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Note on Mathematical Development of Plate Theories

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Abstract

In this article, the history of mathematical development for plate analysis is reviewed with emphasis on providing the plate's governing differential equations in rectangular coordinates. Attention is mainly devoted to theories based on the Kirchhoff, Reissner, Mindlin, and Levinson theories for isotropic plates with uniform thickness. Therefore, it is expected that this review paper should be useful for scientists and researchers in assisting them to understand and classify relevant existing plate's theories quickly.

Mathematics Subject Classification: 35Q74, 74K20

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Notation

a, b	dimensions of a rectangular plate (see Figure 1)
h	thickness of a plate
q	intensity of uniformly distributed load
w	deflection function
x, y, z	rectangular co-ordinates
D	flexural rigidity of a plate, $Eh^3/12(1-\upsilon^2)$
E	modulus of elasticity
M_x, M_y	bending moments per unit length of sections of a plate perpendicular to the *x*- and *y*-axes respectively
M_{xy}	twisting moment per unit length of section of a plate perpendicular to the *x*-axis
Q_x, Q_y	shearing forces parallel to the *z*-axis per unit length of a plate perpendicular to the *x*- and *y*-axes respectively
V_x, V_y	reactions parallel to the *z*-axis per unit length of the boundary of a plate perpendicular to the *x*- and *y*-axes respectively
υ	Poisson's ratio
Δ	Laplace's operator in *x*,*y* co-ordinates

1 Introduction

A very brief survey of the historical background for mathematical development of plate is given in the present section. However, the interested reader should consult the textbook of Szilard [4].

The first impetus to a mathematical statement of plate problems was probably done by Euler in 1776 who studied a free vibration analysis of plate problem based on the membrane theory of very thin plates. In 1802 Chladni first experimentally presented the various modes of free vibrations using the distributed powder to form regular patterns after induction of vibration.

In 1811 the French mathematician named Sophie Germain submitted a paper to the French Academy of Science. She developed and obtained a differential equation for vibration of plates that neglected the warping term of the middle surface in the strain energy expression using the calculus of variations. After corrected Germain's results by adding the missing term, the general differential equation of the free vibration of plates was first presented by Lagrange in 1813. This mentioned equation is well-known and called the Germain-Lagrange plate equation.

In 1823, the first satisfactory theory of bending of plates was derived by Navier, who considered the plate thickness in the general plate equation as a function of flexural rigidity *D* using the modern theory of elasticity. Additionally, he also introduced an exact method to transform the differential equation into algebraic equations with the use of Fourier trigonometric series. Nevertheless, this method yields mathematically exact solutions only for the case of rectangular plate simply supported on four edges.

In 1850, Kirchhoff published an important thesis on the theory of thin plates that stated two independent basic assumptions. These assumptions are now widely accepted in the plate-bending theory and are known as the Kirchhoff's hypotheses. Furthermore, Kirchhoff is considered to be the founder of the extended plate theory which takes into account the combined bending and stretching. Since the Kirchhoff's plate theory neglects the deformation caused by transverse shear, it will lead to considerable errors if applied to moderately thick or thick plates. Therefore, based on the Kirchhoff's theory, it underestimates deflections and overestimates frequencies and buckling loads for the shear deformable plates [2]. Reissner in 1945 and Mindlin in 1951 developed a rigorous plate theory which considers the deformations caused by the transverse shear forces to eliminate the deficiency of the classical plate theory.

A significant contribution and extensive studied in the area of plate bending analysis together with its application have been collected and also summarized in a fundamental monograph of Timoshenko and Woinowsky-Krieger [5] that represented a profound analysis of various plate bending problems. A reference book by Leissa [1] presents a set of available results for the frequencies and mode shapes of free vibrations of plates that could be provided for the design and for a research in the field of plate vibrations.

2 Problem description

The coordinate system to be used is shown in Figure 1. The plate has dimensions of length *a* and width *b* in the directions of *x* and *y*, respectively; and thickness *h* in *z* direction. The middle plane of the plate before bending occurs is taken as the *xy* plane and during bending; points that were in the *xy* plane undergo small displacements normal to the *xy* plane and form the middle surface of the plate. These displacements of the middle surface are then called the deflections (*w*) of a plate.

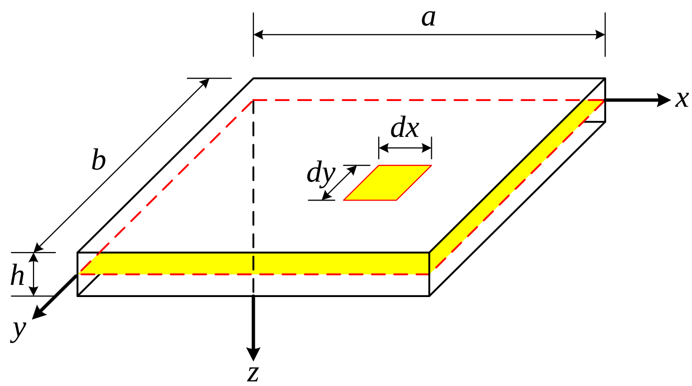

Figure 1 Coordinate system definition

Since the plate is in the deformed state under the applied external load, the resulting action of the stresses is to cause bending and twisting of the plate. Thus, the appropriate stress resultants for the plate theory are bending and twisting moments per unit length. They can be expressed in terms of the deflection as follows:

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \upsilon \frac{\partial^2 w}{\partial y^2}\right), \tag{1}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \upsilon \frac{\partial^2 w}{\partial x^2}\right), \tag{2}$$

$$M_{xy} = -M_{yx} = D(1-\upsilon)\frac{\partial^2 w}{\partial x \partial y}. \tag{3}$$

Consider an infinitesimal element *dxdy* of the plate that presented in Figure 1, the middle plane of the element cut out of the plate is then illustrated in Figure 2 showing all the moments and shearing forces acting on the sides of the element and the load distributed over the upper surface of the plate.

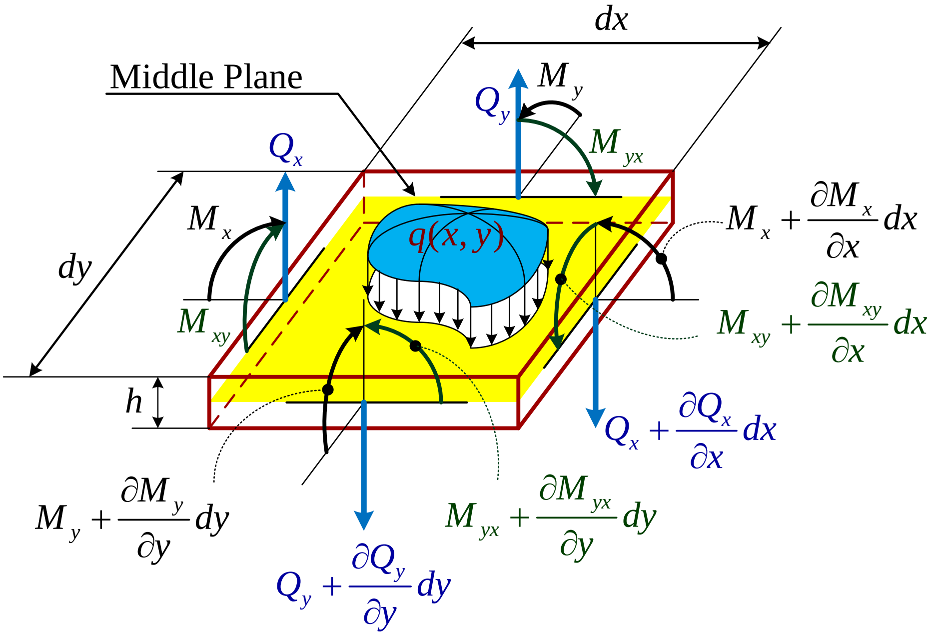

Figure 2 Positive sign conventions for stress resultants

Because the plate must be in the equilibrium states, three conditions for the equilibrium may be written by summing the forces in the *z*-direction and summing the moments about the *x* and *y* axes, setting them equal to zero as follows:

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+q=0\,, \tag{4}$$

$$\frac{\partial M_{xy}}{\partial x}-\frac{\partial M_y}{\partial y}+Q_y=0\,, \tag{5}$$

$$\frac{\partial M_{yx}}{\partial y}+\frac{\partial M_x}{\partial x}-Q_x=0\,. \tag{6}$$

Using (1) to (3), the shearing forces can be written in terms of the deflection, which are

$$Q_x=-D\left(\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w}{\partial x\partial y^2}\right), \tag{7}$$

$$Q_y=-D\left(\frac{\partial^3 w}{\partial y^3}+\frac{\partial^3 w}{\partial y\partial x^2}\right). \tag{8}$$

The above two shearing forces mean the shearing forces acting in an element of the plate. For the shearing forces distributed along the free edge or the support of the plate, they are called the supplemented or Kirchhoff shearing forces or vertical edge reactions that can be determined in the followings below,

$$V_x=Q_x-\frac{\partial M_{xy}}{\partial y}=-D\left[\frac{\partial^3 w}{\partial x^3}+(2-\upsilon)\frac{\partial^3 w}{\partial x\partial y^2}\right], \tag{9}$$

$$V_y=Q_y+\frac{\partial M_{xy}}{\partial x}=-D\left[\frac{\partial^3 w}{\partial y^3}+(2-\upsilon)\frac{\partial^3 w}{\partial x^2\partial y}\right]. \tag{10}$$

However, a more details of discussion for two equations above have been given in Timoshenko and Woinowsky-Krieger [5].

In order to determine the governing differential equation of plate, it can be derived from (4) with using (1) to (3) that results in

$$\Delta\Delta w = \frac{q}{D}, \tag{11}$$

where

$$\Delta w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} . \tag{12}$$

The obtained (11) is represented for the governing differential equation of Kirchhoff's plate theory.

3 Shear deformable plate theories

Although the classical Kirchhoff's plate theory yields sufficiently accurate results for thin plates, its accuracy decreases with growing thickness of the plate. Nowadays many plate theories exist that account for the effects of transverse shear forces. Therefore, at least three well-known plate theories are presented and listed in the following details.

Reissner's Theory

Following the theory of plate introduced by Reissner, there are two assumptions to be used. In the first, the displacement field is varied linearly through the plate thickness. The second is involved with the middle surface where plane sections remain plane but lines originally perpendicular to the middle surface do not remain perpendicular to it after bending.

Utilizing the Castigliano's theorem of least work, three simultaneous partial differential equations can be obtained as

$$D\Delta\Delta w = q - \frac{h^2}{10}\left(\frac{2-\nu}{1-\nu}\right)\Delta q , \tag{13}$$

$$Q_x = -D\frac{\partial \Delta w}{\partial x} + \frac{h^2}{10}\Delta Q_x - \frac{h^2}{10}\left(\frac{1}{1-\nu}\right)\frac{\partial q}{\partial x}, \tag{14}$$

$$Q_y = -D\frac{\partial \Delta w}{\partial y} + \frac{h^2}{10}\Delta Q_y - \frac{h^2}{10}\left(\frac{1}{1-\nu}\right)\frac{\partial q}{\partial y}. \tag{15}$$

Mindlin's Theory

This theory is the displacement-based theory following kinematic assumptions for the in-plane displacements in which thickness stretch is not allowed. It is interesting to note that the mentioned plate theory fails to satisfy the zero-stress condition on the top and bottom surfaces of the plate. Therefore, it is necessary to introduce a correction factor κ^2 that depends on the Poisson's ratios.

Utilization of the minimum potential energy theorem, a set of three differential equations can explicitly be derived as follows:

$$\kappa^2 Gh(\Delta w + \phi) + q = 0\,, \tag{16}$$

$$\frac{D}{2}\left[(1-\nu)\Delta\psi_x + (1+\nu)\frac{\partial\phi}{\partial x}\right] - \kappa^2 Gh\left(\psi_x + \frac{\partial w}{\partial x}\right) = 0\,, \tag{17}$$

$$\frac{D}{2}\left[(1-\nu)\Delta\psi_y + (1+\nu)\frac{\partial\phi}{\partial y}\right] - \kappa^2 Gh\left(\psi_y + \frac{\partial w}{\partial y}\right) = 0\,, \tag{18}$$

where ψ_x and ψ_y represent the rotations at the middle plane and ϕ defines as

$$\phi = \frac{\partial\psi_x}{\partial x} + \frac{\partial\psi_y}{\partial y}\,. \tag{19}$$

It can be noted that two foregoing theories proposed by Reissner and Mindlin are known to be the first-order shear deformable plate theories for which the in-plane displacements *u* and *v* vary linearly. Both theories provide for further development.

Levinson's Theory

In order to avoid the use of shear correction factor κ^2, a higher-order shear theory is based on the assumed in-plane displacement fields to higher-order variations.

Levinson introduced the kinematic assumptions satisfying the requirements for shear-free conditions on the top and bottom surfaces of the plates and also a parabolic distribution of shear stresses across the plate thickness. These assumptions allow the cross sections at the middle plane to warp [3].

By proposing higher-order of in-plane displacement functions to cubic function of *z*, Levinson derived an improved approximation to the theory of shear deformable plate as follows:

$$\frac{2}{3} Gh(\Delta w + \phi) + q = 0\,, \tag{20}$$

$$\frac{2D}{5}\left[(1-\nu)\Delta\psi_x + (1+\nu)\frac{\partial\phi}{\partial x} - \frac{1}{2}\frac{\partial}{\partial x}(\Delta w)\right] - \frac{2}{3} Gh\left(\psi_x + \frac{\partial w}{\partial x}\right) = 0\,, \tag{21}$$

$$\frac{2D}{5}\left[(1-\nu)\Delta\psi_y + (1+\nu)\frac{\partial\phi}{\partial y} - \frac{1}{2}\frac{\partial}{\partial y}(\Delta w)\right] - \frac{2}{3} Gh\left(\psi_y + \frac{\partial w}{\partial y}\right) = 0\,. \tag{22}$$

It can be clearly seen that three unknowns presented in (20) to (22) are the deflection *w*, and the rotations at the middle plane ψ_x and ψ_y similar to those of (16) to (18) for the Mindlin's plate theory.

4 Discussion

In the previous sections, the thin plate and three different shear deformable plates are reviewed. The Kirchhoff's plate theory gives a reasonable result in analyzing the plates which has the ratio of thickness to governing span length (*h/L*) between 1/50 and 1/10. For the moderately thick plates (1/10 < *h/L* < 1/5), the Reissner's plate and Middlin's plate theories can be used properly to obtain the accurate results. If the ratio *h/L* of the plate is larger than 1/5 (*h/L* > 1/5), the plate becomes thick plate and the use of Levinson's theory appears to be more accurate.

Additionally, it is noted that different higher-order of displacement fields can result in obtaining different refined plate theories.

5 Conclusion

The main goal of the present paper is to review some important plate's theories. A classical plate theory as well-known to be the Kirchhoff's plate, two different first-order shear deformable plate theories, namely, the Reissner's plate and Mindlin's plate, and the third-order shear deformable plate of Levinson's theory are then considered. Their governing partial differential equations are also provided.

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