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Dirac Equation with Self Interaction Induced by Torsion: Minkowski Space-Time

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Abstract

A formulation of Dirac equation with non linear self interacting term induced by torsion is studied in Minkowski space-time. The equation is made explicit in Cartesian coordinates by a null tetrad frame whose corresponding van der Waerden matrices are the Pauli matrices. Plane waves are shown to be solution. They are subject to a constraint relation that leave only two of them arbitrary. They are determined in a special case. In the two dimensional space-time one is left with a non linear Dirac equation whose solution is ensured by existing results. Analytical solutions are determined in form of standing waves. In the mass less case there are solutions that propagate with periodic "burst" of amplitude. The paper is also an improvement of a previous study.

Keywords: Non linear Dirac equation; torsion; Minkowski space-time; plane waves; standing waves

1 Introduction

Recently the Dirac equation with self interaction induced by torsion has been re-considered [16]. The argument has been developed by the two spinor formalism and notations of [10]. The procedure according to which that study was performed is in the line of the massive (torsion free) spin field equation in curved space-time of [8]. Accordingly the Dirac equation with torsional self interaction has been obtained by the Euler Lagrange equation from a Lagrangian with an interacting term between the Dirac and the torsion field. Variations have been there performed with respect to the Dirac spinor and the components the torsion spinor. No variation has been done with respect to the metric tensor $g_{\mu\nu}$.

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Such variation would produce an Einstein-Dirac equation whose solution have problematic aspects. Indeed it has been directly checked, in the torsion free Robertson-Walker space-time case, that the Einstein-Dirac equation does not admit standard solutions [15]. It has also been shown that coupling Dirac matter field with electrodynamics and gravitation with torsion is not compatible in spherically symmetric space-time (e. g., [6, 5] and references therein). The resulting equation according to [16] has the form

$$\Big[\nabla_{AB'} + i\frac{3}{4}\alpha^2 J_{AB'}\Big]P^A - i\mu_* Q_{A'} = 0 \tag{1}$$

$$\Big[\nabla_{AB'} + i\frac{3}{4}\alpha^2 J_{AB'}\Big]Q^{B'} + i\mu_* P_A = 0 \tag{2}$$

where $J^{AA'}$ is the spinor current $J^{AA'} = (1/\sqrt{2})(P^A\overline{P}^{A'} + \overline{Q}^A Q^{A'})$, α the interacting parameter between Dirac and torsion field. $\nabla_{AA'}$ is the usual unique torsion free covariant derivative induced by $g_{\mu\nu}$. Equivalently, the equation can be written

$$\nabla_{AB'}P^A - \frac{i}{\sqrt{2}}\Big(m_o + \frac{3}{4}\alpha^2\,\overline{Q}^A P_A\Big)Q_{B'} = 0 \tag{3}$$

$$\nabla_{AB'}Q^{B'} + \frac{i}{\sqrt{2}}\Big(m_o + \frac{3}{4}\,\alpha^2\,Q^{B'}\overline{P}_{B'}\Big)P_A = 0 \tag{4}$$

The equation is the two spinor equivalent (see e. g., [12]) of the 4-spinor standard formulation of Dirac equation with torsion derived from a Lagrangian locally invariant under special coordinate change and Lorentz rotations [9].

The validity of the equation is quite general. Unfortunately the solution does not seem easy even in simple examples of space-time. Explicit solutions could be however of interest in connection, e. g., to neutrino oscillation [1, 17, 14].

In the present paper the equation is studied in Minkowski space-time in Cartesian coordinates. Such study is an improvement of a previous one proposed in [13] for a formulation of Dirac equation with torsion in the line of [3]. The procedure employed here is a canonical one in the two-spinor formalism of Ref. [10]. It is shown that plane waves are solutions if their coefficients, that are determined in a special case, satisfy a suitable constraint. Existence of other kind of solutions is discussed in $1+1$ dimensions. Solutions of L^2-class there exist according to the convergence of a difference scheme proposed in the literature [2, 7]. Also analytical solutions exist. In general they have the form of standing waves whose asymptotic behavior is made explicit and that are shown to propagate in the mass less case.

2 Minkowski space-time

Suppose now $g_{\mu\nu} = diag\ \{1, -1, -1, -1\}$ and choose the null tetrad frame

$$l^\mu = 2^{-\frac{1}{2}}(1, 0, 0, 1), \qquad m^\mu = 2^{-\frac{1}{2}}(0, 1, -i, 0) \tag{5}$$

$$n^\mu = 2^{-\frac{1}{2}}(1, 0, 0, -1), \qquad m^{*\mu} = 2^{-\frac{1}{2}}(0, 1, i, 0) \tag{6}$$

The corresponding Infeld-van der Waerden symbols $g_a^{AB'}$ comes out to be related to the Pauli matrices σ_a. Precisely

$$\left(g_a^{AB'}\right)^\top = g^a_{AB'} = \frac{1}{\sqrt{2}}\sigma_a, \quad a = 0, 1, 2, 3, \quad \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{7}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & o \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{8}$$

By considering that $\nabla_{AA'} = g^a_{AA'}\partial_a$, equations (3), (4) read

$$g^a_{AA'}\partial_a P^A - \frac{i}{\sqrt{2}}\Big[m_0 - \frac{3}{4}\alpha^2 \overline{Q}_A P^A\Big] Q_{A'} = 0 \tag{9}$$

$$g^a_{AA'}\partial_a Q^{A'} + \frac{i}{\sqrt{2}}\Big[m_0 - \frac{3}{4}\alpha^2 Q_{A'} \overline{P}^{A'}\Big] P_{A'} = 0 \tag{10}$$

The equation admits of plane waves solutions:

$$P^A = u^A(k^a)\, e^{ik_a x^a}, \qquad Q_{A'} = v_{A'}(k^a)\, e^{ik_a x^a} \tag{11}$$

Inserting into (9), (10) by using (7), the u^A's and $v_{A'}$'s obey to

$$(k^a\sigma_a)^{A'A} u_A + [m_o + \frac{3}{4}\alpha^2 \overline{v}^A u_A] v^{A'} = 0 \tag{12}$$

$$(k_a\sigma_a)_{AA'} v^{A'} + [m_o + \frac{3}{4}\alpha^2 v^{A'} \overline{u}_{A'}] u_A = 0 \tag{13}$$

where summation over a is understood both in (12) and (13). By solving (12) with respect to $v_{A'}$ and inserting into (13) one finally gets the condition

$$k_0^2 - k_i^2 = |m_o + \frac{3}{4}\alpha^2\, v^{A'}(k)\overline{u}_{A'}(k)|^2 \tag{14}$$

where the property $\sigma_i\sigma_l = \epsilon_{ilh}\sigma_h,\ (i \neq l)$ of the Pauli matrices has been used.

Other constraints on the coefficients $u^A(k)$, $v_{A'}(k)$ can be made explicit by developing (9), (10) with (7), (8) to obtain:

$$(\partial_0 + \partial_3)P^0 + (\partial_1 + i\partial_2)P^1 - i[m_o + \frac{3}{4}\alpha^2 \overline{Q}^A P_A] Q_{0'} = 0 \tag{15}$$

$$(\partial_1 - i\partial_2)P^0 + (\partial_0 - \partial_3)P^1 - i[m_o + \frac{3}{4}\alpha^2 \overline{Q}^A P_A] Q_{1'} = 0 \tag{16}$$

$$(\partial_0 + \partial_3)Q^{0'} + (\partial_1 - i\partial_2)Q^{1'} + i[m_o + \frac{3}{4}\alpha^2 Q^{A'} \overline{P}_{A'}] P_0 = 0 \tag{17}$$

$$(\partial_1 + i\partial_2)Q^{0'} + (\partial_0 - \partial_3)Q^{1'} + i[m_o + \frac{3}{4}\alpha^2 Q^{A'} \overline{P}_{A'}] P_1 = 0 \tag{18}$$

By using (11) in (15)-(18) and taking the complex conjugate of last two resulting equations there follows

$$(k_0 + k_3)u^0 + (k_1 + ik_2)u^1 - [m_o + \frac{3}{4}\alpha^2\overline{v}^A u_A]v_{0'} = 0 \tag{19}$$

$$(k_1 - ik_2)u^0 + (k_0 - k_3)u^1 - [m_o + \frac{3}{4}\alpha^2\overline{v}^A u_A]v_{1'} = 0 \tag{20}$$

$$(k_0 + k_3)\overline{v}^0 + (k_1 + ik_2)\overline{v}^1 + [m_o + \frac{3}{4}\alpha^2\overline{v}^A u_A]\overline{u}_{0'} = 0 \tag{21}$$

$$(k_1 - ik_2)\overline{v}^0 + (k_0 - k_3)\overline{v}^1 + [m_o + \frac{3}{4}\alpha^2\overline{v}^A u_A]\overline{u}_{1'} = 0 \tag{22}$$

By performing in (19)-(22) the substitutions

$$w^A = \overline{v}^A, \quad u^A \to \epsilon u^A, \quad w^A \to \epsilon w^A, \quad \epsilon = (w^A u_A)^{-1/2} \tag{23}$$

and taking the complex conjugate of the last two equations one has

$$\begin{pmatrix} k_0 + k_3 & k_1 + ik_2 & 0 & C \\ k_1 - ik_2 & k_0 - k_3 & -C & 0 \\ 0 & -C & k_0 + k_3 & k_1 - ik_2 \\ C & 0 & k_1 + ik_2 & k_0 - k_3 \end{pmatrix} \begin{pmatrix} u^0\epsilon \\ u^1\epsilon \\ \overline{w^0\epsilon} \\ \overline{w^1\epsilon} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{24}$$

where $C = m_o + (3/4)\alpha^2$. The condition for the existence of non trivial solutions $u^A\epsilon$, $\overline{w^A\epsilon}$ is the vanishing of the determinant of the matrix of the coefficients in (24). This gives

$$\left[k_o^2 - k_1^2 - k_2^2 - k_3^2 - \left(m_o^2 + \frac{3}{4}\alpha^2\right)\right]^2 = 0 \tag{25}$$

that is the condition (14) for $\overline{v}^A u_A = 1$ as it is indeed. One can also check that the matrix in (24) has rank 2. Therefore only two of the four expressions $u^A\epsilon$, $\overline{w^A\epsilon}$ are arbitrary. A property of the plane waves solutions that it is worth to mention is that they identically satisfy the current conservation equation $\nabla_{AA'}J^{AA'} = 0$, that holds in general for the Dirac equation (1), (2) (see e.g., [16]). In the present case that property is easily checked from (11) and the very definition of $J^{AA'}$:

$$\nabla_{AA'}J^{AA'} = \frac{1}{\sqrt{2}} g^a_{AA'}\partial_a(u^A\overline{u}^{A'} + v^{A'}\overline{v}^A) = 0 \tag{26}$$

The lack of linearity does not allow to construct solutions (e.g., of L^2-class) by superposition of plane waves. Other solutions, however, do exist as it will be seen in the following.

3 Explicit value of the coefficients

Suppose $u^1 = v_{1'} = 0$ and $u^0 = \overline{u^0}$, $v_0 = \overline{v_0}$. One has from (20), (21) $k_1 = k_2 = 0$. The surviving equations (19), (22) give then

$$u^0 = \pm\, v_0 \sqrt{\frac{k_0 - k_3}{k_0 + k_3}} \tag{27}$$

By distinguishing according to the plus and minus signs in (27) one obtains

$$u^{0(+)} = \sqrt{\frac{4}{3\alpha^2}\Big[m_o \sqrt{\frac{k_0 - k_3}{k_0 + k_3}} - (k_0 - k_3)\Big]} \tag{28}$$

$$v_0^{(+)} = \sqrt{\frac{4}{3\alpha^2}\Big[m_o \sqrt{\frac{k_0 + k_3}{k_0 k_3}} - (k_0 + k_3)\Big]} \tag{29}$$

$$u^{0(-)} = \mp \sqrt{\frac{4}{3\alpha^2}\Big[-m_o \sqrt{\frac{k_0 - k_3}{k_0 + k_3}} - (k_0 - k_3)\Big]} \tag{30}$$

$$v_0^{(-)} = \pm \sqrt{\frac{4}{3\alpha^2}\Big[-m_o \sqrt{\frac{k_0 - k_3}{k_0 + k_3}} - (k_0 + k_3)\Big]} \tag{31}$$

$$u^{0(\pm)} v_0^{(\pm)} = \frac{4}{3\alpha^2}\Big| m_o \mp \sqrt{k_0^2 - k_3^2}\Big| \tag{32}$$

Therefore the coefficients are completely determined as it must be according to the previous results.

4 Reduction to 1+1 dimensions

As to the problem of the existence of non plane waves solutions a positive answer is given by the following considerations. Suppose

$$P^A \equiv (P^0, P^1), \qquad Q_{A'} \equiv (P^1, P^0) \tag{33}$$

then equations (17), (18) are a duplicate of (15), (16). By further assuming $P^0 \equiv P^0(t,z)$, $P^1 \equiv P^1(t,z)$ in (15) , (18) one is left with

$$(\partial_t + \partial_z) P^0 = i[m_o - (3/4)\alpha^2 (P^0 \overline{P}^1 + P^1 \overline{P}^0)] P^1 \tag{34}$$

$$(\partial_t - \partial_z) P^1 = i[m_o - (3/4)\alpha^2 (P^0 \overline{P}^1 + P^1 \overline{P}^0)] P^0 \tag{35}$$

By summing and subtracting one obtains ($\psi_1 = P^0 + P^1$, $\psi_2 = P^0 - P^1$):

$$\partial_t \psi_1 + \partial_z \psi_2 + im\psi_1 + 2i\lambda(\psi_2 \overline{\psi}_2 - \psi_1 \overline{\psi}_1)\psi_1 = 0 \qquad (m = -m_0) \tag{36}$$

$$\partial_t \psi_2 + \partial_z \psi_1 - im\psi_2 + 2i\lambda(\psi_1 \overline{\psi}_1 - \psi_2 \overline{\psi}_{)}\psi_2 = 0 \qquad (2\lambda = -3\alpha^2/8) \tag{37}$$

that is a well known Dirac equation in 1+1 dimensions (e.g., [2]). The time dependence can be easily separated by the substitutions $\psi_1 \to \psi_1(z)\exp(ikt)$, $\psi_2 \to \psi_2(z)\exp(ikt)$ $(k \in \mathbf{R})$. One is then left with a pair of coupled nonlinear differential equations in the z variable.

The equation (36)-(37) has been studied by a difference scheme that converges in the discrete L^2-norm. Numerical applications of the difference scheme show that, according to initial data, there is a formation of a final state together with a solitary wave [2].

Here standing wave solution can be determined by setting:

$$\psi_1 = A(z)e^{-i\Lambda t}, \qquad \psi_2 = iB(z)e^{-i\Lambda t} \quad (A = \overline{A},\ B = \overline{B}) \tag{38}$$

From (36)-(38) A, B satisfy the coupled non linear equations

$$B' + (m-\Lambda)A - 2\lambda(A^2 - B^2)A = 0 \tag{39}$$
$$A' + (m+\Lambda)B - 2\lambda(A^2 - B^2)B = 0 \tag{40}$$

that give (se e. g., [11, 2, 7]):

$$A = \frac{4}{\alpha\sqrt{3}} \frac{\sqrt{(m_o^2-\Lambda^2)(m_0-\Lambda)}\cosh\left(z\sqrt{m_0^2-\Lambda^2}\right)}{[\Lambda\cosh\left(2z\sqrt{m_0^2-\Lambda^2}\right) - m_0]} \tag{41}$$

$$B = \frac{4}{\alpha\sqrt{3}} \frac{\sqrt{(m_0^2-\Lambda^2)(m_0+\Lambda)}\sinh\left(z\sqrt{m_0^2-\Lambda^2}\right)}{[\Lambda\cosh\left(2z\sqrt{m_0^2-\Lambda^2}\right) - m_0]} \tag{42}$$

$$\begin{aligned} P^0 &\overset{z\to\pm\infty}{\longrightarrow} \frac{2e^{-i\Lambda t}}{\alpha\Lambda\sqrt{3}} \left\{\sqrt{m_0-\Lambda} \pm i\sqrt{m_0+\Lambda}\right\} e^{-|z|\sqrt{m_0^2-\Lambda^2}} \\ &\overset{z\to\pm\infty}{\longrightarrow} 0 \qquad m_0 > |\Lambda| \end{aligned} \tag{43}$$

$$\begin{aligned} P^1 &\overset{z\to\pm\infty}{\longrightarrow} \frac{2e^{-i\Lambda t}}{\alpha\Lambda\sqrt{3}} \left\{\sqrt{m_0-\Lambda} \mp i\sqrt{m_0+\Lambda}\right\} e^{-|z|\sqrt{m_0^2-\Lambda^2}} \\ &\overset{z\to\pm\infty}{\longrightarrow} 0 \qquad m_0 > |\Lambda| \end{aligned} \tag{44}$$

Therefore P^0, P^1 vanishes exponentially for large $|z|$ if $m_0 > |\Lambda|$.

If $0 < m_0 < \Lambda$, by setting $P^0 \equiv P^+$, $P^1 \equiv P^-$ one has

$$\begin{aligned} P^{\pm} = &\frac{2e^{-i\Lambda t}\sqrt{\Lambda^2 - m_0^2}}{\alpha\sqrt{3}\left[\Lambda\cos\left(2z\sqrt{\Lambda^2-m_0^2}\right) - m_0\right]} \times \\ &\times \left\{\sqrt{\Lambda - m_0}\cos\left(z\sqrt{\Lambda^2-m_0^2}\right) \mp i\sqrt{\Lambda+m_0}\sin\left(z\sqrt{\Lambda^2-m_0^2}\right)\right\} \end{aligned} \tag{45}$$

If $\Lambda < -m_0 < 0$ one has

$$P^{\pm} = \frac{-2ie^{-i\Lambda t}\sqrt{\Lambda^2 - m_0^2}}{\alpha\sqrt{3}\left[|\Lambda|\cos\left(2z\sqrt{\Lambda^2-m_0^2}\right) + m_0\right]} \times$$

$$\times\left\{\sqrt{|\Lambda|+m_0}\cos\left(z\sqrt{\Lambda^2-m_0^2}\right)\pm i\sqrt{|\Lambda|-m_0}\sin\left(z\sqrt{\Lambda^2-m_0^2}\right)\right\} \quad (46)$$

If $m_0 = 0$ one obtains by the last results

$$P^0 = \frac{2}{\alpha}\sqrt{\frac{\Lambda}{3}}\,\frac{e^{-i\Lambda(z+t)}}{\cos(2z\Lambda)}, \qquad P^1 = \frac{2}{\alpha}\sqrt{\frac{\Lambda}{3}}\,\frac{e^{i\Lambda(z-t)}}{\cos(2z\Lambda)}, \qquad \Lambda > 0 \quad (47)$$

$$P^0 = \frac{-2i}{\alpha}\sqrt{\frac{|\Lambda|}{3}}\,\frac{e^{-i\Lambda(z+t)}}{\cos(2z\Lambda)}, \quad P^1 = \frac{-2i}{\alpha}\sqrt{\frac{|\Lambda|}{3}}\,\frac{e^{i\Lambda(z-t)}}{\cos(2z\Lambda)}, \quad \Lambda < 0 \quad (48)$$

In the static case ($\Lambda = 0$) one has:

$$P^{\pm} = \frac{2}{\alpha}\sqrt{\frac{m_0}{3}}(\cosh m_0 z \mp i\sinh m_0 z) \quad (49)$$

It is worth noting that in the massless case the solutions $P^0,\ P^1$ propagate in the opposite z-direction by periodically changing the amplitude in the spatial coordinate.

The main object of the present study was to point out that the nonlinear Dirac equation (1)-(4) introduced in [16] is physically meaningful. In Minkowski space-time, it contains a familiar nonlinear Dirac equation whose solution has been considered in the $1+1$ dimensions. Such equation has recently received increasing attention both in physics and mathematics (see, e.g., [7, 4] and references therein).

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