The Modified Friedmann Equations

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Abstract
The Wheeler-DeWitt equation is developed from the Einstein-Hilbert action where the gravity is coupled to the mass-energy and scalar field. The wave function of the universe in Wheeler-DeWitt equation is written in polar form. Subsequently, we obtain the modified Friedmann equation. We then proceed to present the derivation of a more simplified version of modified Friedmann equations where the gravitational part of Einstein-Hilbert action is coupled only to mass-energy. These simplified version of modified first and second Friedmann equations have additional terms which contain the quantum potential energy. We then take an ansatz for the form of amplitude function and find that if the inflationary expansion really happens in the early universe, our universe (for simplicity, spatial curvature $k = 0$) might start to evolve from a non-singularity.

Keywords: Canonical quantum cosmology, de Broglie-Bohm interpretation, Modified Friedmann equations, Wheeler-DeWitt equation

1 Introduction

The Friedmann equation is important in giving the description of the evolution of the universe. It was first derived by Alexander Friedmann in 1922 from the famous Einstein’s field equation. It is obtained by plugging the metric of a spatially homogeneous and isotropic universe into Einstein’s field equation. The metric for an isotropic and homogeneous universe is the Robertson-Walker metric which is given by [1]
\[ ds^2 = -c^2 dt^2 + a^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right). \] \hspace{1cm} (1)

The metric (1) is written using the spherical coordinates system. In addition, the symbols \( c \), \( k \) and \( a = a(t) \) respectively denote the speed of light, spatial curvature and scale factor of the universe. The spatial curvature \( k \) here can take the value \(+1\), \( 0 \) or \(-1\) for a universe of spatially closed, flat or open respectively. Besides this, the scale factor \( a \) has the dimension length. The Einstein’s field equation is needed for determining the geometry of space-time due to a source of energy-momentum and given as follows [1, 2]:

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}, \] \hspace{1cm} (2)

where \( R_{\mu\nu} \), \( g_{\mu\nu} \), \( R \) and \( T_{\mu\nu} \) are Ricci tensor, metric tensor, Ricci scalar and energy-momentum tensor respectively. The symbol \( G \) denotes the Newton’s gravitational constant. The energy-momentum tensor for substance of a perfect fluid is given as follows:

\[ T_{\mu\nu} = (\varepsilon + p) u_{\mu} u_{\nu} + pg_{\mu\nu}, \] \hspace{1cm} (3)

where \( \varepsilon \) and \( p \) are the mass-energy density and pressure respectively. In addition, the symbol \( u_{\mu} \) denotes the four-velocity of substance. After plugging the Robertson-Walker metric (1) into Einstein’s field equation (2), we obtain the following two equations:

\[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{c^2 k}{a^2} = \frac{8\pi G \varepsilon}{3c^2}, \] \hspace{1cm} (4)

and

\[ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2} (\varepsilon + 3p). \] \hspace{1cm} (5)

The notation dot denotes the differentiation with respect to time \( t \). Equation (4) is also called the first Friedmann equation, while equation (5) is referred as the second Friedmann equation or Friedmann’s acceleration equation. We can obtain the scale factor \( a \) as a function of time \( t \) by solving the first Friedmann equation for \( a = a(t) \) as long as we know the dependence of \( \varepsilon \) on \( a \).

Equations (4) and (5) give the description of evolution of the universe in the so-called standard Hot Big Bang model. The pressure and energy density can also be written as follows [3]:
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\[ p = we \]  \hspace{1cm} \text{(6)}

and

\[ \varepsilon = \frac{\lambda_{(n)}}{a^n} = \frac{\lambda_{(n)}}{a^{(n+1)}} \]  \hspace{1cm} \text{(7)}

where \( w \) and \( \lambda_{(n)} \) are constants. We summarize the value of \( n \) and \( w \) for some types of mass-energy in the following table [3, 4, 5]:

**TABLE (1).** Some types of mass-energy and their values of \( w \) and \( n \).

<table>
<thead>
<tr>
<th>Type of Mass-Energy</th>
<th>( w )</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radiation</td>
<td>1/3</td>
<td>4</td>
</tr>
<tr>
<td>Pressure Less Matter</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>Vacuum Energy</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

Physicists also believe that the very early universe underwent an inflationary expansion which is guessed to be governed by the energy density associated with a scalar field \( \phi \). The energy-momentum tensor of the scalar field is as follows [1]:

\[ T^\phi_{\mu\nu} = c^2 \frac{\partial \phi}{\partial q^\mu} \frac{\partial \phi}{\partial q^\nu} + g_{\mu\nu} \bar{L}_\phi, \]  \hspace{1cm} \text{(8)}

where \( \bar{L}_\phi \) and \( q^\mu \) are Lagrangian density for scalar field and a generalized coordinate respectively. The Lagrangian density of a scalar field is given by

\[ \bar{L}_\phi = -\frac{c^2}{2} g^{\mu\nu} \frac{\partial \phi}{\partial q^\mu} \frac{\partial \phi}{\partial q^\nu} - V(\phi), \]  \hspace{1cm} \text{(9)}

where \( V(\phi) \) is the potential energy of scalar field. If the universe is homogeneous and its source of energy-momentum is scalar field, then the Friedmann equation (4) becomes

\[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{c^2 k}{a^2} = \frac{8\pi G}{3c^2} \left( \frac{1}{2} \dot{\phi}^2 + V(\phi) \right), \]  \hspace{1cm} \text{(10)}

The slow rolling of scalar field is said to have occurred in the very early universe where \( \dot{\phi} \parallel 0 \) and \( V(\phi) \) is constant.

In this study, we are going to write down the Einstein-Hilbert action
(gravitational part coupled to mass-energy and scalar field) and reformulate it in the Hamiltonian form. Subsequently, we apply the canonical quantization procedure to the Hamiltonian to obtain the Wheeler-DeWitt equation. To illustrate the de Broglie-Bohm interpretation [6, 7, 8], we write down the wave function of the universe in polar form, \( \psi = A \exp \left( \frac{iS}{\hbar} \right) \). After taking the derivatives and separating into real and imaginary parts, a modified form of Hamilton-Jacobi equation and a continuity equation for probability are obtained.

The modified Hamilton-Jacobi equation differs from the usual classical Hamilton-Jacobi equation by the additional term which is called the quantum potential energy. In this study, we obtain two types of quantum potential energy which are due to the scale factor and scalar field. Subsequently, the terms \( \nabla S \) in the modified Hamilton-Jacobi equation are replaced by the corresponding canonical momenta. This step would lead us to obtain the modified form of Friedmann equations. We regard the usual classical Friedmann equations are valid only for classical systems. In other words, they are applicable only to the late-time universe. On the other hand, the modified Friedmann equations are taken to be valid even when the universe was just after its beginning where the quantum effect is important. The quantum potential energy plays a significant role for the quantum effect that is significant in the very early universe. We also show that the quantum potential energy due to the scale factor is a possible candidate for the cause of inflationary expansion of the very early universe.

De Broglie-Bohm interpretation is an alternative to the orthodox Copenhagen interpretation. Many works have been done using de Broglie-Bohm interpretation to interpret the wave function of the universe [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. It could provide the definite and continuous values of a variable. This is quite different from the Copenhagen interpretation which actually is a probabilistic theory. Therefore, de Broglie-Bohm interpretation is appropriate to be applied to the wave function of the universe in which the universe is evolving continuously with time.

In the next section, we present the de Broglie-Bohm interpretation in the homogeneous minisuperspace models. In section three, we present the derivation of Wheeler-DeWitt equation of the homogeneous and isotropic universe. In section four and five, we obtain the modified form of Friedmann equations and show that in a special case, the modified second Friedmann equation may imply the inflationary expansion of the very early universe. We also show that in the special case, the value of the scale factor does not equal to zero when \( t = 0 \). Finally, we end with our discussions and conclusions in section six.

## 2 The homogeneous minisuperspace cosmological model

First of all, we introduce the Einstein-Hilbert action where the gravity is coupled to the mass-energy and scalar field and it is written as follows [1, 12, 20]:
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\[ S = \int \sqrt{-g} \left[ \frac{Rc^4}{16\pi G} - \varepsilon + L_\phi \right] dr d\theta d\varphi dt, \]  

(11)

where \( g \) is the determinant of the metric. Next, we decompose the space-time of the homogeneous universe into space and time variables with the help of lapse function \( N \) and shift vectors \( N_i \). In the case of homogeneous minisuperspace cosmological models, the shift vectors are taken to be zero \( N_i = 0 \) and Einstein-Hilbert action (11) can be written as follows [9, 10, 19, 21, 22]:

\[ S = \int \left[ \frac{1}{2N} F_{AB} (q) \dot{q}^A \dot{q}^B - NU (q) + \frac{1}{2N} F_{(\phi)} \dot{\phi}^2 - NF_{(\phi)} V (\phi) \right] dt, \]

(12)

where \( F_{AB} \) is the simplified version of the DeWitt metric [21, 22]. As in the standard way, the momenta conjugate to \( q^A, \phi \) and Hamiltonian are obtained respectively as

\[ p_A = \frac{\partial L}{\partial \dot{q}^A} = \frac{F_{AB} \dot{q}^B}{N}, \]

(13)

\[ p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \frac{F_{(\phi)} \dot{\phi}}{N}, \]

(14)

and

\[ H = p_A \dot{q}^A + p_\phi \dot{\phi} - L = N \left[ \frac{1}{2} F_{AB} p_A p_B + U + \frac{1}{2} F_{(\phi)} \dot{\phi}^2 + F_{(\phi)} V (\phi) \right]. \]

(15)

The Hamiltonian constraint gives (for details, see [23])

\[ \frac{1}{2} F_{AB} p_A p_B + U + \frac{1}{2} F_{(\phi)} \dot{\phi}^2 + F_{(\phi)} V (\phi) = 0. \]

(16)

Canonical quantization procedure is then applied to equation (16) to obtain the Wheeler-DeWitt equation:

\[ \hat{H} \left( q^A, \phi; -i\hbar \frac{\partial}{\partial q^A}, -i\hbar \frac{\partial}{\partial \phi} \right) \psi (q, \phi) = 0. \]

(17)

The wave function \( \psi (q, \phi) \) in (17) is a complex function and written in the form:
\[ \psi(q) = R(q, \phi) \cdot \exp \left[ \frac{iS(q, \phi)}{\hbar} \right]. \]

After taking the derivatives, Wheeler-DeWitt equation (17) reduces to the following two equations [9, 10, 19]:

\[ \frac{1}{2} F^{AB} \left( \frac{\partial S}{\partial q^A} \right) \left( \frac{\partial S}{\partial q^B} \right) + Q_{AB} + U + \frac{1}{2} F^{(\phi)} \left( \frac{\partial S}{\partial \phi} \right)^2 + Q_{\phi} + F^{(\phi)} V(\phi) = 0, \] (18)

and

\[ \frac{1}{2} F^{AB} \frac{\partial}{\partial q^A} \left( A^2 \frac{\partial S}{\partial q^B} \right) + \frac{1}{2} F^{(\phi)} \frac{\partial}{\partial \phi} \left( A^2 \frac{\partial S}{\partial \phi} \right) = 0. \] (19)

In de Broglie-Bohm interpretation, equation (18) is called the modified Hamilton-Jacobi equation, whereas equation (19) is the continuity equation for probability. The new additional potential energy are quantum potential energy due to scale factors of three spatial directions \( Q_{AB} = -\frac{1}{2} F^{AB} \frac{\hbar^2}{R} \frac{\partial^2 R}{\partial q^A \partial q^B} \) and quantum potential energy due to scalar field \( Q_{\phi} = -\frac{1}{2} F^{(\phi)} \frac{\hbar^2}{R} \frac{\partial^2 R}{\partial \phi^2} \), which are added to the usual classical Hamilton-Jacobi equation to form the modified Hamilton-Jacobi equation (18). The quantum potential energy is important in manifesting the quantum effects. We could thus identify

\[ p_A = \frac{\partial S}{\partial q^A}, \] (20)

and

\[ p_{\phi} = \frac{\partial S}{\partial \phi}. \] (21)

Some useful differential equations can be established by relating equations (13) and (14) to (20) and (21). These equations are obtained and given as follows:

\[ \frac{\partial S}{\partial q^A} = F_{AB} \dot{q}^B / N, \] (22)

and
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\[ \frac{\partial S}{\partial \Phi} = \frac{F(\Phi)}{N}. \] (23)

Equations (22) and (23) are known as guidance equations. We may want to solve these equations to obtain the evolution of \( q \) with respect to time \( t \).

3 The homogeneous and isotropic cosmological model

In this section, we study a more specific cosmological model which is isotropic and homogeneous universe. First of all, we decompose a homogeneous and isotropic space-time into space and time variables. The corresponding metric is given as follows [21, 22]:

\[ ds^2 = -N^2c^2dt^2 + a^2 \left( \frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right), \] (24)

where \( N = N(t) \) is the lapse function and dimensionless. The scale factor, \( a = a(t) \) here takes the dimension length. The \( q^A \) in the section 2 turns out to be the scale factor \( a \) for this homogeneous and isotropic model. We subsequently compute the Ricci scalar and determinant of metric (24) and give them as follows:

\[ R = \frac{6\ddot{a}}{N^2c^2a} - \frac{6\dot{N}\dot{a}}{N^2c^2a} + \frac{6\dot{a}^2}{N^2c^2a^2} + \frac{6k}{a^2}, \] (25)

\[ g = -\frac{N^2a^6r^4\sin^2 \theta}{1-kr^2}. \] (26)

Next, we substitute the Ricci scalar (25), determinant of metric (26) and Lagrangian density of scalar field (9) into Einstein-Hilbert action (11). Consequently, the Einstein-Hilbert action becomes

\[ S = V_o \left[ \frac{3\dddot{a}^2c^2}{8\pi GN} + \frac{3\dot{a}^2ac^2}{8\pi GN} + \frac{3\dot{N}a\dot{c}^2}{8\pi GN^2} + \frac{3kaNc^4}{8\pi G} - \frac{N\dot{a}^3c}{2N} - \frac{\dot{\phi}^2a^3}{2N} - Na^3V(\phi) \right] dt, \] (27)

where \( V_o = \int \frac{cr^2\sin \theta}{\sqrt{1-kr^2}} drd\theta d\phi \). Integrating the first term in the square bracket by parts with respect to \( t \), Einstein-Hilbert action (27) reduces to
\[ S = V_o \int \left[ -\frac{3\epsilon^2 a \dot{a}^2}{8\pi G N} + \frac{3\kappa a N c^4}{8\pi G} - Na^3 \epsilon + \frac{\phi^2 a^3}{2N} - Na^3 V(\phi) \right] dt, \]  

We set \( V_o \) to one. This is reasonable since for any given value of \( t \), the geometry of our universe is identical everywhere. Hence, the value of \( V_o = \int \frac{cr^2 \sin \theta}{\sqrt{1 - kr^2}} dr d\theta d\phi \) can be set to one by integrating over an appropriate compact region of space \([24]\). Thus, we obtain the Lagrangian \( L \) as

\[ L = -\frac{3\epsilon^2 a \dot{a}^2}{8\pi G N} + \frac{3\kappa a N c^4}{8\pi G} - Na^3 \epsilon + \frac{\phi^2 a^3}{2N} - Na^3 V(\phi). \]  

The momenta conjugate to \( a \) and \( \phi \) are given respectively as

\[ P_a = \frac{\partial L}{\partial \dot{a}} = -\frac{3a \dot{a} c^2}{4\pi G N}, \]  

and

\[ P_\phi = \frac{\partial L}{\partial \dot{\phi}} = \frac{a^3 \dot{\phi}}{N}. \]  

The Hamiltonian \( H \) is obtained as follows:

\[ H = P_a \dot{a} + P_\phi \dot{\phi} - L = N \left( -\frac{2\pi G P_a^2}{3a c^2} + \frac{P_\phi^2}{2a^3} - \frac{3\kappa a c^4}{8\pi G} + a^3 \epsilon + a^3 V(\phi) \right). \]  

The Hamiltonian constraint provides \( H = 0 \). Therefore, we have

\[ -\frac{2\pi G P_a^2}{3a c^2} + \frac{P_\phi^2}{2a^3} - \frac{3\kappa a c^4}{8\pi G} + a^3 \epsilon + a^3 V(\phi) = 0. \]  

Next, the canonical quantization procedure is applied to equation (33) in which the momenta \( P_a \) and \( P_\phi \) are replaced by \( -i\hbar \frac{\partial}{\partial a} \) and \( -i\hbar \frac{\partial}{\partial \phi} \) respectively. Consequently, equation (33) becomes

\[ \frac{2\pi G h^2}{3a c^2} \frac{\partial^2 \psi}{\partial a^2} - \frac{\hbar^2}{2a^3} \frac{\partial^2 \psi}{\partial \phi^2} - \frac{3\kappa a c^4 \psi}{8\pi G} + a^3 \epsilon \psi + a^3 V(\phi) \psi = 0. \]  

Equation (34) is known as the Wheeler-DeWitt equation and it governs the wave function \( \psi \) of the homogeneous and isotropic universe. The wave function contains the information of evolution of the universe.
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We write down the wave function $\psi$ as follows [6, 7, 8]:

$$\psi = A \cdot \exp \left[ \frac{iS}{\hbar} \right],$$

(35)

where $A = A(a, \phi)$ and $S = S(a, \phi)$ are the real amplitude and real phase functions. Next we substitute the wave function (35) into Wheeler-DeWitt equation (34). After taking the derivatives and separating the real and imaginary parts, we obtain the following two equations:

$$-\frac{2\pi G}{3c^2 a} \left( \frac{\partial S}{\partial a} \right)^2 + a' \varepsilon - \frac{3kac^4}{8\pi G} - \left( -\frac{2\pi G \hbar^2}{3c^2 a} \frac{\partial^2 A}{\partial a^2} \right) + \frac{1}{2a'} \left( \frac{\partial S}{\partial \phi} \right)^2 + a' V(\phi) + \left( -\frac{\hbar^2}{2a'} \frac{\partial^2 A}{\partial \phi^2} \right) = 0$$

(36)

and

$$-\frac{2\pi G}{3ac^2} \left[ \frac{\partial}{\partial a} \left( A^2 \frac{\partial S}{\partial a} \right) \right] + \frac{1}{2a^3} \left[ \frac{\partial}{\partial \phi} \left( A^2 \frac{\partial S}{\partial \phi} \right) \right] = 0.$$  

(37)

Equation (37) is viewed as the continuity equation for probability. In de Broglie-Bohm interpretation, equation (36) is regarded as the modified version of Hamilton-Jacobi equation. Hence from (22) and (23), we have the following two equations:

$$\frac{\partial S}{\partial a} = -\frac{3a a c^2}{4\pi G N},$$

(38)

and

$$\frac{\partial S}{\partial \phi} = \frac{a^2 \dot{\phi}}{N}.$$  

(39)

Subsequently, substituting (38) and (39) into (36) and taking a gauge choice of $N = 1$, we obtain an equation as follows:

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G \varepsilon}{3c^2} - \frac{k c^2}{a^2} + \frac{8\pi G V(\phi)}{3c^2} + \frac{4\pi G \dot{\phi}^2}{3c^2} + \frac{8\pi G Q}{3c^2 a^2} + \frac{8\pi G Q_\phi}{3c^2 a^4}.$$  

(40)

Equation (40) is a modified form of Friedmann equation where the usual app-
licable only to the late-time universe. The last two terms in equation (40) are related to the quantum potential energy which will play the role for quantum effect. The quantum effect is significant in the very early universe. Hence, we expect that the last two terms of equation (40) are important for the very early universe. However, these two terms should become negligible in the late-time universe. We also notice that if the amplitude of the wave function of the universe is constant, then equation (40) would reduce to the usual classical Friedmann equation.

5 A more specific model

In this section, we show that besides the scalar field, the quantum potential energy due to the scale factor is also a possible candidate for the cause of inflationary expansion of the very early universe. First of all, we consider a special case where the action (11) reduces to [12, 20, 25, 26]

$$S = \int \sqrt{-g} \left[ \frac{R c^4}{16\pi G} - \varepsilon \right] dr d\theta d\phi cdt .$$

(41)

Hence, the corresponding amplitude and phase of the wave function are given by $A = A(a)$ and $S = S(a)$ respectively. Consequently, the modified first Friedmann equation and continuity equation for probability reduce to

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G \varepsilon}{3c^2} - \frac{kc^2}{a^2} + \frac{8\pi G Q_a}{3c^2 a^3} ,$$

(42)

and

$$\frac{d}{da} \left( A^2 \frac{dS}{da} \right) = 0 .$$

(43)

The quantum potential energy $Q_a$ is related to the amplitude function $A = A(a)$ and given as follows:

$$Q_a = \frac{-2\pi G \hbar^2}{3c^2 a} \frac{d^2 A}{A \, da^2} .$$

(44)

For information, the reference [5] has shown the derivation of the modified Friedmann equation from action (41) for a spatially flat universe ($k = 0$). Next, we proceed to write the energy density $\varepsilon$ in (42) as $\frac{\lambda(a)}{a^3}$ (please refer to equation (7)) and take $\frac{8\pi G Q_a}{3c^2 a^3} = \beta$. Consequently, equation (42) becomes
The modified Friedmann equations

\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G \lambda_{(n)}}{3c^2 a^n} - \frac{kc^2}{a^2} + \beta. \]  

(45)

Let us now differentiate both sides of equation (45) with respect to time \( t \). We thus obtain the following equation:

\[ \frac{\ddot{a}}{a} = \left( \frac{\dot{a}}{a} \right)^2 - \frac{4\pi G n \lambda_{(n)}}{3c^2 a^n} + \frac{kc^2}{a^2} + \frac{a\dot{\beta}}{2\dot{a}}. \]  

(46)

Referring to equation (7), we could write down \( n \) in equation (46) in term of \( \omega \) as

\[ n = 3(1 + \omega). \]  

(47)

Besides this, we notice that the first term of the right hand side of equation (46) is just given by (45). Hence, by substituting equation (45) and (47) into (46), we obtain [5]

\[ \frac{\ddot{a}}{a} = -\frac{4\pi G \lambda_{(n)}}{3c^2 a^n} - \frac{12\pi G \omega \lambda_{(n)}}{3c^2 a^n} + \frac{a\dot{\beta}}{2\dot{a}} + \beta. \]  

(48)

With the help of equations (6) and (7), equation (48) can be rewritten as follows:

\[ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3\rho) + \frac{a\dot{\beta}}{2\dot{a}} + \beta. \]  

(49)

Equation (49) is the modified second Friedmann equation (modified Friedmann’s acceleration equation). Next, we proceed to find the amplitude of the wave function, \( A \) by writing it as a function of the scale factor \( a \). Finding the amplitude of the wave function, \( A \) is to obtain the quantum potential energy which is believed to play the important role for the very early universe.

Next, the amplitude of the wave function of the universe is assumed to take the following simple form:

\[ A = Ka^m, \]  

(50)

where \( K \) and \( m \) are constant and real number respectively. We have to keep in mind that the wave function of the universe is very likely to be in the form of sinusoidal as the universe is evolving. Hence, equation (50) can be used to only describe the amplitude of the wave function for a particular short duration of time by taking an appropriate value of \( m \). In other words, different values of \( m \) are needed for different times in the evolution of the universe since the amplitude of wave function is changing as the universe is evolving. In addition, we may add a
constant $C$ to the right hand side of equation (50) such that $A = Ka^n + C$. However, the initial condition at which $\psi = 0$ when $a = 0$ requires $C = 0$. Besides this, the different value of constant $C$ in the amplitude function $A$ does not bring any change to the value of quantum potential energy $Q_a$ for the subsequent evolution of the universe. The quantum potential energy $Q_a$ is then computed and given as follows:

$$Q_a = \frac{2\pi G h^2 (m-1)m}{3c^2a^3}. \quad (51)$$

Subsequently, $\beta$ is given as $\beta = \frac{16\pi^2 G^2 h^3 (m-1)m}{9c^4 a^6}$. Hence, equation (45) can be written as follows:

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G \lambda_0}{3c^2a^n} - \frac{ke^2}{a^2} + \frac{16\pi^2 G^2 h^3 (m-1)m}{9c^4 a^6}. \quad (52)$$

The last term of equation (52) is regarded as the term of quantum correction. Computing $\dot{\beta}$ and substituting it into equation (49), we have

$$\ddot{a} = -\frac{4\pi G}{3c^2} (\varepsilon + 3p) - \frac{32\pi^2 G^2 h^3 (m-1)m}{9c^4 a^6}. \quad (53)$$

Equations (52) and (53) are the modified first and second Friedmann equations which are obtained from inserting the amplitude (50) into (45) and (49).

We proceed to solve a special case of equation (52) in which $k = 0$. The spatial curvature $k = 0$ is chosen purely due to simplicity. The equation (52) for $k = 0$ and $n = 3$ is given as follows:

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G \lambda_3}{3c^2a^3} + \frac{16\pi^2 G^2 h^3 (m-1)m}{9c^4 a^6}. \quad (54)$$

We thus have

$$\int_{a_0}^{a} \frac{1}{\sqrt{\frac{8\pi G \lambda_3}{3c^2a^3} + \frac{16\pi^2 G^2 h^3 (m-1)m}{9c^4 a^6}}} \, da = \int_{b_0}^{b} \, dt. \quad (55)$$

Integrating (55), we obtain
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\[
2 \left( \frac{8\pi G \lambda_1 a^3}{3c^2} + \frac{16\pi^2 G^2 \hbar^2 (m-1)m}{9c^4} \right)^{1/2} - 2 \left( \frac{8\pi G \lambda_1 a_0^3}{3c^2} + \frac{16\pi^2 G^2 \hbar^2 (m-1)m}{9c^4} \right)^{1/2} = \frac{t - t_0}{24 \pi G \lambda_1 / 3c^2}
\]

(56)

We conclude that

\[
t = 2c_1 \left( \frac{c^2 a_1^3}{24\pi G \lambda_1} \right)^{1/2}
\]

(57)

where \( c_1 \) is a positive constant. In the matter-dominated universe, the value of the scale factor, \( a \) is large. Hence, the value of term of quantum correction in (57) is relatively small and negligible. Equation (57) can then be reduced to

\[
t = 2c_1 \left( \frac{c^2 a_1^3}{24\pi G \lambda_1} \right)^{1/2}
\]

which is just same as the solution of classical Friedmann equation (4). Subsequently, we write down the equation (52) for \( k = 0 \) and \( n = 4 \) as follows:

\[
\left( \frac{a}{a} \right)^2 = \frac{8\pi G \lambda_1}{3c^2 a^4} + \frac{16\pi^2 G^2 \hbar^2 (m-1)m}{9c^4 a^6}.
\]

(58)

We then repeat the same step as from (55) to (57) for equation (58), we would have the following equation [27]:

\[
t = \frac{c_2 a}{\sqrt{\frac{8\pi G \lambda_1}{3c^2} a^2 + \frac{16\pi^2 G^2 \hbar^2 (m-1)m}{9c^4}}} - \frac{16c_2 \pi^2 G^2 \hbar^2 (m-1)m}{9c^4} \log \left( \sqrt{\frac{8\pi G \lambda_1}{3c^2} a^2 + \frac{16\pi^2 G^2 \hbar^2 (m-1)m}{9c^4}} + a \sqrt{\frac{8\pi G \lambda_1}{3c^2}} \right) + \frac{a_0}{2} \sqrt{\frac{8\pi G \lambda_1}{3c^2}}^{1/2}
\]

(59)
where $c_2$ is a positive constant. In the radiation-dominated universe, the value of the scale factor is small and the term of quantum correction is now becoming relatively important.

The last term of modified second Friedmann equation (53) is playing the important role for the early universe (such as the radiation-dominated universe) at which the scale factor is small. However, this last term becomes negligible in the late-time universe where the value of the scale factor is large. If we assume that the early inflationary expansion ($\ddot{a} > 0$) really happens, then the last term of equation (53) should take a positive value in order to obtain the condition $\ddot{a} > 0$. This is achieved if $(m-1)m < 0$ or $0 < m < 1$. Hence, we have the condition $(m-1)m < 0$ for the early universe. As mentioned in above, the value of $m$ would vary as the universe evolves. Therefore here, we actually assume that the value of $m$ is constant for a short duration of time after the universe began. On the other hand, we also realize that in order to have a real value of time in equation (59), we must have

$$\frac{8\pi G\lambda_{(4)}}{3c^2}a^2 + \frac{16\pi^2G^2\hbar^2 (m-1)m}{9c^4} > 0. \quad (60)$$

However, we have the condition $(m-1)m < 0$ for the early universe. Therefore,

$$\frac{8\pi G\lambda_{(4)}}{3c^2}a^2 > \left| \frac{16\pi^2G^2\hbar^2 (m-1)m}{9c^4} \right|. \quad (61)$$

The inequality (61) is valid if the scale factor $a$ has a minimum value and this implies $a \neq 0$. Besides this, we also find that by setting $t = 0$ in equation (59), the value of the scale factor is not equal to zero.

6 Discussions and conclusions

We regard quantum theory as a universal theory. In other words, it can be used to explain physical phenomenon ranging from microscopic to macroscopic scale. In this study, the amplitude of the wave function at a particular short duration of time is taken to be approximately described by equation (50). The value of $m$ would vary as the universe is evolving. We notice that if $m = 1$ or $m = 0$, equations (52) and (53) would reduce to the usual classical first and second Friedmann equations.

Besides this, the last term of equation (53) becomes important only when the value of the scale factor $a$ is small. In other words, it plays the important role in the very early universe.
On the other hand, the last term of equation (53) becomes unimportant and negligible in the late-time universe. In addition, the inflationary expansion \((\ddot{a} > 0)\) of early universe could happen if the last term of equation (53) takes a positive value. This is achieved if \((m-1)m < 0\). The possibility for the last term of (53) to have a positive value is especially important for the early universe. Many physicists believe that our universe underwent an inflationary expansion \((\ddot{a} > 0)\) in the very early universe. However, the Standard Cosmology (the usual classical second Friedmann equation (5)) could not explain this. In Standard Cosmology, the right hand side of equation (5) is taking a negative value (for ordinary matter). Therefore, a scalar field is thought to take the responsibility for the inflationary expansion. However here, we have showed that without the scalar field, there is still a possibility for us to have an early inflationary expansion from the modified second Friedmann equation (53).

By imposing the condition \((m-1)m < 0\) to equation (59) and assuming the value of \(m\) as constant for a short duration of time after the universe began, we find that the value of the scale factor \(a\) cannot be zero as shown in inequality (61). Besides this, we also find that from equation (59), the value of the scale factor is finite when \(t = 0\). The result obtained here implies that our universe started to evolve from a physical system of finite size and subsequently underwent an inflationary expansion. The early universe investigated here is assumed to be dominated by radiation and quantum potential energy where quantum potential energy is responsible for the inflationary expansion of early universe.

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