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Are $f(R)$ Theories of Gravity Necessary for Cosmology?

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Abstract

The $f(R)$ theories of gravity and their cosmological response over a space - time manifold of real dimension N are studied. As seen that the $f(R)$ theories are nothing more than cosmological models with exotic matter source and they are exactly equivalent to the Einstein's theory with cosmological constant, that is $R - 2\Lambda$.

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1 Introduction

A gravitation theory whose is a scalar field source was given by Brans and Dicke via Mach's principle [1]. Consider a space - time manifold of real N dimension with Levi - Civita connection: $\Gamma = \Gamma(g, \partial g)$, where g is the metric on the space - time manifold. If $R = R(g, \partial g, \partial^2 g)$ is Ricci scalar of this connection and ϕ is a smooth scalar over the manifold, then one can define a Lagrangian

$$\mathcal{L} = (-g)^{1/2} \left[F(\phi, R) - \omega(\phi) g^{\mu\nu} \mathcal{D}_\mu \phi \mathcal{D}_\nu \phi - V(\phi)\right] + \mathcal{L}_{matter}(g^{\mu\nu}), \qquad (1)$$

where $F(\phi, R)$ is the interacting term between R and ϕ, $\omega(\phi)$ is Brans - Dicke parameter, $V(\phi)$ is a potential function of the ϕ, $\mathcal{L}_{matter}$ is matter Lagrangian.

Therefore the field equations with respect to the metric and the scalar field read, respectively,

$$
\begin{aligned}
-\frac{1}{2}T_{\mu\nu} &= \frac{\partial F}{\partial R}R_{\mu\nu} - \frac{1}{2}F(\phi,R)g_{\mu\nu} - \mathcal{D}_\mu\mathcal{D}_\nu\frac{\partial F}{\partial R} + g_{\mu\nu}\Box\frac{\partial F}{\partial R} \\
&\quad +\omega(\phi)\left(\frac{1}{2}g_{\mu\nu}\mathcal{D}_\lambda\phi\mathcal{D}^\lambda\phi - \mathcal{D}_\phi\mu\mathcal{D}_\nu\phi\right) + \frac{1}{2}V(\phi)g_{\mu\nu}, \qquad (2)
\end{aligned}
$$

$$
\frac{\partial F}{\partial \phi} + \frac{d\omega}{d\phi}\mathcal{D}_\lambda\phi\mathcal{D}^\lambda\phi + 2\omega(\phi)\Box\phi - \frac{dV}{d\phi} = 0, \qquad (3)
$$

where $T_{\mu\nu}$ is the energy - momentum tensor of the metric:

$$
T_{\mu\nu} = -\frac{2}{(-g)^{1/2}}\frac{\delta\mathcal{L}_{matter}}{\delta g^{\mu\nu}}, \quad g = \det[g_{\mu\nu}]. \qquad (4)
$$

Selecting the Brans - Dicke parameter as $\omega(\phi) = 0$, the Lagrangian becomes

$$
\mathcal{L} = (-g)^{1/2}\left[F(\phi,R) - V(\phi)\right] + \mathcal{L}_{matter}(g^{\mu\nu}), \qquad (5)
$$

and so the field equation with respect to the ϕ one is rewritten as

$$
\frac{\partial F}{\partial \phi} - \frac{dV}{d\phi} = 0. \qquad (6)
$$

As easily seen that it must be $\phi = \phi(R)$. A similar ansatz can be seen in O'Hanlon's Lagrangian served in ref. [3]. Thus, one can write $F(\phi,R) = f(R)$ and in result the Lagrangian depend only on a general function of the scalar curvature R and the metric g:

$$
\mathcal{L} = -\frac{1}{16\pi G}(-g)^{1/2}f(R) + \mathcal{L}_{matter}(g^{\mu\nu}), \qquad (7)
$$

where $f(R)$ has to satisfy condition $\frac{d^2f}{dR^2} \neq 0$, on the contrary, it would be $f(R) = R - 2\Lambda$, that is it would reduction to the Einstein's theory. The field and the continuity equations of the Lagrangian in eq. (7) are, respectively, as follow

$$
\begin{aligned}
f'R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \mathcal{D}_\mu\mathcal{D}_\nu f' + g_{\mu\nu}\Box f' &= -8\pi G T_{\mu\nu} \qquad (8) \\
\mathcal{D}_\mu\left[(-g)^{1/2}g^{\mu\nu}\right] &= 0, \qquad (9)
\end{aligned}
$$

where $f' = df/fR$ and $\mathcal{D}$ is the covariant derivative of the Levi - Civita. This ansatz is known as the metric formalism of the $f(R)$ theory of the gravity. The field equation is of fourth order with respect to the coordinates. If one sets

$$
F(\phi,R) = -\frac{\phi R}{16\pi} \quad and \quad \omega(\phi) = -\frac{\omega_0}{16\pi\phi}, \qquad (10)
$$

the field equation with respect to the metric of the Brans - Dicke type Lagrangian given in eq. (2) becomes

$$\begin{aligned} -\frac{8\pi}{\phi}T_{\mu\nu} &= G_{\mu\nu}(g) - \frac{\omega_0}{\phi^2}\left(\frac{1}{2}g_{\mu\nu}\mathcal{D}_\lambda\phi\mathcal{D}^\lambda\phi - \mathcal{D}_\phi\mu\mathcal{D}_\nu\phi\right) \\ &\quad -\frac{1}{\phi}\left(\mathcal{D}_\mu\mathcal{D}_\nu\phi - g_{\mu\nu}\Box\phi\right) - \frac{8\pi}{\phi}V(\phi)g_{\mu\nu}, \end{aligned} \tag{11}$$

where $G_{\mu\nu}(g) = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor. If the field equation of the metric formalism given in eq. (8) is rearranged as

$$G_{\mu\nu}(g) - \frac{1}{f'}\left(\mathcal{D}_\mu\mathcal{D}_\nu f' - g_{\mu\nu}\Box f'\right) - \frac{1}{2}\left(\frac{f(R)}{f'} - R\right)g_{\mu\nu} = -\frac{8\pi G}{f'}T_{\mu\nu}. \tag{12}$$

Considering eq. (11) one shows that the metric formalism is indeed equivalent the Brans - Dicke theory with $\omega_0 = 0$.

There are some approaches for the gravity based on geometrical constructions. If ϵ is the dual base 1 - form and ∇ is a connection on a frame bundle over a space - time manifold, then curvature and torsion 2 - forms and non - metriticity 1 - forms are written as, respectively, $\mathcal{R} = \mathrm{d}^\nabla\Gamma$, $\mathcal{T} = \mathrm{d}^\nabla\epsilon$ and $\mathcal{Q} = \mathrm{d}^\nabla g$, where d is exterior derivative, g is the metric and $\mathrm{d}^\nabla = \mathrm{d}+\Gamma$. Hence a classification with respect to these forms can be written from [6] and [7] such as in Table 1.

Einstein - Hilbert:	$\mathcal{Q} = 0$	$\mathcal{R} \neq 0$	$\mathcal{T} = 0$
Weyl:	$\mathcal{Q} \neq 0$	$\mathcal{R} \neq 0$	$\mathcal{T} = 0$
Telleparalel:	$\mathcal{Q} = 0$	$\mathcal{R} = 0$	$\mathcal{T} \neq 0$
Einstein - Cartan:	$\mathcal{Q} = 0$	$\mathcal{R} \neq 0$	$\mathcal{T} \neq 0$
Metric - Affine:	$\mathcal{Q} \neq 0$	$\mathcal{R} \neq 0$	$\mathcal{T} \neq 0$

Table 1: Some Gravity Models

In this case, the $f(R)$ theories have three different formalisms in the perspective of the geometric appointment of the space - time manifold such that metric, Palatini and metric - affine. Here the metric - affine formalism is out of this paper, only we will consider the metric and the Palatini formalisms. In this context, the torsion has to vanish $\mathcal{Q} = 0$. Thus, the Table 1 is rearranged as in Table 2.

For non vanishing non - metriticity the connection must be independent on metric, that is $\nabla g \neq 0$, and so the Ricci scalar becomes $R = R(\Gamma, \partial\Gamma)$. Thus, the Lagrangian in eq. (7) is rewritten as

$$\mathcal{L} = -\frac{1}{16\pi G}(-g)^{1/2}f\left(R(\Gamma,\partial\Gamma)\right) + \mathcal{L}_{matter}(g^{\mu\nu}), \tag{13}$$

Metric:	$\mathcal{Q}=0$	$\mathcal{R}\neq 0$	$\mathcal{T}=0$
Palatini:	$\mathcal{Q}\neq 0$	$\mathcal{R}\neq 0$	$\mathcal{T}=0$

Table 2: Metric and Palatini Formalisms

and the field equation with respect to the metric g and the continuity equation with respect to the connection Γ become as follow, respectively,

$$f'\mathcal{R}_{\mu\nu}-\frac{1}{2}f(\mathcal{R})g_{\mu\nu} = -8\pi G T_{\mu\nu} \tag{14}$$

$$\nabla_\mu\left[(-g)^{1/2}f'g^{\mu\nu}\right] = 0, \qquad \nabla_\mu g^{\mu\nu}\neq 0, \tag{15}$$

where ∇_μ is covariant derivative independent on the metric. This is known as Palatini formalism of the $f(R)$ theories of the gravity. The field equation is of first order with respect to the connection. Considering the conformal transformation of the metric with respect to the any scalar smooth function, which will be given by eq. (25), the field equation in Palatini formalism becomes such that

$$G_{\mu\nu}(g)+\frac{1}{f'^2}\left(\frac{N-1}{N-2}\right)\left(\mathcal{D}_\mu f'\mathcal{D}_\nu f'-\frac{1}{2}g_{\mu\nu}\mathcal{D}^\lambda f'\mathcal{D}_\lambda f'\right)$$
$$-\frac{1}{f'}\left(\mathcal{D}_\mu\mathcal{D}_\nu f'-g_{\mu\nu}\Box f'\right)-\frac{1}{2}\left(\frac{f(R)}{f'}-R\right)g_{\mu\nu}=-\frac{8\pi G}{f'}T_{\mu\nu}. \tag{16}$$

Comparing the Brans - Dicke type theory, distinctly it will be

$$\phi=f'(R), \quad and \quad V(\phi[f'(R)])=\frac{1}{16\pi}(f(R)-Rf'). \tag{17}$$

Then the Palatini formalism of the $f(R)$ theories is indeed equivalent to the Brans - Dicke theory with $\omega_0=-\frac{(N-1)}{(N-2)}$.

2 Conformal Transformation of Metric

The connection in Palatini formalism is independent on the metric. However, considering the continuity equation in this formalism given in eq. (15), indeed it seen that this is a fake situation. Namely, if one considers a different connection $\widetilde{\nabla}$ over the same space - time manifold, then a new metric $\widetilde{g}$ appears in eq. (15), so that this metric satisfies $\widetilde{\nabla}_\mu\widetilde{g}^{\mu\nu}=0$ and the continuity equation becomes such that

$$\widetilde{\nabla}_\mu\left[(-\widetilde{g})^{1/2}\widetilde{g}^{\mu\nu}\right]=0. \tag{18}$$

In this case, the connection $\widetilde{\nabla}$ must be compatible by the metric $\widetilde{g}$. Then the coefficients of the connection $\widetilde{\nabla}$ is written as in Levi - Civita connection, so that

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\widetilde{g}^{\lambda\rho}\left[-\partial_\rho \widetilde{g}_{\mu\nu} + \partial_\mu \widetilde{g}_{\mu\rho} + \partial_\nu \widetilde{g}_{\rho\mu}\right]. \tag{19}$$

Consider a conformal transformation of the metric g as follow

$$\widetilde{g}^{\mu\nu} = (f')^{\frac{N}{2}-1} g^{\mu\nu}. \tag{20}$$

Then, the coefficients of the connection $\widetilde{\nabla}$ in eq. (19) are written as

$$\Gamma^{\lambda}_{\mu\nu} = \{^{\lambda}_{\mu\nu}\} + \frac{(N-2)}{4f'}\left[-g^{\lambda\rho}g_{\mu\nu}\mathcal{D}_\rho f' + \delta^{\lambda}_{\nu}\mathcal{D}_\mu f' + \delta^{\lambda}_{\mu}\mathcal{D}_\nu f'\right], \tag{21}$$

where $\{^{\lambda}_{\mu\nu}\}$ are the coefficients of the Levi - Civita connection in the term of the metric g:

$$\{^{\lambda}_{\mu\nu}\} = \frac{1}{2}g^{\lambda\rho}\left[-\partial_\rho g_{\mu\nu} + \partial_\mu g_{\mu\rho} + \partial_\nu g_{\rho\mu}\right], \tag{22}$$

and $\mathcal{D} = \mathrm{d} + \{\}$ is the connection which is compatible by the metric and at the same time covariant derivative with respect to the metric g. Then, the covariant derivative of the connection $\widetilde{\nabla}$ is written as $\widetilde{\nabla} = \mathrm{d} + \Gamma$ and using eq. (21) one rewrites as

$$\widetilde{\nabla} = \mathcal{D} + \mathcal{C} = \mathrm{d} + \Gamma + \mathcal{C}, \tag{23}$$

where $\mathcal{C}$ labels the coefficient in the parenthesis in eq. (21):

$$\mathcal{C}^{\lambda}_{\mu\nu} = \frac{(N-2)}{4f'}\left[-g^{\lambda\rho}g_{\mu\nu}\mathcal{D}_\rho f' + \delta^{\lambda}_{\nu}\mathcal{D}_\mu f' + \delta^{\lambda}_{\mu}\mathcal{D}_\nu f'\right]. \tag{24}$$

Hence, the Ricci tensor of the connection $\widetilde{\nabla}$ becomes

$$\mathcal{R}_{\mu\nu} = R_{\mu\nu} + \left(\frac{N-1}{N-2}\right)\frac{1}{f'^2}\mathcal{D}_\mu f'\mathcal{D}_\nu f' - \frac{1}{f'}\mathcal{D}_\mu\mathcal{D}_\nu f' - \frac{1}{(N-2)f'}g_{\mu\nu}\Box f'. \tag{25}$$

Therefore under the conformal transformation of the metric in the Palatini formalism the continuity equation given in eq. (15) is obtained as

$$\nabla_\mu\left[(-g)^{1/2}f'g^{\mu\nu}\right] = \frac{N}{2}(-g)^{1/2}g^{\mu\nu}f''\mathcal{D}_\mu\mathcal{R} = 0 \tag{26}$$

In result, since $f''(\mathcal{R}) \neq 0$, the scalar curvature is covariant constant $\mathcal{D}_\mu\mathcal{R} \equiv \partial_\mu\mathcal{R} \equiv 0$. In this case, since it will be $f'(\mathcal{R}) = constant$, then one gets $\mathcal{D}f' = 0 \equiv \mathcal{D}f'' = 0 \equiv \Box f' = 0$. Thus the field equation given in eq. (8) in metric formalism of the $f(R)$ theories of the gravity becomes

$$f'R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} = -8\pi G T_{\mu\nu}, \tag{27}$$

that is it isn't different than the field equation in Palatini formalism, and so the connection which its coefficients is given in (21) is reduced to the Levi - Civita connection:

$$\Gamma^{\lambda}_{\mu\nu} = \{^{\lambda}_{\mu\nu}\} = \frac{1}{2} g^{\lambda\rho} \left[-\partial_\rho g_{\mu\nu} + \partial_\mu g_{\mu\rho} + \partial_\nu g_{\rho\mu} \right]. \tag{28}$$

In this case, nevertheless the curvature in Palatini formalism becomes the same in the metric one:

$$\mathcal{R}_{\mu\nu} = R_{\mu\nu}. \tag{29}$$

3 Discussion: Cosmological Response

WMAP (Wilkinson Microwave Anisotropy Probe) and Ia supernovae observations for CMB (Cosmic Microwave Background) have shown that our universe expands accelerating [11], [9], [8], [10]. Considering the existence of matter and energy distribution in the universe, the Einstein's theory becomes inadequate to explain the accelerating expand. Therefore the models intended to explain the acceleration can be classified under two groups: **i)** Cosmological models with exotic matter source; i.e dark matter which is zero pressure and positive energy density, dark energy which is negative pressure and positive energy density, vacuum energy (cosmological constant) [4] and **ii)** to make some modifications adding some terms of the curvature; i.e. the power of the Ricci scalar, the quadratic square of the Ricci and Riemann curvature tensors [5]. Here the modification to the Einstein' s theory is seen in some early works, i.e [1], [2], [12].

Consider the Friedmann - Robertson - Walker universe on $(N-1)$ - sphere embedded in real N dimension given by the metric

$$ds^2 = dt^2 - a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right), \tag{30}$$

where $a(t)$ is scale multiplier and $k = -1, 0, +1$ is geometrical factor for open (hyperbolic), flat and closed (elliptic) universes, respectively, and $d\Omega^2$ is the solid angle of the $(N-1)$ - sphere given by

$$d\Omega^2 = \sum_{m}^{N-2} \left(\prod_{i=1}^{m} sin^2\theta_{i-1} \right) d\theta_m^2, \quad (sin\theta_0 = 1). \tag{31}$$

Since we indicated upper that the metric and Palatini formalisms of the $f(R)$ theories of the gravity are equivalent to each other because of the covariant constant scalar curvature generated by the conformal transformation of the metric, there exists a unique field equation such as given in eq. (27):

$$f' R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} = -8\pi G T_{\mu\nu}. \tag{32}$$

If this equation is rearranged as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\frac{8\pi G}{f'}(T_{\mu\nu} + \tau_{\mu\nu}), \tag{33}$$

then an effective energy - momentum tensor appears such that

$$\tau_{\mu\nu} = -\frac{1}{16\pi G}\left(f - f'R\right) g_{\mu\nu}. \tag{34}$$

If one considers this effective energy - momentum tensor belonging a perfect fluid, then one can write

$$\tau_{\mu\nu} = \left(\rho_{eff} + p_{eff}\right) u_\mu u_\nu - p_{eff} g_{\mu\nu}, \quad (u^0 = u_0 = 1, u^\lambda u_\lambda = 1), \tag{35}$$

where u is the proper velocity, ρ_{eff}, p_{eff} are the effective density and the pressure of the perfect fluid, respectively, also the state equation of this fluid is such that

$$p_{eff} = \varpi_{eff}\rho_{eff}, \tag{36}$$

where ϖ_{eff} is the state parameter of the fluid. Hence, the contraction of the eq. (33) together with eq. (35) gives

$$-\frac{N}{16\pi G}\left(f - f'R\right) = (1 - (N-1)\varpi_{eff})\rho_{eff}, \tag{37}$$

and also the continuity equation becomes

$$\dot{\rho}_{eff} + (N-1)\left(1 + \varpi_{eff}\right) H\rho_{eff} = 0, \tag{38}$$

where $(\dot{\ })$ denotes the derivative with respect to the time, $(\dot{\ }) = d/dt$, and $H = \frac{\dot{a}}{a}$ is Hubble parameter. The derivative with respect to the time of the eq. (37), together with eq. (38), is obtained as

$$\dot{R} = \frac{1}{Rf''}(N-1)\left(1 + \varpi_{eff}\right) H\rho_{eff}. \tag{39}$$

Since the scalar curvature is constant, $\dot{R} = 0$, it must be $\varpi_{eff} = -1$. If one writes them into the continuity equation given by eq. (38), then one gets $\rho_{eff} = Constant$. Thus, the equation (37) becomes

$$f(R) = Constant \times R - 16\pi G\rho_{eff}, \tag{40}$$

that is $Constant = 1$ and $\Lambda = 8\pi G\rho_{eff}$, and so one gets the Einstein' s theory. This situation shows that the $f(R)$ theories of the gravity are nothing more than the Einstein's theory with cosmological models whose are the exotic matter source given like in references.

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