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On the Symmetric Properties for the Generalized Twisted q-Tangent Numbers and Polynomials

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Abstract

We study the symmetry for the generalized twisted q-tangent numbers $T_{n,\chi,q,\zeta}$ and polynomials $T_{n,\chi,q,\zeta}(x)$. We obtain some interesting identities of the power sums and the generalized twisted q-tangent polynomials $T_{n,\chi,q,\zeta}(x)$ using the symmetric properties for the p-adic invariant integral on $\mathbb{Z}_p$.

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1 Introduction

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $g \in UD(\mathbb{Z}_p)$ the fermionic p-adic invariant q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{see [2]}.$$

Note that

$$\lim_{q \to 1} I_{-q}(g) = I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x). \tag{1.1}$$

If we take $g_n(x) = g(x+n)$ in (1.1), then we see that

$$I_{-1}(g_n) = (-1)^n I_{-1}(g) + 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} g(l). \tag{1.2}$$

Let a fixed positive integer d with $(p, d) = 1$, set

$$X = X_d = \varprojlim_{N} (\mathbb{Z}/dp^N\mathbb{Z}), X_1 = \mathbb{Z}_p, \quad X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$
$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \le a < dp^N$. It is easy to see that

$$I_{-1}(g) = \int_X g(x) d\mu_{-1}(x) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x). \tag{1.3}$$

Let $T_p = \cup_{N \ge 1} C_{p^N} = \lim_{N \to \infty} C_{p^N}$, where $C_{p^N} = \{\zeta | \zeta^{p^N} = 1\}$ is the cyclic group of order p^N. For $\zeta \in T_p$, we denote by $\phi_\zeta : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto \zeta^x$. In [6], we introduced the generalized twisted q-tangent numbers $T_{n,\chi,q,\zeta}$ and polynomials $T_{n,\chi,q,\zeta}(x)$ attached to χ. Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1 (\text{mod } 2)$. The generalized twisted q-tangent numbers $T_{n,\chi,q,\zeta}$ attached to χ are defined by the generating function:

$$\frac{2 \sum_{a=0}^{d-1} \chi(a)(-1)^a \zeta^a q^a e^{2at}}{\zeta^d q^d e^{2dt} + 1} = \sum_{n=0}^{\infty} T_{n,\chi,q,\zeta} \frac{t^n}{n!}, \text{ cf. [6]}. \tag{1.4}$$

We consider the generalized twisted q-tangent polynomials $T_{n,\chi,q,\zeta}(x)$ attached to χ as follows:

$$\left(\frac{2 \sum_{a=0}^{d-1} \chi(a)(-1)^a \zeta^a q^a e^{2at}}{\zeta^d q^d e^{2dt} + 1} \right) e^{xt} = \sum_{n=0}^{\infty} T_{n,\chi,q,\zeta}(x) \frac{t^n}{n!}, \text{ cf. [6]}. \tag{1.5}$$

Theorem 1.1 *([6]) For positive integers n and $\zeta \in T_p$, we have*

$$T_{n,\chi,q,\zeta}(x) = \int_X \chi(y)\phi_\zeta(y)q^y(2y+x)^n d\mu_{-1}(y).$$

Corollary 1.2 *([6]) For positive integers n and $\zeta \in T_p$, we have*

$$T_{n,\chi,q,\zeta} = \int_X \chi(y)\phi_\zeta(y)q^y(2y)^n d\mu_{-1}(y).$$

Theorem 1.3 *([6]) For positive integers n and $\zeta \in T_p$, we have*

$$T_{n,\chi,q,\zeta}(x) = \sum_{l=0}^{n} \binom{n}{l} T_{l,\chi,q,\zeta} x^{n-l}.$$

2 Symmetry for the generalized twisted tangent polynomials

In this section, we assume that $\zeta \in T_p$. We obtain some interesting identities of the power sums and the generalized twisted q-tangent polynomials $T_{n,\chi,q,\zeta}(x)$ using the symmetric properties for the p-adic invariant integral on $\mathbb{Z}_p$. If n is odd from (1.2), we obtain

$$I_{-1}(g_n) + I_{-1}(g) = 2\sum_{k=0}^{n-1}(-1)^k g(k) \text{ (see [2]).} \tag{2.1}$$

It will be more convenient to write (2.1) as the equivalent integral form

$$\int_{\mathbb{Z}_p} g(x+n)d\mu_{-1}(x) + \int_{\mathbb{Z}_p} g(x)d\mu_{-1}(x) = 2\sum_{k=0}^{n-1}(-1)^k g(k). \tag{2.2}$$

Substituting $g(x) = \chi(x)(\zeta q)^x e^{2xt}$ into the above, we have

$$\begin{aligned}&\int_X \chi(x+n)\zeta^{x+n}q^{(x+n)}e^{(2x+2n)t}d\mu_{-1}(x) + \int_X \chi(x)\zeta^x q^x e^{2xt}d\mu_{-1}(x)\\ &= 2\sum_{j=0}^{n-1}(-1)^j\chi(j)\zeta^j q^j e^{2jt}.\end{aligned} \tag{2.3}$$

For $k \in \mathbb{Z}_+$, let us define the power sums $\mathcal{T}_{k,\chi,q,\zeta}(n)$ as follows:

$$\mathcal{T}_{k,\chi,q,\zeta}(n) = \sum_{l=0}^{n}(-1)^l\chi(l)\zeta^l q^l(2l)^k. \tag{2.4}$$

After some elementary calculations, we obtain

$$\begin{aligned}&\zeta^{-n}q^{-n}e^{-2nt}\int_X \chi(x)\zeta^{x+n}q^{(x+n)}e^{(2x+2n)t}d\mu_{-1}(x)\\&=\frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a\zeta^a q^a e^{2at}}{\zeta^d q^d e^{2dt}+1}.\end{aligned}\tag{2.5}$$

From the above, we get

$$\begin{aligned}&\int_X \chi(x)\zeta^{x+nd}q^{(x+nd)}e^{(2x+2nd))t}d\mu_{-1}(x)+\int_X \chi(x)\zeta^x q^x e^{2xt}d\mu_{-1}(x)\\&=\frac{2\int_X \chi(x)\zeta^x q^x e^{2xt}d\mu_{-1}(x)}{\int_X \zeta^{ndx}q^{ndx}e^{2ndtx}d\mu_{-1}(x)}.\end{aligned}\tag{2.6}$$

By substituting Taylor series of e^{2xt} into (2.3), we obtain

$$\begin{aligned}&\zeta^{nd}q^{nd}\sum_{k=0}^{m}\binom{m}{k}(2nd)^{m-k}\int_X \chi(x)\zeta^x q^x(2x)^k d\mu_{-1}(x)\\&+\int_X \chi(x)\zeta^x q^x(2x)^m d\mu_{-1}(x)=2\mathcal{T}_{m,\chi,q,\zeta}(nd-1).\end{aligned}\tag{2.7}$$

By using (2.6) and (2.7), we arrive at the following theorem:

Theorem 2.1 *Let n be odd positive integer. Then we obtain*

$$\frac{2\int_X \chi(x)\zeta^x q^x e^{2xt}d\mu_{-1}(x)}{\int_X \zeta^{ndx}q^{ndx}e^{2ndtx}d\mu_{-1}(x)}=\sum_{m=0}^{\infty}\left(2\mathcal{T}_{m,\chi,q,\zeta}(nd-1)\right)\frac{t^m}{m!}.$$

Let w_1 and w_2 be odd positive integers. Then we set

$$\begin{aligned}&S(w_1,w_2)=\\&\frac{\int_X\int_X \chi(x_1)\chi(x_2)\zeta^{(w_1x_1+w_2x_2)}q^{(w_1x_1+w_2x_2)}e^{(2w_1x_1+2w_2x_2+w_1w_2x)t}d\mu_{-1}(x_1)d\mu_{-1}(x_2)}{\int_X \zeta^{w_1w_2dx}q^{w_1w_2dx}e^{2w_1w_2dxt}d\mu_{-1}(x)}.\end{aligned}\tag{2.8}$$

By Theorem 2.1 and (2.8), after elementary calculations, we have

$$S(w_1,w_2)=\left(\frac{1}{2}\sum_{m=0}^{\infty}T_{m,\chi,q^{w_1},\zeta^{w_1}}(w_2x)w_1^m\frac{t^m}{m!}\right)\left(2\sum_{m=0}^{\infty}\mathcal{T}_{m,\chi,q^{w_2},\zeta^{w_2}}(w_1d-1)w_2^m\frac{t^m}{m!}\right).$$

By using Cauchy product in the above, we have

$$S(w_1,w_2)=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}T_{j,\chi,q^{w_1},\zeta^{w_1}}(w_2x)w_1^j\mathcal{T}_{m-j,\chi,q^{w_2},\zeta^{w_2}}(w_1d-1)w_2^{m-j}\right)\frac{t^m}{m!}.\tag{2.9}$$

From the symmetry of $S(w_1, w_2)$ in w_1 and w_2, we also see that

$$S(w_1,w_2)=\left(\frac{1}{2}\sum_{m=0}^{\infty}T_{m,\chi,q^{w_2},\zeta^{w_2}}(w_1x)w_2^m\frac{t^m}{m!}\right)\left(2\sum_{m=0}^{\infty}\mathcal{T}_{m,\chi,q^{w_1},\zeta^{w_1}}(w_2d-1)w_1^m\frac{t^m}{m!}\right).$$

Thus we have

$$S(w_1,w_2)=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}T_{j,\chi,\zeta^{w_2}}(w_1x)w_2^j\mathcal{T}_{m-j,\chi,\zeta^{w_1}}(w_2d-1)w_1^{m-j}\right)\frac{t^m}{m!} \tag{2.10}$$

By comparing coefficients $\dfrac{t^m}{m!}$ in the both sides of (2.9) and (2.10), we arrive at the following theorem:

Theorem 2.2 *Let w_1 and w_2 be odd positive integers. Then we obtain*

$$\begin{aligned}&\sum_{j=0}^{m}\binom{m}{j}w_1^{m-j}w_2^jT_{j,\chi,q^{w_2},\zeta^{w_2}}(w_1x)\mathcal{T}_{m-j,\chi,q^{w_1},\zeta^{w_1}}(w_2d-1)\\&=\sum_{j=0}^{m}\binom{m}{j}w_1^jw_2^{m-j}T_{j,\chi,q^{w_1},\zeta^{w_1}}(w_2x)\mathcal{T}_{m-j,\chi,q^{w_2},\zeta^{w_2}}(w_1d-1),\end{aligned}$$

where $T_{k,\chi,q,\zeta}(x)$ and $\mathcal{T}_{m,\chi,q,\zeta}(k)$ denote the generalized twisted q-tangent polynomials and the alternating sums of powers, respectively.

By Theorem 2.2 and Theorem 1.3, we have the following corollary.

Corollary 2.3 *Let w_1 and w_2 be odd positive integers. Then we have*

$$\begin{aligned}&\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{m-k}w_2^jx^{j-k}T_{k,\chi,q^{w_2},\zeta^{w_2}}\mathcal{T}_{m-j,\chi,q^{w_1},\zeta^{w_1}}(w_2d-1)\\&=\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^jw_2^{m-k}x^{j-k}T_{k,\chi,q^{w_1},\zeta^{w_1}}\mathcal{T}_{m-j,\chi,q^{w_2},\zeta^{w_2}}(w_1d-1).\end{aligned}$$

Now we will derive another interesting identities for the generalized twisted q-tangent polynomials using the symmetric property of $S(w_1, w_2)$. By (2.8), after elementary calculations, we have

$$\begin{aligned}&S(w_1,w_2)\\&=\sum_{j=0}^{w_1d-1}(-1)^j\chi(j)\zeta^{w_2j}q^{w_2j}\int_X\chi(x_1)\zeta^{w_1x_1}q^{w_1x_1}e^{\left(2x_1+w_2x+\frac{2jw_2}{w_1}\right)(w_1t)}d\mu_{-1}(x_1)\\&=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_1d-1}(-1)^j\chi(j)\zeta^{w_2j}q^{w_2j}T_{n,\chi,q^{w_1},\zeta^{w_1}}\left(w_2x+\frac{2jw_2}{w_1}\right)w_1^n\right)\frac{t^n}{n!}\end{aligned} \tag{2.11}$$

By using the symmetry property in (2.11), we also have

$$
\begin{aligned}
&S(w_1, w_2) \\
&= \sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) \zeta^{w_1 j} q^{w_1 j} \int_X \chi(x_2) \zeta^{w_2 x_2} q^{w_2 x_2} e^{\left(2x_2 + w_1 x + \frac{2jw_1}{w_2}\right)(w_2 t)} d\mu_{-1}(x_1) \\
&= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{w_2-1} (-1)^j \chi(j) \zeta^{w_1 j} q^{w_1 j} T_{n,\chi,q^{w_2},\zeta^{w_2}} \left(w_1 x + \frac{2jw_1}{w_2} \right) w_2^n \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.12}
$$

By comparing coefficients $\frac{t^n}{n!}$ in the both sides of (2.11) and (2.12), we have the following theorem.

Theorem 2.4 *Let w_1 and w_2 be odd positive integers. Then we have*

$$
\begin{aligned}
&\sum_{j=0}^{w_1 d-1} (-1)^j \chi(j) \zeta^{w_2 j} q^{w_2 j} w_1^n T_{n,\chi,q^{w_1},\zeta^{w_1}} \left(w_2 x + \frac{2jw_2}{w_1} \right) \\
&= \sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) \zeta^{w_1 j} q^{w_1 j} w_2^n T_{n,\chi,q^{w_2},\zeta^{w_2}} \left(w_1 x + \frac{2jw_1}{w_2} \right).
\end{aligned}
$$

If we take $x = 0$ in Theorem 2.4, we also derive the interesting identity for the generalized twisted q-tangent numbers as follows:

$$
\begin{aligned}
&\sum_{j=0}^{w_1 d-1} (-1)^j \chi(j) \zeta^{w_2 j} q^{w_2 j} w_1^n T_{n,\chi,q^{w_1},\zeta^{w_1}} \left(\frac{2jw_2}{w_1} \right) \\
&= \sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) \zeta^{w_1 j} q^{w_1 j} w_2^n T_{n,\chi,q^{w_2},\zeta^{w_2}} \left(\frac{2jw_1}{w_2} \right).
\end{aligned}
$$

Letting $q \to 1, \zeta \to 1$ in Theorem 2.4, we can immediately have the generalized multiplication theorem for the generalized tangent polynomials(see, [5]).

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