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Canonical Exterior Formalism on Group Manifold for (3+1) Dimensional Supergravity

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Abstract

This paper describes the geometric structure of the model constructed from the application of the canonical exterior formalism (CEF), also known as canonical covariant formalism, on group manifold starting from a classical superconformal theory described by the Wess-Zumino-Witten action. From a geometric point of view we construct and analyze the constraints and field equation. The main features of the diferent, linear and non linear gravity, are reviewed. In particular, the supersymmetric extension of the Jackiw-Teitelboim (3+1) gravity is considered in detail within the CEF. In that context the role of the several fields are analyzed. Finally, this supergravity model is treated in the second order formalism. Using this formalism CEF is more beneficial and simple, it can be seen the covariance at each step of the development. The Dirac brackets are written in terms of second-class constraints provided by the CEF, which facilitates the quantification of that canonical model.

Keywords: Canonical exterior formalism; Group manifold; (3+1) dimensional supergravity

1 Introduction

From several years ago the interest of the researchers in studying the different dimensional gravity and supergravity models has been made evident. These

models were studied from different points of view. The first linear model of gravity was proposed by Jackiw-Teitelboim (JT), obtained by dimensionally reducing the usual Einstein-Hilbert (EH) action in 3D [1, 2]. From the gauge theoretic formulation, those models have the property of possessing a topological and gauge invariant [3, 4, 5, 6].

Initially, the supersymmetric extension of the Jackiw-Teitelboim (1+1) linear gravity within the CEF on group manifold was constructed.

The CEF for gravity and supergravity on group manifold was proposed by D'Auria, Lerda, Nelson and Regge and it is founded on the Hamiltonian theory for constrained systems first developed by Dirac [7]. The CEF was extended to any dimension and next applied to supergravity coupled to super Yang-Mills fields and Wess-Zumino matter supermultiplet. In this context the role of the several fields was well defined. The constraints and field equation were found and analyzed from a geometric point of view. This supergravity model is treated in the second order formalism.

Frequently, the linear gravity or supergravity models are used as a theoretical laboratory to studying properties also present in supergravities of greater dimensions. Perhaps one reason is the presence of "black-holes" in this models [8, 9, 10].

It That sense following works are iconic: 1) A class of linear gravity theories is based on the Riemann scalar curvature R (JT) [1, 2, 11, 12]. 2) 2D gravitational and vector gauge theories by of 3D topologically massive models (EH) [13]. 3) The geometrical structure of the different models in the framework are generally the Sitter or anti-de Sitter groups.

In particular, various models of linear gravity theories involving non-geometrical fields acting as Lagrange multipliers can be constructed. As mentioned, all these models have the remarkable property of possessing a topological and gauge invariant formulation [14, 15, 16]. It is important to clarify that these "string inspired" models are based on the extended Poincaré group. So, is possible to prove that "black-hole" solution appears in this kind of models. Thus, their study acquires interest from quantum. We distinguish three models from the gauge invariant and quantum point of view, the original model based on SO(2,1) group, the "string-inspired" models based on the extended Poincaré group and the gravity model in 3D, in which is possible to prove that "black-hole" solution appears, and so its study, becomes interesting. Lately, the aforementioned research has engendered much further works [17, 18, 19, 20, 21].

In the framework of gauge theory, these models are based on the n-dimensional graded de Sitter group, whose associated graded algebra is the well known osp (n,1/1). The supersymmetric action in components can be written in terms of the field variables V^a, $\omega = \frac{1}{2}\epsilon_{ab}\omega^{ab}$, and ξ, i.e., the zweibein, the spin connection and the gravitino gauge fields respectively.

A class of linear gravity theories based on the Riemann scalar curvature R

was proposed as a simple model, which requires an additional non-geometrical field ϕ in the action.

$$I_1 = \int d^2x \; \phi(R - \lambda) \; , \tag{1}$$

where ϕ is an invariant world scalar acting as a Lagrangian multiplier enforcing the field equation.

$$R - \lambda = 0 \; , \tag{2}$$

By introducing the additional field generalizations and modifications of (1) and (2) allow us to consider different dynamics for the fields R and ϕ is obtained.

The action of a similar model, which is "string inspired", is defined as:

$$I_2 = \int d^2x \; (\phi R - \lambda) \; , \tag{3}$$

In this case the equation of motion is:

$$R = 0 \; , \tag{4}$$

whose flat metric $g_{\mu\nu} = h_{\mu\nu}$ reads:

$$\partial_\mu \partial_\nu \phi = \frac{1}{2} \lambda h_{\mu\nu} \; . \tag{5}$$

Then, the "black-hole" solution appears, and the model becomes interesting from the quantum point of view.

Generalizing, this geometrical Hamiltonian formalism can be adapted to describe non polynomial supergravity, whose Lagrangian contain terms of higher order in the Riemann curvature. In previous work [22], different supersymmetric extension of the linear gravity and supergravity model within the canonical covariant formalism on group manifold were constructed.

Other model, Chern-Simons model, by means of the Kaluza-Klein like *ansatz*, decomposing the three-dimensional metric into a two-dimensional metric. The dimensional reduction of the invariant action produces a two-dimensional topological theory. This was done, a $U(1)$ gauge field $A = A_\mu \, dx^\mu$ and a scalar field ϕ, [19, 20]. We can show is that two types of local classical solutions exist, which were then extended at global level with the purpose of constructing the Carter-Penrose diagrams, symmetry breaking and kink solutions. It is interesting to note that the kink make possible an space whose geometry is asymptotically anti-de Sitter. Another point is to note that at small distances the scalar curvature is positive and it vanishes at an intermediate point. Because of this the effect of the kink is analogous to a geometric gravitational force and it can be proved that the resulting action is formally similar to the

action of the dilaton model. At a global level, the action is written by using target space coordinates, the use of such coordinates brings some advantages from classical as well as quantum point of view [17, 18].

Finally, the conformal supergravity is the proper framework for the description of superstring theories, [23, 24, 25, 26, 27]. This idea is easy to visualize in 2D, since two is the dimension of the world-sheet (WS) spanned by a one-dimensional object while propagating in an external space-time, named target manifold (M_{target}). In low-dimensional, the manifolds play an important role because they are responsible for the fundamental geometric structure of superstring theory. In this context the vielbein and the gravitino are necessary to make the graded algebra local. In a two-dimensional manifold, the action reduces to a pure divergence in both cases, gravity or supergravity, for which the gravitational field is a non-dynamical one.

The gravitational field must be interpreted as a Lagrangian multiplier for the corresponding constraints giving the vanishing condition of the matter fields, so the gravitational formalism reduces to a theory of boundary conditions and only its topology is the matter of interest. Then, the superstring WS becomes a Riemann surface by means of a wick rotation of the time variable, so can be treated by using all the results the algebraic geometry.

Considering this last role of conformal supergravity, the motivation of the present paper is essentially to study from a mathematical-physics point of view the supersymmetric models in 4D in the framework of the CEF on group manifold. The first advantage is that this formalism is covariant in all its steps, and there is a direct relation between the form brackets provided by the CEF and the Dirac brackets, which facilitates the canonical quantization.

Assuming the algebraic, complexity in a more general context, it is interesting to analyze a 4D model in which the fields acquire dynamic.

2 Geometrical structure and Definitions

The geometric structure of the model is that of N = 1, D = 4 conformal supergravity. Let us consider the problem by starting from the first-order (CEF) on group manifold. In first-order exterior formalism the dynamics is described by the three 1-form gauge fields ω^{ab} (spin connection), V^a (zweibein) and ξ (gravitino), and the three 0-form fields η^J, η_a and η. The three 0-form fields are non-geometrical objects and are introduced for obtaining rheonomic equations of motion, equations compatible with the Bianchi identities. The Lagrangian density is a bosonic 4-form functional of the dynamical fields and their exterior derivatives. For a supergroup G and a bosonic gauge subgroup $H \subset G$, the phisical superspace is defined by the coset manifold $M^4 = G/H$, and all the fields must be considered only as reduce forms, i.e. forms defined

on M^4 [28, 29, 30, 31, 32, 33, 34].

The Lagrangian density is written as:

$$
\begin{aligned}
\mathcal{L} \;=\;& \eta\, R^a \wedge\, \omega_{ab} \wedge\, V^b + \frac{4}{\lambda^2}\, \eta^J R \wedge\, V^a \wedge\, V^b\, \epsilon_{ab} + \frac{4i}{\lambda}\, \eta\, \overline{\rho}\, \gamma^5 \wedge\, \overline{\xi} \wedge\, \xi \\
-\;& \frac{1}{2}\, \eta^J\, \overline{\xi} \wedge\, \xi \wedge\, V^a \wedge\, V^b \epsilon_{ab} - \frac{i}{2\lambda}\, \eta^J\, \overline{\xi}\, \gamma^5 \wedge\, \xi \wedge\, V^a \wedge\, V^b\, \epsilon_{ab} \\
-\;& \frac{4i}{\lambda}\, \eta\, \overline{\xi}\, \gamma^b \wedge\, \xi\, \gamma^d \wedge\, V^a \wedge\, V^b\, \epsilon_{ab}\, \epsilon_{cd} \\
-\;& (1 + \frac{i}{\lambda}\, \gamma^5)\, \eta^J\, \xi\, \wedge\, dV^a \wedge\, V^b \epsilon_{ab} - \frac{2i}{\lambda}\, \eta\, \gamma^b\, \xi\, \gamma^d\, \wedge\, dV^a \wedge\, V^c \epsilon_{ab}\, \epsilon_{cd} \\
-\;& (1 + \frac{i}{\lambda}\, \gamma^5)\, \eta^J\, d\xi\, \wedge\, V^a \wedge\, V^b \epsilon_{ab} \\
-\;& \frac{4i}{\lambda}\, \eta\, \gamma^b\, d\xi\, \gamma^d\, \wedge\, V^a \wedge\, V^c \epsilon_{ab}\, \epsilon_{cd}
\end{aligned}
\tag{6}
$$

The curvature 2-forms corresponding to the gauge fields are R (Riemann curvature), R^a (torsion) and ρ (gravitino curvature), and they are given by:

$$R = d\omega\ , \tag{7}$$

$$R^a = dV^a - \omega \wedge\, V_b \epsilon^{ab} - \frac{i}{4}\, \overline{\xi}\, \gamma^a \wedge\, \xi\ , \tag{8}$$

$$\rho = d\xi\ - \frac{i}{2}\, \omega \wedge\, \overline{\xi}\, \gamma^5\ . \tag{9}$$

The Lagrangian density is linear in the curvatures taking the general form $\mathcal{L} = R^A(\mu) \wedge\, \nu_A(\mu) + \Lambda(\mu)$.

In the group manifold approach the third principle states that the functional coefficients $\Lambda(\mu)$ and $\nu_A(\mu)$ must satisfy: $\lfloor A \rfloor\, \Lambda + D\ \nu_A = 0$, for the vacum solution $R^a = 0$. The coefficients do not depend on the spin connection and they must be invariant under transformation of the bosonic gauge symmetry group H.

We define the 2-form canonical conjugate momenta π_Σ to each one of the dynamical field variables $\mu^\Sigma = (\omega^{ab}, V^a, \xi, \eta^J, \eta_a, \eta_\alpha)$ for the compound index Σ. By means of the functional variation of the Lagrangian with respect to the "velocities" $d\mu^\Sigma$, the canonical conjugate momenta remain defined as follows:

$$\pi_\Sigma = \frac{\delta\, \mathcal{L}}{\delta\, (d\mu^\Sigma)}\ . \tag{10}$$

Explicitly, the moments associated with the six fields of the model are:

$$\begin{aligned}\pi_a(V) &= \eta\,\omega_{ab}\wedge\,V^b-(1+\frac{i}{\lambda}\,\gamma^5)\,\eta^J\,\xi\,\wedge\,V^b\,\epsilon_{ab}\\ &- \frac{2i}{\lambda}\,\eta\,\xi\,\gamma^b\,\gamma^d\,\wedge\,V^c\,\epsilon_{ab}\,\epsilon_{cd}\,,\end{aligned}\tag{11}$$

$$\pi(\omega)=\frac{4}{\lambda^2}\,\eta^J\,V^a\,\wedge\,V^b\,\epsilon_{ab}\,,\tag{12}$$

$$\begin{aligned}\pi_\alpha(\overline{\xi}) &= \frac{4i}{\lambda}\,\eta\,\gamma^5\,\overline{\xi}\wedge\,\xi-(1+\frac{i}{\lambda}\,\gamma^5)\,\eta^J\,V^a\wedge\,V^b\,\epsilon_{ab}\\ &- \frac{4i}{\lambda}\,\eta\,\gamma^b\,\gamma^d\,V^a\wedge\,V^c\,\epsilon_{ab}\,\epsilon_{cd}\,,\end{aligned}\tag{13}$$

$$\pi(\eta_a)=0\,,\tag{14}$$

$$\pi(\eta^J)=-\frac{4}{\lambda^5}\,\omega\,,\tag{15}$$

$$\pi(\eta_\alpha)=0\,,\tag{16}$$

The momenta associated with the three gauge fields are 2-forms, and the others three momenta associated with the auxiliary fields are 1-form or zero.

In the CEF must define the operation between pairs of canonical conjugate variables to replace the role of the classical Poisson brackets. Thus, the graded form-brackets between pairs of canonical conjugate variables is defined by:

$$\Big(\mu^\Sigma\,,\,\pi_\Lambda\Big)=(-1)^{a+1+A}\delta^\Sigma_\Lambda\,,\tag{17}$$

where a and A are respectively the degree and the Fermi grading of the 1-form μ^Σ. For the present case the form-brackets between pairs of canonical variables are:

$$\Big(\pi_a(V)\,,\,V^b\Big)=\delta^b_a\,,\tag{18}$$

$$\Big(\pi(\omega)_{ab}\,,\,\omega^{cd}\Big)=\delta^{cd}_{ab}\,,\tag{19}$$

$$\Big(\pi(\xi)\ ,\ \overline{\xi}\Big) = -\Big(\pi(\overline{\xi})\ ,\ \xi\Big) = -1\ . \tag{20}$$

$$\Big(\pi(\eta_a)\ ,\ \eta^b\Big) = \delta_a^b\ , \tag{21}$$

$$\Big(\pi(\eta^J)\ ,\ \eta^K\Big) = \delta_J^K\ , \tag{22}$$

$$\Big(\pi(\eta_\alpha)\ ,\ \eta^\beta\Big) = \delta_\alpha^\beta\ , \tag{23}$$

The set of primary constraints is:

$$\begin{aligned}\Phi_a(V) &= \pi_a(V) - \eta\,\omega_{ab}\ \wedge\ V^b + (1+\frac{i}{\lambda}\,\gamma^5)\,\eta^J\,\xi\ \wedge\ V^b\,\epsilon_{ab}\\ &+ \frac{2i}{\lambda}\,\eta\,\xi\,\gamma^b\,\gamma^d\ \wedge\ V^c\,\epsilon_{ab}\,\epsilon_{cd} \approx 0\ ,\end{aligned} \tag{24}$$

$$\Phi(\omega) = \pi(\omega) - \frac{4}{\lambda^2}\,\eta^J\,V^a\ \wedge\ V^b\,\epsilon_{ab} \approx 0\ , \tag{25}$$

$$\begin{aligned}\Phi_\alpha(\overline{\xi}) &= \pi_\alpha(\overline{\xi}) - \frac{4i}{\lambda}\,\eta\,\gamma^5\,\overline{\xi}\wedge\ \xi + (1+\frac{i}{\lambda}\,\gamma^5)\,\eta^J\,V^a\wedge\ V^b\,\epsilon_{ab}\\ &+ \frac{4i}{\lambda}\,\eta\,\gamma^b\,\gamma^d\,V^a\wedge\ V^c\,\epsilon_{ab}\,\epsilon_{cd} \approx 0\ ,\end{aligned} \tag{26}$$

$$\Phi(\eta_a) = \pi(\eta_a) \approx 0\ , \tag{27}$$

$$\Phi(\eta^J) = \pi(\eta^J) + \frac{4}{\lambda^5}\,\omega\ \approx 0\ , \tag{28}$$

$$\overline{\Phi}(\eta_\alpha) = \pi(\eta_\alpha) \approx 0\ , \tag{29}$$

Utilizing the graded form-brackets, definition and properties, it is possible obtain the form-brackets (Φ_A, Φ_B) for pairs of constraints. All the primary constraints are second class ones, that is: $(\Phi_A, \Phi_B) \neq 0$.

The bosonic 4-form extended Hamiltonian H_T describing the dynamics of the system. In the CEF H_T is the conserved first-class dynamical quantity, defined as:

$$\begin{aligned} H_T &= H_{can} + \Lambda^{\Sigma} \wedge \Phi_{\Sigma} \\ &= d\mu^{\Sigma} \wedge \pi_{\Sigma} - \mathcal{L} + \Lambda^{\Sigma} \wedge \Phi_{\Sigma} \, . \end{aligned} \tag{30}$$

with: $H_{can} = d\mu^{\Sigma} \wedge \pi_{\Sigma} - \mathcal{L}$.

The set of superform Λ^{Σ} is the arbitrary Lagrange multipliers, which can be easily determined. In analogy to the classical mechanics, we introduce the fundamental equation of motion in the CEF:

$$\frac{df}{dt} = (f, H) + \frac{\partial f}{\partial t} \, . \tag{31}$$

which involving the graded form-brackets:

$$dA = (A, H_T) + \partial A \, . \tag{32}$$

This fundamental equation of motion must be taken into account for to write the Hamiltonian equations for pairs of canonical variables, and $A = A(\mu, \pi)$ is a generic polynomial in the canonical variables μ^{Σ} and π_{Σ}. Therefore, for the canonical variables we have:

$$\partial\mu^{\Sigma} = \partial\pi_{\Sigma} = 0 \, , \tag{33}$$

and also for the constraints:

$$\partial\Phi_{\Sigma} = 0 \, . \tag{34}$$

Considering the equation (32) for the canonical variables we can write the following Hamiltonian equations:

$$d\mu^{\Sigma} = \left(\mu^{\Sigma} \, , \, H_T\right) \, , \tag{35}$$

$$d\pi^{\Sigma} = \left(\pi^{\Sigma} \, , \, H_T\right) \, . \tag{36}$$

Taking into account the expression for H_T and equations (35) and (36), the Lagrange multipliers Λ^{Σ} can be explicitly evaluated. Remember that in the framework of group manifold the canonical Hamiltonian $H_{can} = d\mu^{\Sigma} \wedge \pi_{\Sigma} - \mathcal{L}$ is only function of the field variables μ^{Σ}. By straightforward calculation, we find the following general results:

$$\Lambda^{\Sigma} = d\mu^{\Sigma} \, . \tag{37}$$

The canonical Hamiltonian is given by:

$$\begin{aligned} H_{can} &= -\frac{i}{4}\,\eta\,\omega_{ab}\,\wedge\,\overline{\xi}\,\gamma^a\,\wedge\,\xi\,\wedge\,V^b \\ &+ \frac{1}{2}\,\eta^J\,\overline{\xi}\,\wedge\,\xi\,\wedge\,V^a\,\wedge\,V^b\,\epsilon_{ab} \\ &+ \frac{i}{2\lambda}\,\eta^J\,\overline{\xi}\,\gamma^5\,\wedge\,\xi\,\wedge\,V^a\,\wedge\,V^b\,\epsilon_{ab} \\ &+ \frac{4i}{\lambda}\,\eta\,\overline{\xi}\,\gamma^b\,\wedge\,\xi\,\gamma^d\,\wedge\,V^a\,\wedge\,V^b\,\epsilon_{ab}\,\epsilon_{cd}\,. \end{aligned} \tag{38}$$

The field equations of motion in the CEF are given by the consistency conditions on the primary constraints, $d\Phi^A = \left(\Phi^A\,,\,H_T\right)\,\approx 0$. After some algebraic manipulation they read:

$$\begin{aligned} d\Phi^a(V_a) &= \eta\,\omega\,\wedge\,V^a\wedge\,V^b\,\epsilon_{ab} + \eta^J\,\xi\,\wedge\,V^a\wedge\,V^b\,\epsilon_{ab} \\ &- i\,\eta\,\overline{\xi}\wedge\,\gamma^b\,\xi\wedge\,V^a\,\epsilon_{ab} - d\eta\,\wedge\,\overline{\xi}\wedge\,\xi \\ &+ 2\,(1+\frac{i}{\lambda}\,\gamma^5)\,\eta^J\,\xi\,\wedge\,dV^b\,\epsilon_{ab} - \frac{4i}{\lambda}\,\eta\,\xi\,\gamma^b\,\gamma^d\,\wedge\,dV^c\,\epsilon_{ab}\,\epsilon_{cd} \\ &- 2\,(1+\frac{i}{\lambda}\,\gamma^5)\,\eta^J\,d\xi\wedge V^b\,\epsilon_{ab} - \frac{8i}{\lambda}\,\eta\,\gamma^b\gamma^d\,d\xi\wedge\,V^c\,\epsilon_{ab}\,\epsilon_{cd}\,. \end{aligned} \tag{39}$$

$$\begin{aligned} d\Phi(\omega) &= d\eta\,\wedge\,V^a\wedge\,V^b\,\epsilon_{ab} - \frac{2i}{\lambda}\,\xi\,\eta\,\wedge\,V^a\,\wedge\,V^b\,\epsilon_{ab} \\ &+ \frac{4}{\lambda^2}\,d\eta\,\wedge\,V^a\,\wedge\,V^b\,\epsilon_{ab}\,. \end{aligned} \tag{40}$$

$$\begin{aligned} d\Phi(\overline{\xi}) &= \frac{i}{2}\,\eta_a\,\gamma^a\,\xi\,\wedge\,V^b\,\wedge\,V^c\,\epsilon_{bc} + \frac{i}{\lambda}\,\eta^J\,\gamma^5\,\xi\,\wedge\,V^a\,\wedge\,V^b\,\epsilon_{ab} \\ &+ i\,\gamma^b\,d\eta\,\wedge\,V^a\,\wedge\,\xi\,\epsilon_{ab} + \frac{4i}{\lambda}\,\gamma^5\,d\eta\,\wedge\,V^a\,\wedge\,V^b\,\epsilon_{ab} \\ &- (1+\frac{i}{\lambda}\,\gamma^5)\,\eta^J dV^a\wedge V^b\epsilon_{ab} - \frac{2i}{\lambda}\,\eta\,\gamma^b\gamma^d\wedge dV^a\wedge V^c\,\epsilon_{ab}\epsilon_{cd}\,. \end{aligned} \tag{41}$$

For simplicity, and by way of examples, are given the equations for only three cases.

Since the Lagrangian density is rheonomic, the solution for the curvature R^a , R , ρ associated it the gauge fields must be compatible with the Bianchi identities. Also, the parametrization of these curvatures are obtained directly from the field equations (39)-(41), and they may be written as follows:

$$R^a = 0 \ . \tag{42}$$

$$R = -\frac{1}{8}\,\lambda^2\, V^a \ \wedge\ V^b\, \epsilon_{ab} - \frac{1}{8}\,\lambda\,\bar{\xi}\,\gamma^5 \wedge\ \xi \ . \tag{43}$$

$$\rho = -\frac{1}{4}\,\lambda\,\bar{\xi}\,\gamma^b\,\gamma^5 \ \wedge\ V^b\,\epsilon_{ab} \ . \tag{44}$$

From the above equations it can be observed that the curvatures R ans ρ, both non zero, are proportional to the cosmological constant λ.

We can see how the auxiliary 0-form fields η^a , η_J , η can be interpreted from the CEF. For these non-geometrical fields, the curvatures are substituted by the covariant exterior derivatives. Thus, the 1-form field equations for η^a , η_J , η are also rheonomic and they are written as follows:

$$\mathcal{D}\eta_a = \eta^J\ V^b\,\epsilon_{ab} + i\,\eta\,\gamma^b\,\xi\,\epsilon_{ab} \ . \tag{45}$$

$$\mathcal{D}\eta^J = -\frac{1}{4}\,\lambda^2\,\eta_a\,V_b\,\epsilon^{ab} + \frac{i}{2}\,\lambda\,\eta\,\xi \ . \tag{46}$$

$$\mathcal{D}\eta = \frac{1}{4}\,\lambda\,(\gamma^b\,\gamma^5\,\eta)\,V^a\,\epsilon_{ab} + \frac{1}{4}\,(\eta^J + \frac{1}{2}\,\lambda\,\eta_a\,\gamma^a\,\gamma^5)\,\xi \ . \tag{47}$$

It is important to note that although the CEF is covariant in all their steps, it is not a proper Hamiltonian formalism because the extended Hamiltonian H_T is not a true generator of time evolutions. The form-brackets do not contain the same information as the Poisson brackets, indeed the latter contains more information than the form-brackets. Currently, the CEF can be related with the Hamiltonian formalism in components, and so the form-brackets are related to the Poisson brackets but not in a trivial way. The relation is given by the integral relationship [26, 29]:

$$(-1)^{a+1}\ \int_\Sigma\ \alpha\ \wedge (A,B) \wedge\ \beta = \int\int_{\Sigma\times\Sigma}\ \alpha(x)\ \wedge\ [A(x)\,,\,B(y)]\ \wedge\ \beta(y)\ , \tag{48}$$

where a is the degree of the form A and $\alpha\,,\beta$ are text forms.

Once the space time decomposition is done and the surface Σ is well defined, the ordinary Poisson brackets are obtained by expanding the forms $A(x)$ and $B(y)$, in the holonomic bases dx^i, dy^j, and then the Poisson brackets between fields and momenta components can be used.

Before to conclude this section a further consideration about the CEF must be done. As it was said, all the primary constraints provided by the CEF are second-class ones, and so they are not related with the gauge symmetry of the model. Furthermore, the different Lagrangian densities that can be used imply that there is not a unique set of canonical conjugate momenta and consequently there is not a unique set of primary constraints in the CEF. On the other hand, in the second-order formalism the second-class constraints must be eliminated, and this is achieved by defining the Dirac brackets from the Poisson brackets. The Dirac brackets for generic functional are obtained by means of the following expression:

$$[F\,,\,G]^D = [F\,,\,G] - [F\,,\,\Psi_A]\,C^{AB}\,[\Psi_B\,,\,G]\;, \tag{49}$$

where $C^{AB}[\Psi_B\,,\,\Psi_C] = \delta^A_C$ for the compound indices A, B, C. To compute the Dirac brackets (49) we must consider the restriction to Σ of all the second-class constraints (24)-(29).

One property of the Dirac brackets for first class function is $[F\,,\,G]^D \approx [F\,,\,G]$, and particularly for the Hamiltonian $\mathcal{H}$ is:

$$[F\,,\,\mathcal{H}]^D \;\approx\; [F\,,\,\mathcal{H}]\;. \tag{50}$$

This means that the same equations of motion are obtained by using the Poisson or the Dirac brackets. Thus, the rate of change in time of a given functional F of the canonical variables is also:

$$\dot{F} = [F\,,\,\mathcal{H}]^D\;. \tag{51}$$

For any functional of the canonical variables we have:

$$[\Psi_A\,,\,\mathcal{H}]^D = 0\;. \tag{52}$$

Therefore, we can set $\Psi_A = 0$ either before or after evaluating the Dirac brackets. Once the Dirac brackets are evaluated, the transition to quantum theory is realized as usual in a canonical formalism by replacing classical fields by quantum field operators acting on some Hilbert space.

3 Motion equations in the CEF

In the CEF the field equations of motion are give by consistency conditions on the primary constraints, i.e.:

$$\begin{aligned} d\Phi^{\Sigma} &= \left(\Phi^{\Sigma}\ ,\ H_T\right) \\ &= \left(\Phi^{\Sigma}\ ,\ H_{can}\right) + \Lambda^{\Lambda}\left(\Phi^{\Sigma}\ ,\ \Phi_{\Lambda}\right) \\ &= \left(\Phi^{\Sigma}\ ,\ d\mu^{\Sigma}\ \wedge\ \pi_{\Sigma}\right) - \left(\Phi^{\Sigma}\ ,\ \mathcal{L}\right) + \Lambda^{\Lambda}\left(\Phi^{\Sigma}\ ,\ \Phi_{\Lambda}\right) \approx 0\ . \end{aligned} \tag{53}$$

Now, in this dimension the vielbein and the gravitino are dynamical fields, so it is important to consider their equations of motion. That is, the equations of motion for the supergravity background fields V^a and ξ will be considered. As known, the superstress-energy tensor and the supercurrent are respectively defined by making the variation of the action with respect to the supervielbein (V^a, ξ). In contrast, since the variable ω are introduced to enforce the rheonomic parametrization. The three 0-form fields η^J, η_a and η are non-geometrical objects and are introduced for obtaining rheonomic equations of motion, equations compatible with the Bianchi identities.

The explicit expressions for the form-brackets between constraints and H_{can}, $(\Phi^a\ ,\ H_{can})$ and $(\Phi_\alpha\ ,\ H_{can})$, must be considered to get the equations of motion of the model. Equally the explicit expressions for the form-brackets between constraints are replaced (see Ref. [31]). The equations of motion for each dynamical field can be obtained by replacing the above expressions in equation (53) as appropriate, and explicitly we have:

$$\begin{aligned} d\Phi^a(V_a) &= \eta\,\omega\ \wedge\ V^a \wedge\ V^b\,\epsilon_{ab} + \eta^J\,\xi\ \wedge\ V^a \wedge\ V^b\,\epsilon_{ab} \\ &- i\,\eta\,\overline{\xi} \wedge\ \gamma^b\,\xi \wedge\ V^a\,\epsilon_{ab} - d\eta\ \wedge\ \overline{\xi} \wedge\ \xi \\ &+ 2\,(1 + \frac{i}{\lambda}\,\gamma^5)\,\eta^J\,\xi\ \wedge\ dV^b\,\epsilon_{ab} - \frac{4i}{\lambda}\,\eta\,\xi\,\gamma^b\,\gamma^d\ \wedge\ dV^c\,\epsilon_{ab}\,\epsilon_{cd} \\ &- 2\,(1 + \frac{i}{\lambda}\,\gamma^5)\,\eta^J\,d\xi \wedge V^b\,\epsilon_{ab} - \frac{8i}{\lambda}\,\eta\,\gamma^b\gamma^d\,d\xi \wedge V^c\,\epsilon_{ab}\epsilon_{cd} \\ &+ \text{weakly zero terms} = 0\ . \end{aligned} \tag{54}$$

$$
\begin{aligned}
d\Phi(\overline{\xi}) &= \frac{i}{2}\,\eta_a\,\gamma^a\,\xi\,\wedge\,V^b\,\wedge\,V^c\,\epsilon_{bc} + \frac{i}{\lambda}\,\eta^J\,\gamma^5\,\xi\,\wedge\,V^a\,\wedge\,V^b\,\epsilon_{ab} \\
&+ i\,\gamma^b\,d\eta\,\wedge\,V^a\,\wedge\,\xi\,\epsilon_{ab} + \frac{4i}{\lambda}\,\gamma^5\,d\eta\,\wedge\,V^a\,\wedge\,V^b\,\epsilon_{ab} \\
&- (1+\frac{i}{\lambda}\,\gamma^5)\,\eta^J dV^a\wedge V^b\epsilon_{ab} - \frac{2i}{\lambda}\,\eta\,\gamma^b\,\gamma^d\wedge dV^a\wedge V^c\,\epsilon_{ab}\,\epsilon_{cd} \\
&+ \text{weakly zero terms} = 0\,. \qquad (55)
\end{aligned}
$$

Are taken for simplicity, and by way of examples, the equations of motion only for the two dynamic fields, V^a and ξ_α.

The previous equations of motion defined over the superspace can be decomposed into three independent sectors corresponding to the inner-inner direction $V^a\wedge V^b$, the inner-outer directions $V^a\wedge\xi$ and the outer-outer direction $\xi\wedge\xi$. It is easy to seen that the other coefficients are cancel automatically when the rheonomic parametrization is introduced. In turn, in the first of the above equations the cancelation of the component inner-inner arises to the following condition:

$$\mathcal{D}_+\,\Omega^A_- - \mathcal{D}_-\,\Omega^A_+ + \mathcal{D}\,\lambda^A + if^{ABC}\,\lambda_B\,\lambda_C = 0\,. \qquad (56)$$

Analogously, by considering the second equation it can be seen that in the case of the coefficients of the first three components cancel automatically, gives rise to the following conditions:

$$y_A\,\Omega^A_+ + y_A\,\Omega^A_- - y_A\,\mathcal{D}_+\,\Omega^A_- - 2if^{ABC}\,\lambda_A\,\lambda_B\,y_C = 0\,. \qquad (57)$$

$$y_A\,\Omega^A_+ + y_A\,\mathcal{D}\,\lambda^A + y_A\,\lambda^A + if^{ABC}\,\lambda_A\,\lambda_B\,y_C = 0\,. \qquad (58)$$

Now, considering the different projections for the Maurer-Cartan equation, the following conditions are found:

$$\mathcal{D}_o\,\Omega^A_+ - \mathcal{D}_+\,\lambda^A - f^{ABC}\,\Omega^B_+\,\lambda^C = 0\,, \qquad (59)$$

$$\mathcal{D}_o\,\Omega^A_- - \mathcal{D}_-\,\lambda^A - f^{ABC}\,\Omega^B_-\,\lambda^C = 0\,, \qquad (60)$$

and the cancelation of the coefficient of $V^a\wedge V^b$ gives rise to the Bianchi identity, i.e:

$$\mathcal{D}_+\,\Omega^A_- - \mathcal{D}_-\,\Omega^A_+ - \tau\,\lambda^A + f^{ABC}\,\Omega^B_+\,\Omega^C_- = 0\,. \qquad (61)$$

The coefficient of $\xi \wedge \xi$ cancels automatically. Therefore, the conclusion is that the motion field equations (54) and (55) for the fields V^a and ξ_α, are reduced to the two differential equations and the remaining conditions are all geometrical ones.

4 Second order formalism and constraints.

The second order formalism is necessary when the model is considered from the quantum point of view to separate the dynamical degrees of freedom from those gauge degrees. The space-time decomposition chosing a time direction in the manifold M^4, the manifest covariance is lost. The notation and conventions are: tanget space indices are denoted by $a, b = 1, 2, 3, 4$, space-time indices by $\mu, \nu = 0, 1, 2, 3$, space indices $i, j = 1, 2, 3$; $\eta_{ab} = (+-)$; $\epsilon_{ab} = V_{a\mu} V_{b\nu} \epsilon^{\mu\nu}$; $g^4_{\mu\nu} = V_{a\mu} V^a_\nu$. In the space-time decomposition it is convenient to introduce the shift and lapse functions N^i and $N^\perp$, which determine the components of the metric tensor. The zweibein 1-form is written $V_a = V_{a\mu} dx^\mu$, where the holonomic components are $V_{a\mu} = (V^2_{ai}, V_{a0})$. The normal n_a satisfies $n_a n^a = -1$, $n_a V^a_i = 0$, $n_a = -N^\perp V^0_a$ and $(-g^{(2)})^{1/2} = N^\perp g^{1/2}$.

The time variable is chosen so that the 1-form dt^0 can be detached. More precisely, we consider fields and forms defined on a spacelike $x^0 = t = t^0$ four-dimensional "surface" Σ, by defining the injection map. Then the associated pullback χ^* acts on any form by setting $t = t^0$ and $dt^0 = 0$.

In order to obtain the final form of the generator of time evolution in the CEF, the metricity condition must be considered. The general equation relating the spin connection ω^{ab} with the spin connection Ω^{ab} writes:

$$\omega^{ab}_i = \Omega^{ab}_i + (n^b V^{aj} - n^a V^{bj}) K_{ij} \ , \tag{62}$$

where the extrinsic curvature K_{ij} was introduced. Similarly, from the metricity condition the following equation holds:

$$\partial_i n^a + \Omega^{ab}_i n_b = 0 \ . \tag{63}$$

The second-order formalism is obtained by solving the torsion field equation, given the following results for the spin connection:

$$\omega_\mu(V, \zeta) = \omega_\mu(V) + \kappa_\mu(\zeta) \ . \tag{64}$$

with $\omega_\mu(V) = -\epsilon^{\nu\rho} \ V^a_\mu \ \partial_\nu \ V_{a\rho}$ and $\kappa_\mu(\zeta) = \frac{i}{4} \ \zeta_\mu \ \gamma^5 \ \gamma^\nu \ \zeta_\nu$, is the contorsion tensor.

All the quantities provided by the CEF, such as the total Hamiltonian, the constraints and the field equations, must be projected on the "surface" Σ. The canonical conjugate momenta π_A are written in terms of the spatial components of the holonomic basis. The Poisson brackets between pairs of canonical variables remain defined as usual. On the other hand, the CEF plays, with respect to the first order canonical component formalism, an analogous role to that played by the first order canonical component formalism with respect to the second order formalism. Therefore, in the CEF all the primary constraints remain at least weakly zero in the canonical component formalism.

Then, we assume that the restrictions to Σ of the constraints (25)-(29) are strongly equal to zero, and the remaining constraint, (24), are weakly zero quantities:

$$\chi^* \, \Phi_a = \Psi_a \approx 0 \; . \tag{65}$$

The bosonic 4-form (23) provides by the CEF can be written as follows:

$$\int \; H_T = \int \; dx^0 \; \tilde{\mathcal{H}} \; , \tag{66}$$

where the time variable is chosen so that the 1-form dx^0 can be detached.

The remaining bosonic 4-form integrated in 4D is the proper Hamiltonian generator of time evolutions:

$$\tilde{\mathcal{H}} = \int \; dx \; (\frac{1}{2} \; \omega_{0ab} \; \mathcal{H}^{ab} + V_{0a} \; \mathcal{H}^a + \bar{\xi}_{0\alpha} \;\; \mathcal{H}^\alpha) \; . \tag{67}$$

The antisymmetric weakly zero quantity $\mathcal{H}$ that appears in (67) is the generator of local Lorentz rotations, that in context of the CEF naturally appears when the space-time decomposition is carried out. Contrarily, starting from the component Hamiltonian formalism, the generator of local Lorentz rotations must be introduced *ad hoc* by demanding the closure of the constraint algebra.

It can be proven that the constraints $\mathcal{H}^a \, , \mathcal{H}^{ab}$ and $\mathcal{H}^\alpha$ are the first-class constraints that close the superalgebra:

$$[\mathcal{H}_A(x) \, , \, \mathcal{H}_B(y)] = \Lambda^C_{AB} \; \mathcal{H}_C(x) \; \delta(x-y) \; , \tag{68}$$

where $\Lambda^C_{AB} = R^C_{AB} - C^C_{AB}$ are the structure functions for curvatures R^C_{AB} and structure constant C^C_{AB} of the graded Lie algebra.

And their explicit expressions are:

$$\begin{aligned} \mathcal{H}^a \; dx^3 \;\; &= \;\; -d\eta^a \; V_b \; V_c \; \epsilon^{bc} + \eta^J \; \bar{\xi} \; \xi \; V_b \; \epsilon^{ab} + \omega \; \bar{\xi} \; \xi \; \epsilon \; \eta^a \\ &- \;\; i \; \bar{\xi} \; \gamma_b \; \eta \; \epsilon^{ab} \; V_c \; V_d \; \epsilon^{cd} + \omega \; \Phi_b \; \epsilon^{ab} \;\; \approx 0 \; . \end{aligned} \tag{69}$$

$$\mathcal{H}^{ab}\, dx^3 = \Phi^a\, V^b - \Phi^b\, V^a \approx 0\ . \tag{70}$$

$$\begin{aligned}
\mathcal{H}^{\alpha}\, dx^3 &= -\frac{4i}{\lambda}\, \gamma^5\, d\eta^{\alpha}\, V_a\, V_b\, \epsilon^{ab} + \frac{i}{2}\, \eta_a\, \gamma^a\, \xi^{\alpha}\, V_b\, V_c\, \epsilon^{bc} \\
&+ \frac{i}{\lambda}\, \eta^J\, \gamma^5\, \xi^{\alpha}\, V_a\, V_b\, \epsilon^{ab} + \frac{2i}{\lambda}\, \omega\, \eta^{\alpha}\, V_a\, V_b\, \epsilon^{ab} \\
&+ i\, \gamma^b\, \eta^{\alpha}\, \overline{\xi}\, \xi\, V^a\, \epsilon_{ab} + \frac{i}{2}\, \gamma^a\, \xi^{\alpha}\, \Phi_a\ \approx 0\ .
\end{aligned} \tag{71}$$

5 Conclusions

In this paper the dynamics of constrained system was analyzed by applying the geometrical prescriptions of the CEF on the supergroup manifold for the supersymmetric extension of the 4D Jackiw-Teiteiboim model.

As it was remarked, the CEF is not a proper canonical formalism because it does not propagate data defined on an initial surface as it is usually required by a standard mechanical system. However, as it can be seen from the above construction, the CEF is a powerful method at classical level, which enables to find the equations of motion and the constraints in a very simple way without introducing complex calculations. Also, the CEF is covariant in all its steps because of the use of exterior algebra, and all the primary constraints are second-class ones. From this affirmation, the Dirac brackets are easily defined by projecting these constraints on the surface Σ.

The relation between the form-brackets and the usual Poisson brackets was expressed by means of a non-trivial integral relationship. It is also important to note, that the torsion equation allows to obtain the second-order canonical starting from the first-order one. The torsion equation $R^a = 0$, in the Riemannian gravity case, must be considered as an strongly equal to zero constraint, and so the spin connection is solved in terms of the zweibein and the spinor field. With the purpose to get the second-order formalism, and following to the Dirac prescriptions, the space-time decomposition in M^4 was performed, losing the explicit covariance of all the equations. Thus the total Hamiltonian of the CEF is considered an appropriate Hamiltonian, generator of time evolutions. The primary constraint obtained in the CEF plays an important role in the construction of the first-class constraints.

Concluding, all the Hamiltonian gauge symmetries remain determined and the apparent gauge degrees of freedom can be eliminated leaving only the physical ones.

If the model is studied from a quantum point of view this steps are necessary. As a final conclusion, and for the study of dynamic aspects of the model, the CEF can be used as an interesting formal resource to derive the set of constraints and equations of motion due to their intrinsic geometrical language.

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