Poincaré invariance of these equations. We show that there are two infinite sets of Poincaré invariant linear dynamical equations; one set contains all the equations with the even powers of a second-order operator and the other set contains all the equations of the odd powers of this operator. The well-known Klein-Gordon equation is a member of the second group and it is the only second-order partial differential equation in this set. The result has far reaching consequences as it clearly demonstrates that any *local* quantum field theory (QFT) describing free, massive, spin-zero particles must be based on this equation; a field theory is considered to be local when its effects quickly diminish with distance from a point where the theory is formulated [9]. The fact that there are two infinite sets of higher-order partial differential equations and that all of them are Poincaré invariant is important for constructing higher derivative (non-local) relativistic field theories.

The basic idea of constructing higher derivative (HD) field theories was originally put forward by Podolski [18], who generalized an electromagnetic theory by introducing the HD. It was then extended by Green [19], Coelho de Souza and Rodrigues [18] and others [21-24] to QFT. In more recent work, a field theory with derivatives of arbitrary order was constructed for a scalar field by Weldon [23]. The advantage of this method is that it can formally be applied to dynamical equations of any order. Therefore, the method is

especially suitable for the two infinite sets of Poincaré invariant HD dynamical equations obtained in this paper.

We use the method to develop two HD quantum field theories (see Sec. 5.2 and 6.2). Then, we generalize the results to all fundamental HD dynamical equations in both infinite sets, which contain two different types of fundamental, higher derivative, linear dynamical equations, with the original Klein-Gordon equation being a member of one of these sets. For the set with the Klein-Gordon equation, it is shown that all physically acceptable solutions to the higher derivative equations are the same as those obtained for the Klein-Gordon equation. As a result, none of the higher order equation in this set can be considered as new and the original Klein-Gordon is the only fundamental dynamical equation available to construct local quantum field theories. Now for the other set, it is demonstrated that all physically acceptable solutions to the higher derivative equations are the same as the solutions to the lowest order fundamental equation, which is still the higher order equation in this set. This means that the lowest order equation is the only new fundamental equation in the set, and that this equation can be used to construct higher derivative (non-local) quantum field theories. It is suggested that possible applications of such theories may range from string theories and modeling of dark energy to non-localities in quantum mechanics.

This paper is organized as follows: the Poincaré group, the resulting eigenvalue equations, and definitions of elementary particles are presented in Sec. 2; two infinite sets of Poincaré invariant dynamical equations are derived in Sec. 3; the Klein-Gordon equation is discussed in Sec. 4; the relationship of the Poincaré invariant dynamical equations to the Klein-Gordon equation and their quantization are presented and discussed in Sec. 5 and 6; the invariant frequencies for selected particles are given in Sec. 7; and conclusions are presented in Sec. 8.

2 Poincaré group, eigenvalue equations and elementary particles

To derive dynamical equations for particles on a space-time manifold, we begin by defining carefully what we mean by an elementary particle. We are interested in quantum particles, objects which may be described by a function of space-time variables that we associate with a space-time, metric manifold. An elementary particle is any object that may be described by a function existing in a Hilbert space and transforming like an irreducible representation (irrep) of some symmetry group. The symmetry group may be chosen arbitrarily, defining a particular type of particle. In this paper, we are interested in the symmetry group consisting of all the coordinate transformations leaving the

space-time metric of a given manifold invariant. We limit ourselves to flat space-time and employ the Minkowski metric.

The Minkowski metric can be written as $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$, where the spatial coordinates x , y and z , and time t are all measured in the same units because the speed of light has been defined as $c = 1$. The group of this metric is the Poincaré group, whose structure is given by the following semi-direct product: $P = H_p \otimes_s T(3+1)$, where $T(3+1)$ is an invariant subgroup of space-time translations and H_p is a non-invariant subgroup consisting of the remaining transformations and the identity transformation [2,23]. In this paper, we consider the so-called proper orthochronous group $P_+^\uparrow$ that is a subgroup of P .

To identify an elementary particle, we require that a scalar, analytical wave function $\psi(x^\mu)$ transforms as one of the irreps of the invariant subgroup $T(3+1)$. Since the transformation properties of the function are preserved in the irreps of the semi-direct product of the group, it can be shown that a necessary condition that ψ represents an elementary particle in any inertial frame of reference is the following set of eigenvalue equations [13]

$$i\partial^\mu\psi = k^\mu\psi, \quad (1)$$

where $\partial^\mu = \partial/\partial x_\mu = (\partial_t, -\nabla)$ and $k^\mu = (\omega, k^i)$, with $\partial_0 = \partial_t = \partial/\partial t$, $\mu = 0, 1, 2$ and 3 , and $i = 1, 2$ and 3 . Since $i\partial^\mu$ is a Hermitian operator, k^μ must be real numbers. The irreps of $T(3+1)$ and thus $P_+^\uparrow$ may be labelled by real numbers. By contrast, if we label the irreps by a parameter called mass by Wigner [1], some values of this parameter may become imaginary [2]. We will now use the set of eigenvalue equations given by Eq. (1) to derive Poincaré invariant dynamical equations in Minkowski space-time.

3 Poincaré invariant dynamical equations

A physical theory is called fundamental in space-time with the Minkowski metric when its dynamical equations are invariant with respect to all transformations of coordinates that leave the metric invariant. Since these transformations form a representation of the Poincaré group, we refer to an invariant dynamical equation as a Poincaré invariant (or fundamental) equation. We now present the main results of our search for Poincaré invariant dynamical equations using the eigenvalue equations (see Eq. 1) as the starting point.

Let us consider two inertial observers who use sets of coordinates x^μ and x'^μ to describe the state of the particle that is given by the following two scalar functions: $\psi(x^\mu)$ and $\psi'(x'^\mu)$. In space-time with the Minkowski metric the coordinates x^μ and x'^μ are related to each other by the Lorentz transformation Λ_ν^μ , which can be used to obtain $x'^\mu = \Lambda_\nu^\mu x^\nu$. With this coordinate transformation, it is easy to show that $(\partial'^\mu \partial'_\mu)\psi'(x'^\mu) = (\partial^\mu \partial_\mu)\psi(x^\mu)$. The operator

$(\partial^\mu \partial'_\mu)$ is one of two Casimir operators for the Poincaré group [2] and the only one needed to develop possible higher order equations for scalar functions. We shall discuss more general functions for which both Casimir operators may be used to generate new Poincaré invariant higher order equations in another paper.

Using the above results, the simplest Poincaré invariant dynamical equation that can be derived from the eigenvalue equations is

$$-\partial^\mu \partial_\mu \psi = k^\mu k_\mu \psi = (\omega^2 - k^2) \psi , \quad (2)$$

which is commonly known as the Klein-Gordon (KG) equation [3,4]. We can write this equation in a more compact form by introducing a special frame of reference with $k^i = 0$, and in that frame defining $\omega = \omega_0$. We call ω_0 the invariant frequency since it is the same for all inertial observers [13]. This allows us to write $\omega^2 = \omega_0^2 + k^2$, where $k^2 \equiv k^i k_i$, and obtain

$$(\partial^\mu \partial_\mu + \omega_0^2) \psi = 0 . \quad (3)$$

We note that classical mass and the Planck constant do not appear and are in fact not needed since ω_0 can be determined experimentally (see Sec. 6).

In developing higher order dynamical equations for particles or fields we rely upon the fundamental equation (1), which is a set of eigenvalue equations defining an elementary particle. The above method used to obtain the KG equation can now be applied again and the following fundamental dynamical equations are derived

$$(\partial^\mu \partial_\mu)^2 \psi - \omega_0^4 \psi = 0 , \quad (4)$$

and

$$(\partial^\mu \partial_\mu)^3 \psi + \omega_0^6 \psi = 0 . \quad (5)$$

After repeating the procedure m and n -times, the resulting Poincaré invariant dynamical equations are

$$[(\partial^\mu \partial_\mu)^m - \omega_0^{2m}] \psi = 0 , \quad (6)$$

and

$$[(\partial^\mu \partial_\mu)^n + \omega_0^{2n}] \psi = 0 , \quad (7)$$

where ω_0 is a constant, and m is any positive *even* integer and n is any positive *odd* integer. Based on our approach presented here, the above equations exhaust all possibilities of obtaining the fundamental linear equations for scalar state functions with no interactions. It must be noted that these higher order differential equations have solutions other than elementary state functions of the eigenvalue equations (see Eq. 1), possibly introducing new physical phenomena in the form of fields.

An interesting result obtained here is that there are two distinct sets of Poincaré invariant dynamical equations: one infinite set with the even powers of $(\partial^\mu \partial_\mu)$ and the 'minus' sign, and the other infinite set with the odd powers of $(\partial^\mu \partial_\mu)$ and the 'plus' sign. In the remaining parts of this paper, we shall refer to Eqs (6) and (7) as the even and odd order fundamental dynamical equations, respectively.

4 Klein-Gordon equation

The Klein-Gordon (KG) equation is typically obtained by using the relativistic relationship between energy, momentum and mass, and by substituting differential operators for the energy and momentum [3,4]. The equation can also be derived by another method that is based on group theory originally introduced by Bargmann and Wigner [26]. Both methods are different than the approach presented in this paper. In our approach, the KG equation is formally derived from the eigenvalue equations, which are established by the structure of the group of the metric [13], and the two fundamental principles described in Sec. 1.

The well-known difficulties, such as negative energy states and the probabilistic interpretation, associated with the KG equation are eliminated in QFT by the procedure of field quantization. The so-called single particle interpretation of the KG equation can formally be introduced by using the true particle operators [27] or by limiting the theory to situations where states with energy greater than twice the particle's rest mass are ignored [4]. However, such single-particle interpretations are not commonly accepted; see for example the recent work by Wharton [28] and his novel interpretation of the KG equation.

Our results presented in Sec. 3 showed that the original KG equation belongs to the infinite set of fundamental dynamical equations given by Eq. (7) and it is obtained by taking $n = 1$

$$(\partial^\mu \partial_\mu)\psi + \omega_0^2 \psi = 0 . \quad (8)$$

This equation was derived from the eigenvalue equations (see Eq. 1), whose solutions form a complete set in any coordinate system. Moreover, each solution of the eigenvalue equations is also a solution of the above equation. However, there are solutions of Eq. (8) that are not solutions of the eigenvalue equations. These extra solutions are the KG field, which has been 'quantized' to gain a meaningful interpretation in terms of single particles as reported in standard QFT textbooks [3,4].

Equation (8) is the only second-order fundamental dynamical equation for ψ being a scalar and analytical state function. The result has already been known (e.g., Bargmann and Wigner [26], and also Daughty [9] and references

therein), however, what is new here is the fact that the KG equation is the only one that can formally be derived from the eigenvalue equations without involving Einstein's energy momentum relationship and promoting the classical physical quantities to operators, which is a standard procedure used in QFT textbooks [3,4,27].

Since the eigenvalue equations are not Poincaré invariant equations for the scalar state function, the KG equation is the only one that is available to construct *local* quantum field theories for free and spin-zero elementary particles. Here, we follow a standard approach and refer to a field theory as local when its effects quickly decrease with distance from the point in space-time at which the theory is formulated.

The procedure of quantization of the KG equation is well-known and given in most QFT textbooks [3,4,27]. The coefficients of Fourier expansions ψ are treated as operators, and the negative and positive energy solutions correspond to the frequency $\omega_{\pm} = \pm\sqrt{k^2 + \omega_0^2}$, respectively. In the approach presented here, these solutions are consequences of the metric and our definition of an elementary particle (see Sec. 2). However, it must also be pointed out that the solutions with ω representing the energy E and the wave vector $\vec{k}$ representing the momentum $\vec{p}$ do satisfy the energy-momentum relationship of Special Theory of Relativity (STR). The solutions with $\omega_{\pm}$ describe spinless and massive elementary particles such as the pions, kaons and Higgs bosons [7].

Standard local quantum field theories are based either on first or second-order differential equations [3,4,9,27], however, higher derivatives (HD) do arise in various contexts of such theories. For example, in the electro-weak theory the HD arise in the gauge-fixing terms [23]. They also appear in Higgs theories [7], in quantum corrections for black holes [27], in quantum theories of gravitation [23,30,31], in modified Einstein's gravitation [32,33], and in general covariant Horava-Lifshitz gravity [34,35]. A third-order derivative also appears in the classical Lorentz-Dirac equation, which describes a radiating point-like charge [36].

5 Fundamental Higher Derivative Equations with Even Order

5.1 Relationship to Klein-Gordon equation

The higher derivative (HD) fundamental dynamical equations with even powers are given by Eq. (6). It is easy to show that this equation can be written as

$$\left[(\partial^{\mu}\partial_{\mu})^m - \omega_0^{2m}\right]\psi = \left[(\partial^{\mu}\partial_{\mu}) + \omega_0^2\right][\text{other terms}]\psi = 0, \quad (9)$$

which is valid for m being any positive even integer.

Our main result obtained here is that all Poincaré invariant HD dynamical equations with even powers do explicitly display the original KG equation. This is an interesting result, however, it must be kept in mind that the structure of the Poincaré invariant HD equations is richer than the KG equation, hence, they may account for physical phenomena that the original KG equation cannot (see below).

5.2 Quantization of fundamental equation with $m = 2$

We now use the higher derivative fundamental equation given by Eq. (6) with $m = 2$ to develop the corresponding quantum field theory. The resulting fourth-order fundamental equation can be written as

$$\left[(\partial^\mu \partial_\mu)^2 - \omega_0^4 \right] \psi(t, \vec{x}) = 0 , \quad (10)$$

where $\psi(t, \vec{x})$ is a scalar wave function. The Lagrangian $\mathcal{L}$ for this equation is

$$\mathcal{L} = \frac{1}{2} \left[(\partial^\mu)^2 \psi (\partial_\mu)^2 \psi + \omega_0^4 \psi^2 \right] . \quad (11)$$

This Lagrangian is a special case of the most general form of the Poincaré invariant Lagrangians for scalar fields given by Weldon [23] (see his Eq. (1.2)).

Making a Fourier transform in space

$$\psi(t, \vec{x}) = \int \frac{d^3 k}{(2\pi)^3} \phi(t) e^{i\vec{k} \cdot \vec{x}} , \quad (12)$$

we obtain

$$\frac{d^4 \phi}{dt^4} + 2k^2 \frac{d^2 \phi}{dt^2} + (k^4 - \omega_0^4) \phi = 0 , \quad (13)$$

with the corresponding Lagrangian

$$\mathcal{L} = \frac{1}{2} \left[\left(\frac{d^2 \phi}{dt^2} \right)^2 - 2k^2 \left(\frac{d\phi}{dt} \right)^2 + (k^4 - \omega_0^4) \phi^2 \right] . \quad (14)$$

To find solutions of Eq. (13), we take $\phi(t) = A e^{i\omega t}$, where A and ω are constants to be determined. The allowed solutions are those with the following frequencies: $\omega_{1\pm} = \pm \sqrt{k^2 + \omega_0^2}$ and $\omega_{2\pm} = \pm \sqrt{k^2 - \omega_0^2}$. Thus, we can write

$$\phi(t) = \sum_{r=1}^2 \left(a_r e^{-i\omega_r t} + b_r e^{i\omega_r t} \right) , \quad (15)$$

where a_r and b_r represent A for different values of ω , and ω_r is either ω_1 or ω_2 .

Having obtained the solution for $\phi(t)$, we now promote ϕ to be an operator $\hat{\phi}$. In addition, we introduce $\hat{\phi}_1(t)$ given by

$$\hat{\phi}_1(t) = \hat{\phi}(t) = \sum_{r=1}^2 \left(\hat{a}_r e^{-i\omega_r t} + \hat{b}_r e^{i\omega_r t} \right) , \quad (16)$$

and $\hat{\phi}_2(t)$ defined as

$$\hat{\phi}_2(t) = \frac{d\hat{\phi}}{dt} = \sum_{r=1}^2 \left[\hat{a}_r (-i\omega_r) e^{-i\omega_r t} + \hat{b}_r (i\omega_r) e^{i\omega_r t} \right] . \quad (17)$$

We calculate the momenta $\hat{\Pi}_1$ and $\hat{\Pi}_2$ by using Eq. (2.12) given by Weldon [21], and obtain

$$\hat{\Pi}_1 = \sum_{r=1}^2 \left[\omega_r (2k^2 + \omega_r^2) \left(-i\hat{a}_r e^{-i\omega_r t} + i\hat{b}_r e^{i\omega_r t} \right) \right] , \quad (18)$$

and

$$\hat{\Pi}_2 = - \sum_{r=1}^2 \omega_r^2 \left(\hat{a}_r e^{-i\omega_r t} + \hat{b}_r e^{i\omega_r t} \right) . \quad (19)$$

The following commutation relationships are obeyed: $[\hat{\Pi}_j, \hat{\phi}_l] = -i\delta_{jl}$, $[\hat{\phi}_j, \hat{\phi}_l] = 0$ and $[\hat{\Pi}_j, \hat{\Pi}_l] = 0$, where $j = 1$ and 2 , and $l = 1$ and 2 .

Lagrange mechanics was generalized to higher derivatives by Ostrogradski [37], who defined the Hamiltonian $\hat{H}$ as

$$\hat{H} = -\hat{\mathcal{L}} + \hat{\Pi}_1 \hat{\phi}_2 + \hat{\Pi}_2 \frac{d^2 \hat{\phi}_1}{dt^2} , \quad (20)$$

which becomes

$$\hat{H} = 2 \sum_{r=1}^2 \left[\omega_r^2 (k^2 + \omega_r^2) \left(\hat{a}_r \hat{b}_r + \hat{b}_r \hat{a}_r \right) \right] . \quad (21)$$

Introducing

$$\frac{1}{R_r} = 4\omega_r (k^2 + \omega_r^2) , \quad (22)$$

and using $[\hat{a}_r, \hat{b}_s] = \delta_{rs} R_s$, we obtain

$$\hat{H} = \sum_{r=1}^2 \omega_r \left(\frac{\hat{b}_r \hat{a}_r}{R_r} + \frac{1}{2} \right) , \quad (23)$$

with the following commutators being satisfied: $[\hat{H}, \hat{a}_r] = -\omega_r \hat{a}_r$, $[\hat{H}, \hat{b}_r] = \omega_r \hat{b}_r$ and $[\hat{H}, \hat{\phi}_n] = -id\hat{\phi}_n/dt$, where $n = 1$ and 2 .

Let us now assume that the field operators $\hat{\phi}_1$ and $\hat{\phi}_2$ are self-adjoint, then we have $\hat{b}_r = \hat{a}_r^\dagger$ with $[\hat{a}_r, \hat{a}_s^\dagger] = \delta_{rs} R_s$. Moreover, we consider only positive and physically acceptable (real) ω_r . Since ω_{1+} is the only frequency that satisfies these conditions, we write the Hamiltonian as

$$\hat{H} = \sum_{s=1}^2 \omega_s \left(\frac{\hat{a}_s^\dagger \hat{a}_s}{R_s} + \frac{1}{2} \right), \quad (24)$$

where $\omega_s \equiv \omega_{1+}$, and $\hat{a}_s^\dagger$ and $\hat{a}_s$ are the creation and annihilation operators corresponding to this frequency.

Following Weldon [23], we define the natural vacuum $|\text{vac}\rangle$ of the Fock space as $\hat{a}_s |\text{vac}\rangle = 0$ with the zero-point energy $\hat{H} |\text{vac}\rangle = E_0 |\text{vac}\rangle$, where $E_0 = \omega_s/2$. For a one-particle state, we have

$$\hat{H} \hat{a}_s^\dagger |\text{vac}\rangle = (\omega_s + E_0) \hat{a}_s^\dagger |\text{vac}\rangle, \quad (25)$$

with the norm being given by

$$\langle \text{vac} | \hat{a}_s \hat{a}_s^\dagger | \text{vac} \rangle = R_s. \quad (26)$$

5.3 Discussion

Our quantization of the fourth-order Poincaré invariant dynamical equation shows that this equation has four solutions corresponding to $\omega_{1\pm} = \pm \sqrt{k^2 + \omega_0^2}$ and $\omega_{2\pm} = \pm \sqrt{k^2 - \omega_0^2}$. The resulting solutions are consequences of the metric. Moreover, the eigenvalue equations (see Eq. 1) require that $\omega_{1\pm}$ and $\omega_{2\pm}$ are real. Assuming that $\omega_0^2 > 0$, the two solutions given by $\omega_{1\pm}$ are real and they describe spinless, massive particles of ordinary matter (see Sec. 4); specifically, the solutions with ω_{1-} and ω_{1+} represent particles (positive energy) and antiparticles (negative energy), respectively.

Now, $\omega_{2\pm}$ can only be real if $k^2 > \omega_0^2$, however, for typical physical conditions ω_0^2 is larger than k^2 , which means that $\omega_{2\pm}$ is complex. The problem can be formally solved by postulating that ω_0 is imaginary, namely, $\omega_0 = i\tilde{\omega}_0$. This gives $\omega_{2\pm} = \pm \sqrt{k^2 + \tilde{\omega}_0^2}$, which is real and consistent with the eigenvalue equations, however, as a result of our assumption $\omega_{1\pm}$ becomes complex. Since ω_0 represents mass of an elementary particle, the imaginary ω_0 cannot describe any ordinary matter. Instead, the solutions with $\tilde{\omega}_0$ represent hypothetical particles whose masses are imaginary and whose travel velocities are superluminal. Such particles are known as tachyons [38] but so far there is no experimental evidence for their existence. Nevertheless, tachyonic fields become important in some versions of string theory [39-41] and they have also been used in attempts to explain dark energy [42-45].

The above results can also be related to the energy-momentum relationship established by STR. With ω_1 representing the energy E , the wave vector $\vec{k}$ representing the momentum $\vec{p}$, and $\omega_0^2 > 0$, the solutions $\omega_{1\pm}$ do satisfy the STR energy-momentum relationship, and they describe spinless and massive elementary particles such as the pions, kaons and Higgs bosons [7]. However, the solutions $\omega_{2\pm}$ are not consistent with the energy-momentum relationship of STR unless $\omega_0^2 < 0$ is assumed and tachyons are introduced [37]. Another interesting alternative for the solutions $\omega_{2\pm}$ is recently formulated Einstein's Special Relativity beyond the speed of light [46].

The results obtained in Sec. 5.2 can be easily generalized to any value of m , and the general solutions correspond to

$$\omega_{L\pm} = \pm \sqrt{k^2 + \omega_0^2 e^{i2l\pi/L}} , \quad (27)$$

where $l = 0, 1, 2, \dots, L-1$. The solutions with $\omega_{1\pm}$ and $\omega_{2\pm}$ are obtained for $L = 1$ or $l = 0$, and $L = 2$ or $l = 1$. However, for all other values of L , the frequencies $\omega_{L\pm}$ are complex conjugates, which means that they are inconsistent with our requirement that $\omega_{L\pm}$ must be real (see Sec. 2).

This is an interesting result as it clearly shows that every fundamental dynamical equation of the set of infinite Poincaré invariant equations with $m > 2$ (see Eq. 6) has the solutions corresponding to $\omega_{1\pm}$ and $\omega_{2\pm}$, which are the same solutions as those obtained for the fundamental dynamical equation with $m = 2$. As already stated in Sec. 2, the requirement imposed by the eigenvalue equations is that the values of all ω 's must be real, which means that the complex conjugate solutions cannot be considered in the approach presented here. Hence, we have reached an important conclusion that among the infinite set of fundamental dynamical equations with the even values of m , the invariant equation with $m = 2$ should be preferentially used in constructing HD quantum field theories.

The explicit form of the fundamental dynamical equation with $m = 2$ is

$$(\partial^\mu \partial_\mu)^2 \psi - \omega_0^4 \psi = 0 . \quad (28)$$

The fact that the solutions with $\omega_{1\pm}$ to this equation are the same as those given by the original KG equation (see Sec. 4) should not be surprising because our results presented in Sec. 5.1 established the relationship between both equations. On the other hand, the order and form of the above equation are different than the KG equation and, as a result, Eq. (28) also allows for tachyonic solutions with $\omega_{2\pm}$. These solutions are relevant to the faster than the speed of light Special Relativity [46] and they may also be of interests in string theories, and in some recent attempts to explain the nature of dark energy. Moreover, the solutions may play some role in accounting for non-localities of quantum mechanics.

6 Fundamental Higher Derivative Equations with Odd Orders

6.1 Relationship to Klein-Gordon equation

The HD fundamental dynamical equations given by Eq. (7) look similar to the original KG equation. Let us now demonstrate that each equation of the infinite set of fundamental equations explicitly displays the KG equation. To show this, we factorize Eq. (7) by using the following general formula

$$\begin{aligned} & \left[(\partial^\mu \partial_\mu)^m - \omega^{2m} \right] \psi = \left[(\partial^\mu \partial_\mu) + \omega^2 \right] \\ & \times \left[(\partial^\mu \partial_\mu)^{n-1} + \sum_{k=1}^{n-2} (-1)^k \omega_0^{2k} (\partial^\mu \partial_\mu)^{n-1-k} + \omega_0^{2(n-1)} \right] = 0, \end{aligned} \quad (29)$$

which is valid for n being any odd integer greater than 1.

This well-known algebraic formula guarantees that each Poincaré invariant HD dynamical equation derived with $n > 1$ contains the original KG equation, which is obtained when $n = 1$.

6.2 Quantization of fundamental equation with $n = 3$

The method of quantizing the fourth-order fundamental equation presented in Sec. 5.1 can be now easily extended to the six-order fundamental dynamical equations obtained from Eq. (7) by taking $n = 3$. The result is

$$\left[(\partial^\mu \partial_\mu)^3 + \omega_0^6 \right] \psi(t, \vec{x}) = 0, \quad (30)$$

with the corresponding Lagrangian

$$\mathcal{L} = \frac{1}{2} \left[(\partial^\mu)^3 \psi (\partial_\mu)^3 \psi - \omega_0^6 \psi^2 \right]. \quad (31)$$

After making Fourier transforms in space (see Eq. 18), the six possible solutions must have the following frequencies: $\omega_1^\pm = \pm \sqrt{k^2 + \omega_0^2}$, $\omega_2^\pm = \pm \sqrt{k^2 + \omega_0^2 e^{2\pi i/3}}$, and $\omega_3^\pm = \pm \sqrt{k^2 + \omega_0^2 e^{-2\pi i/3}}$, which give

$$\hat{\phi}_1(t) = \hat{\phi}(t) = \sum_{r=1}^3 \left(\hat{a}_r e^{-i\omega_r t} + \hat{b}_r e^{i\omega_r t} \right), \quad (32)$$

$$\hat{\phi}_2(t) = \frac{d\hat{\phi}}{dt} = \sum_{r=1}^3 \left[\hat{a}_r (-i\omega_r) e^{-i\omega_r t} + \hat{b}_r (i\omega_r) e^{i\omega_r t} \right], \quad (33)$$

and

$$\hat{\phi}_3(t) = \frac{d^2 \hat{\phi}}{dt^2} = \sum_{r=1}^3 \left[\hat{a}_r (-\omega_r^2) e^{-i\omega_r t} + \hat{b}_r (-\omega_r^2) e^{i\omega_r t} \right] . \quad (34)$$

The momenta $\hat{\Pi}_1$, $\hat{\Pi}_2$ and $\hat{\Pi}_3$ are given by

$$\hat{\Pi}_1 = \sum_{r=1}^3 \left[\omega_r (3k^4 - 3k^2 \omega_r^2 + \omega_r^4) \left(i\hat{a}_r e^{-i\omega_r t} - i\hat{b}_r e^{i\omega_r t} \right) \right] , \quad (35)$$

$$\hat{\Pi}_2 = \sum_{r=1}^3 \left[\omega_r^2 (\omega_r^2 - 3k^2) \left(\hat{a}_r e^{-i\omega_r t} + \hat{b}_r e^{i\omega_r t} \right) \right] , \quad (36)$$

and

$$\hat{\Pi}_3 = \sum_{r=1}^3 \left[(-i\omega_r^2) \left(\hat{a}_r e^{-i\omega_r t} - \hat{b}_r e^{i\omega_r t} \right) \right] . \quad (37)$$

The commutation relationships obeyed by the function $\hat{\phi}_l$ and the momenta $\hat{\Pi}_j$, where $l = 1, 2$ and 3 , and $j = 1, 2$ and 3 , are the same as those given below Eq. (25). The method of evaluating the Hamiltonian is also the same (see Eq. 26), and the resulting Hamiltonian is

$$\hat{H} = \sum_{r=1}^3 \left[-3\omega_r (k^4 - 2k^2 \omega_r^2 + \omega_r^4) \left(\hat{a}_r \hat{b}_r + \hat{b}_r \hat{a}_r \right) \right] . \quad (38)$$

We now define

$$\frac{1}{R_r} = -6\omega_r (k^4 - 2k^2 \omega_r^2 + \omega_r^4) , \quad (39)$$

and obtain

$$\hat{H} = \sum_{r=1}^3 \omega_r \left(\frac{\hat{b}_r \hat{a}_r}{R_r} + \frac{1}{2} \right) , \quad (40)$$

with the same commutators being satisfied as those given below Eq. (23).

With the assumption that the field operators $\hat{\phi}_1$, $\hat{\phi}_2$ and $\hat{\phi}_3$ are self-adjoint, we get $\hat{b}_r = \hat{a}_r^\dagger$. By taking into account only real and positive frequency $\omega_s \equiv \omega_{1+}$, we obtain the same results as those given by Eqs (24) through (26).

6.3 Discussion

The solutions of the sixth-order Poincaré invariant dynamical equation correspond to $\omega_{1\pm} = \pm\sqrt{k^2 + \omega_0^2}$, $\omega_{2\pm} = \pm\sqrt{k^2 + \omega_0^2 e^{2\pi i/3}}$, and $\omega_{3\pm} = \pm\sqrt{k^2 + \omega_0^2 e^{-2\pi i/3}}$. The obtained solutions are consequences of the metric. The requirement imposed by the eigenvalue equations (see Eq. 1) that the solutions must be real is obeyed only by $\omega_{1\pm}$ if $\omega_0^2 > 0$. However, $\omega_{2\pm}$ and $\omega_{3\pm}$ are complex conjugate. We may introduce $\omega_0 = i\sqrt{2}\tilde{\omega}_0$ and write $\omega_{2\pm} = \pm\sqrt{k^2 + (1 + i\sqrt{3})\tilde{\omega}_0^2}$ and

$\omega_{3\pm} = \pm\sqrt{k^2 + (1 - i\sqrt{3})\tilde{\omega}_0^2}$, which show that $\omega_{2\pm}$ and $\omega_{3\pm}$ are also complex conjugate with $\omega_0^2 < 0$. Since neither $\omega_{2\pm}$ nor $\omega_{3\pm}$ satisfies the requirement imposed by the eigenvalue equations, we shall not consider these solutions.

It must also be pointed out that the solutions with $\omega_1^\pm$ are consistent with STR and that they describe spinless and massive elementary particles of ordinary matter (see Sec. 4). However, the solutions with $\omega_{2\pm}$ and $\omega_{3\pm}$ are complex conjugate and they are not consistent with STR.

The results obtained for the sixth-order fundamental equation can be easily generalized to higher odd values of n , and it can be shown that the general solutions are given by

$$\omega_{L\pm} = \pm\sqrt{k^2 + \omega_0^2 e^{i2l\pi/L}}, \quad (41)$$

where $l = 0, 1, 2, \dots, L-1$.

The solution with $\omega_{1\pm}$ is obtained for $L = 1$ or $l = 0$, and this solution is the same as that found for the original KG equation (see Sec. 4). The result is actually consistent with that obtained in Sec. 6.1, where we demonstrated that all Poincaré invariant dynamical equations with odd values $n \geq 3$ are related the original KG equation. Now, the fundamental equations with $n > 3$ have other solutions but all of them are complex conjugates, which means that they are inconsistent with the requirement imposed by the eigenvalue equations. As a result, all complex conjugate solutions are not acceptable physical solutions in our approach presented in this paper.

Hence, we have reached an interesting conclusion that all the HD equations of the infinite set of fundamental dynamical equations with odd values of $n \geq 1$ have the same physically acceptable solutions as the original KG equation. This simply means that there are no new HD fundamental dynamical equations in this set that could be used to construct HD quantum field theories.

7 Invariant frequencies for selected particles

According to Standard Model [3], there are no elementary particles with spin zero; however, the most recent data collected by the LHC experiment at CERN seem to indicate that spin-zero Higgs boson does exist [47]. In quantum field theories (QFT), the pions π^0 and $\pi^\pm$, and the kaons K^0 , $K^\pm$ and $\bar{K}^0$, are considered as spin zero particles, and described by scalar state functions despite the fact that they are composites made out of quarks [3,4,9,27].

The derived Poincaré invariant dynamical equations have only one parameter, which is the invariant frequency ω_0 [48]. Since this parameter represents the physical property of elementary particles known as mass, it is important to know its value for different particles [13]. The values of ω_0 for the pions and kaons are given in Table 1. The values of mass m in $[MeV/c^2]$ were

Particle	m (MeV/c^2)	ω_0 (s^{-1})
Pion π^0	134.9766(6)	$2.050656(9) \times 10^{23}$
Pion $\pi^\pm$	139.57018(35)	$2.1204445(55) \times 10^{23}$
Kaon K^0	497.614(24)	$7.56009(36) \times 10^{23}$
Kaon $K^\pm$	493.677(16)	$7.50027(55) \times 10^{23}$

Table 1: The classical mass m and the invariant frequency ω_0 for the pions and kaons; the values of m were taken from [45].

obtained from [49] and we computed ω_0 [1/s] by using the conversion factor $1eV = 1.602176487 \times 10^{-10} J$ and $\hbar = 1.054571628 \times 10^{-34} J s$ after the CODATA 2008 paper [50].

8 Conclusions

We considered space-time with the Minkowski metric and used the structure of the proper orthochronous Poincaré group $P_+^\uparrow$ of this metric to obtain the eigenvalue equations, which guarantee that a scalar and analytical state function transforms as the irreps of the invariant subgroup of space-time translations $T(3+1)$ of $P_+^\uparrow$. We used the eigenvalue equations to derive Poincaré invariant (fundamental) dynamical equations for a scalar and analytical state function, which represents free, spinless and massive elementary particles such as the pions, kaons and Higgs bosons. Since all Poincaré invariant dynamical equations depend on only one physical parameter, which is the so-called invariant frequency ω_0 and represents the physical property of elementary particles known as mass, we computed values of this frequency for the pions and kaons that are considered here as elementary particles.

The main results obtained in this paper are: (i) the eigenvalue equations allow deriving two infinite sets of the Poincaré invariant dynamical equations; (ii) in one of these sets, there is an infinite number of *even order* fundamental higher derivative (HD) dynamical equations; (iii) in the other set, there is an infinite number of *odd order* fundamental HD dynamical equations, with the lowest order equation being the original Klein-Gordon equation; (iv) the original Klein-Gordon equation is the only fundamental dynamical equation available to construct local quantum field theories; (v) all the fundamental HD dynamical equations in both sets can be factored to explicitly display the Klein-Gordon equation; (vi) all physically acceptable solutions to the *odd order* fundamental HD dynamical equations are the same as the solutions of the Klein-Gordon equation; (vii) all physically acceptable solutions to the *even order* fundamental HD dynamical equations are the same as the solutions of the lowest even order equation in this set; (viii) the lowest even order equation is the only new fundamental HD equation that can be used to construct HD

quantum field theories; (ix) possible applications of these HD theories may range from string theories and modeling of dark energy to non-localities in quantum mechanics.

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