Advanced Studies in Theoretical Physics
Vol. 9, 2015, no. 2, 71 - 75
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/astp.2015.412153

Antibound State for the Discrete Schrödinger Equation

Ana-Magnolia Marin-Ramirez

Faculty of Exact and Natural Sciences, University of Cartagena
Campus San Pablo, Avenue of Consulado
Cartagena de Indias, Bolivar, Colombia

Ruben-Dario Ortiz-Ortiz

Faculty of Exact and Natural Sciences, University of Cartagena
Campus San Pablo, Avenue of Consulado
Cartagena de Indias, Bolivar, Colombia

Eilberth-Enrique Caro-Zuñiga

Faculty of Exact and Natural Sciences, University of Cartagena
Campus San Pablo, Avenue of Consulado
Cartagena de Indias, Bolivar, Colombia

Abstract

In this paper we construct a resonance of the discrete Schrödinger equation in presence of a potential. To achieve this, we use Whittaker-Kotelnikov interpolation, Fourier transform and Neumann series.

Mathematics Subject Classification: 39A12, 35P15

Keywords: discrete Schrödinger equation, resonance, Neumann series

1 Introduction

It was constructed the asymptotic for natural frequencies of the Schrödinger equation using the (WKB) method [4]. It was encountered a formula explicitly to the eigenvalue that appears below the essential spectrum of the discrete equation Klein-Gordon [3]. It was found a resonance for the discrete shallow water equation in the case of an underwater trench [1]. It was proposed a simple method for the construction of an asymptotic of a small negative eigenvalue for the Schrödinger equation in the presence of a shallow potential well [2]. It was encountered exact solutions describing trapped water waves over an underwater ridge of small height in the shallow water and resonances (antibound states) over an underwater trench [5].

In this paper, we will construct a resonance of the discrete Schrödinger equation, we use the Whittaker-Kotelnikov interpolation, Fourier transform and Neumann series to find a solution that characterizes resonance.

2 Preliminary Notes

We consider the discrete Schrödinger equation

$$-\frac{1}{h^2}\left(\varphi_{j+1}-2\varphi_j+\varphi_{j-1}\right)+\varepsilon V_j\varphi_j=E\varphi_j, \tag{1}$$

where $\varphi(jh)=\varphi_j$ with $j\in\mathbb{Z}$, $h>0$ and $\varepsilon\to 0^+$. V_j is a discrete potential with

$$V_j=0,\ \text{for}\quad |j|\geq R,\ \text{for some } R\in\mathbb{R}^+$$

Definition 2.1 *A solution φ_j of equation (1) is called a discrete resonance, if it satisfies*

$$\varphi_j\propto e^{\beta|jh|}\qquad |j|\to\infty \tag{2}$$

with $\beta>0$, and $E=-\beta^2$.

3 Main Results

The main result is as follows

Theorem 1 *Let $\sum V_j>0$, then for ε sufficiently small, the equation (1) has a discrete resonance for $E=-\beta^2$, where*

$$\beta=\frac{h\varepsilon}{2}\sum V_j+O(\varepsilon^2) \tag{3}$$

Proof. We consider the equation (1) with $E=-\beta^2$. Applying Whittaker-Kotelnikov interpolation and the Fourier transform, we obtain

$$\tilde{\varphi}_h(p)=2\pi C_1\delta(p-p_+)+2\pi\delta(p-p_-).$$

where the zeros of $\frac{4}{h^2}\sin^2\left(\frac{hp}{2}\right)+\beta^2=0$, are $p_\pm=\frac{2\pi k}{h}\pm\frac{2i\sinh^{-1}(\frac{\beta h}{2})}{h}$.

Then $\varphi_h(x)=C_1e^{ip_+x}+C_2e^{ip_-x}$. Now, considering the equation (1), we obtain

$$A(p)=-\frac{\varepsilon}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}}W(p-p^{'})\tilde{\varphi}_h(p^{'})dp^{'}, \tag{4}$$

where $W(p)=\frac{h}{2\pi}\sum_j V_j e^{-ijhp}$. The above expression has the form

$$\left(\frac{4}{h^2}\sin^2(\frac{hp}{2})+\beta^2\right)\tilde{\varphi}_h(p)=A(p). \tag{5}$$

We are looking for the resonance in the following form

$$\varphi_h(x)=\frac{1}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}}e^{ipx}\frac{A(p)}{\frac{4}{h^2}\sin^2(\frac{hp}{2})+\beta^2}dp+C_1e^{ip_+x}+C_2e^{ip_-x}. \tag{6}$$

We will take the of integration contour in the complex plane

$$\begin{aligned}\Gamma_+ &= \left[-\frac{\pi}{h},-1\right]\cup\left\{p+qi:p^2+q^2=1,q>0\right\}\cup\left[1,\frac{\pi}{h}\right],\\ \Gamma_- &= \left[-\frac{\pi}{h},-1\right]\cup\left\{p+qi:p^2+q^2=1,q<0\right\}\cup\left[1,\frac{\pi}{h}\right].\end{aligned}$$

Applying the Cauchy residue theorem to the equation (6), we obtain

$$\varphi_h(x)=\frac{1}{2\pi}\int_{\Gamma_+}e^{ipx}\frac{A(p)}{\frac{4}{h^2}\sin^2(\frac{hp}{2})+\beta^2}dp+\left(C_1+\frac{\pi A(p_+)}{\beta\sqrt{1+\frac{h^2\beta 2}{4}}}\right)e^{ip_+x}+C_2e^{ip_-x}$$

for $x>0$. Considering the right hand side of the equation and $C_1=-\frac{\pi A(p_+)}{\beta\sqrt{1+\frac{h^2\beta^2}{4}}}$

$$\varphi_h(x)=\frac{e^{-x}}{2\pi}\int_{\Gamma_+-i}e^{ipx}\frac{A(p+i)}{\frac{4}{h^2}\sin^2(\frac{h(p+i)}{2})+\beta^2}dp+C_2e^{ip_-x}.$$

Since the last integral is bounded, then $\varphi_h(x)=C_2e^{ip_-x}+O(e^{-x})$ when $x\to+\infty$. Similarly, $\varphi_h(x)=C_1e^{ip_+x}+O(e^x)$, when $C_2=-\frac{\pi A(p_-)}{\beta\sqrt{1+\frac{h^2\beta^2}{4}}}$ and $x\to-\infty$. Therefore,

$$\varphi_h(x)=\frac{1}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}}e^{ipx}\frac{A(p)}{\frac{4}{h^2}\sin^2(\frac{hp}{2})+\beta^2}dp+C_1e^{ip_+x}+C_2e^{ip_-x}. \tag{7}$$

The Fourier transform of (7) has de form

$$\tilde{\varphi}_h(p) = \frac{A(p)}{\frac{4}{h^2}\sin^2(\frac{hp}{2}) + \beta^2} + 2\pi C_1\delta(p-p_+) + 2\pi\delta(p-p_-). \tag{8}$$

Replacing (8) into (4), we obtain

$$A(p) = -\frac{\varepsilon}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \frac{W(p-p^{'})A(p^{'})}{\frac{4}{h^2}\sin^2(\frac{hp^{'}}{2}) + \beta^2}dp^{'} - \varepsilon C_1 W(p-p_+) - \varepsilon C_2 W(p-p_-). \tag{9}$$

Applying the Cauchy residue theorem to the equation (9), we obtain

$$A(p) = -\frac{\varepsilon}{2\pi}\int_{\Gamma_+} \frac{W(p-p^{'})A(p^{'})}{\frac{4}{h^2}\sin^2(\frac{hp^{'}}{2}) + \beta^2}dp^{'} - \varepsilon C_2 W(p-p_-). \tag{10}$$

We define the operator $T_\beta : H \to H$ as

$$\left[T_\beta A(\zeta)\right](z) = \frac{1}{2\pi}\int_{\Gamma_+} \frac{W(z-\zeta)A(\zeta)}{\frac{4}{h^2}\sin^2(\frac{h\zeta}{2}) + \beta^2}d\zeta, \qquad z \in B_{\frac{\pi}{h}},$$

where H is the space of bounded analytic functions in $B_{\frac{\pi}{h}} = \left\{z \in \mathbb{C} : |\mathrm{Im}\, z| < \frac{\pi}{h}\right\}$ with the norm $\|A\| = \sup_{z\in B_{\frac{\pi}{h}}} |A(z)|$. Equation (10) can be rewritten

$$\left[1 + \varepsilon T_\beta A(\zeta)\right](z) = -\varepsilon C_2 W(z-p_-)$$

where 1 is the identity operator. As T_β is analytic and bounded, then it is contraction operator, then we can take its inverse,

$$A(z) = -\varepsilon C_2 \left[1 + \varepsilon T_\beta\right]^{-1} W(z-p_-)$$

Using Neumann series at $z = p_-$, with $C_2 = -\frac{\pi A(p_-)}{\beta\sqrt{1+\frac{h^2\beta^2}{4}}}$, we obtain

$$\beta = \frac{\varepsilon\pi}{\sqrt{1+\frac{h^2\beta^2}{4}}} W(z-p_-)\mid_{z=p_-} + O(\varepsilon^2)$$

■

Acknowledgements. The authors express their deep gratitude to Universidad de Cartagena for partial financial support

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Received: December 19, 2014; Published: January 25, 2015