On the Twisted Modified
$q$-Daehee Numbers and Polynomials

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Abstract

The $p$-adic $q$-integral(or $q$-Volkenborn integration) was defined by Kim(see [9,10]). From $p$-adic $q$-integrals’ equations, we can derive various $q$-extension of Bernoulli numbers and polynomials(see [1-21]). In [4], D.S.Kim and T.Kim have studied Daehee numbers and polynomials and their applications. For the twisted Daehee numbers and polynomials are investigate in [17]. In [11], Kim-Lee-Mansour-Seo introduced the $q$-analogue of Daehee numbers and polynomials which are called $q$-Daehee numbers and polynomials. In [16], Park investigated twisted version of Daehee polynomials as numbers with $q$-parameter, which related with usual Bernoulli numbers and polynomials. Lim considered in [13], the modified $q$-Daehee numbers and polynomials which are different from the $q$-Daehee numbers and polynomials of Kim-Lee-Mansour-Seo. For the twisted version of Daehee polynomials, In this paper, we give some useful properties and identities of twisted modified $q$-Daehee numbers and polynomials related with twisted $q$-Bernoulli numbers and polynomials.

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1 Introduction

Let $p$ be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will respectively denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers and the completion $s$ of algebraic closure of $\mathbb{Q}_p$. The $p$-adic norm is defined $|p|_p = \frac{1}{p}$.

When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q - 1|_p < p^{\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. Throughout this paper, we use the notation:

$$[x]_q = \frac{1 - q^x}{1 - q}.$$

Note that $\lim_{q \to 1} [x]_q = x$ for each $x \in \mathbb{Z}_p$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{\left[p^N\right]_q} \sum_{x=0}^{p^N-1} f(x) q^x \quad \text{(see [9, 10]).}$$

(1)

As is well known, the Stirling number of the first kind is defined by

$$x^{(n)} = x(x-1) \cdots (x-n+1) = \sum_{l=0}^{n} S_1(n, l) x^l,$$

(2)

and the Stirling number of the first kind is given by the generating function to be

$$(e^t - 1)^m = m! \sum_{l=m}^{\infty} S_2(l, m) \frac{t^l}{l!} \quad \text{(see [8]).}$$

(3)

Unsigned Stirling numbers of the first kind are given by

$$x^n = x(x+1) \cdots (x+n-1) = \sum_{l=0}^{n} |S_1(n, l)| x^l.$$

(4)

Note that if we place $x$ to $-x$ in (2), then

$$(-x)^{(n)} = (-1)^n x^n = \sum_{l=0}^{n} S_1(n, l)(-1)^l x^l$$

$$= (-1)^n \sum_{l=0}^{n} |S_1(n, l)| x^l.$$

(5)
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Hence, $S_1(n, l) = |S_1(n, l)|(-1)^{n-l}$.

Using integration (1), the $q$-Daehee polynomials $D_{n,q}(x)$ are defined and studied by Kim et al. (see [11]), the generating function to be

$$1 - q + \frac{1 - q \log(1 + t)}{1 - q - qt} (1 + t)^x = \sum_{n=0}^{\infty} D_{n,q}(x) \frac{t^n}{n!}. \quad (6)$$

And the modified $q$-Daehee polynomials are defined and studied by the author. The generating function to be

$$\frac{q - 1 \log(1 + t)}{\log q} (1 + t)^x = \sum_{n=0}^{\infty} D_n(x|q) \frac{t^n}{n!} \quad (see \ [13]). \quad (7)$$

From (1), we have the following integral identity.

$$qI_q(f_1) - I_q(f) = \frac{q - 1}{\log q} f'(0) + (q - 1) f(0), \quad (8)$$

where $f_1(x) = f(x + 1)$ and $f'(x) = \frac{d}{dx} f(x)$.

In special case, we apply $f(x) = e^{tx}$ on (8), we have the modified $q$-Bernoulli number $B_n(q)$ as follows:

$$\int_{\mathbb{Z}} q^{-x} e^{xt} d\mu_q(x) = \frac{q - 1}{\log q} \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n(q) \frac{t^n}{n!} \quad (see \ [13]). \quad (9)$$

Indeed if $q \to 1$, we have $\lim_{q \to 1} B_n(q) = B_n$. The $n$th modified $q$-Bernoulli polynomials and the generating function to be

$$\frac{q - 1}{\log q} \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x|q) \frac{t^n}{n!}. \quad (10)$$

When $x = 0$, $B_n(0|q) = B_n(q)$ are the $n$th $q$-Bernoulli numbers (see [13]).

From (8) and (10), we have

$$B_n(x|q) = \int_{\mathbb{Z}} q^{-y} (x + y)^n d\mu_q(y).$$

and

$$B_n(x|q) = \sum_{l=0}^{n} \binom{n}{l} B_l(q)x^{n-l}.$$ 

We define the twisted modified $q$-Bernoulli numbers by the generating function as follows:

$$\sum_{n=0}^{\infty} B_{n,\xi}(x) \frac{t^n}{n!} = \frac{q - 1}{\log q} \frac{\xi t}{e^{\xi t} - 1}, \quad (11)$$
where $|t|_p \leq p^{-\frac{1}{p-1}}$.

If we apply $f(x) = q^{-x} e^{\xi tx}$ in (8), we have

$$
\int_{\mathbb{Z}_p} q^{-x} e^{\xi tx} d\mu_q(x) = \sum_{n=0}^{\infty} B_{n,\xi} (q) \frac{t^n}{n!}.
$$

(12)

The $n$th twisted modified $q$-Bernoulli polynomials $B_{n,\xi}(x|q)$ are given by,

$$
B_{n,\xi}(x|q) = \int_{\mathbb{Z}_p} q^{-x} \xi^n x^n d\mu_q(x) = \frac{q-1}{\log q} \frac{\xi t}{e^{\xi t} - 1} e^{xt}.
$$

(13)

The generating function of Daehee polynomials are introduced by Kim as follows:

$$
\sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!} = \frac{\log(1 + t)}{t} (1 + t)^x = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x + y)_n d\mu_0(x) \quad \text{(see [11])}. (14)
$$

When $x = 0$, $D_n(0) = D_n$ are called the Daehee numbers.

For $n \in \mathbb{N}$, let $T_p$ be the $p$-adic locally constant space defined by

$$
T_p = \bigcup_{n \geq 1} C_{p^n} = \lim_{n \to \infty} C_{p^n},
$$

where $C_{p^n} = \{ \omega \mid \omega^{p^n} = 1 \}$ is the cyclic group of order $p^n$.

We assume that $q$ is an indeterminate in $\mathbb{C}_p$ with $|1 - q|_p < p^{-\frac{1}{p-1}}$. Then we define the $q$-analog of a falling factorial sequence as follows:

$$(x)_{n,q} = x(x - q)(x - 2q) \cdots (x - (n-1)q) \quad (n \geq 1), \quad (x)_{0,1} = 1.
$$

Note that

$$
\lim_{q \to 1} (x)_{n,q} = (x)_n = \sum_{l=0}^{n} S_1(n,l) x^l.
$$

From the viewpoint of a generalization of the modified $q$-Bernoulli polynomials, we consider the twisted modified $q$-Bernoulli polynomials defined to be

$$
\sum_{n=0}^{\infty} D_{n,\xi}(x|q) \frac{t^n}{n!} = (1 + q \xi t)^{\frac{x}{q}} \frac{q-1}{q \log q (1 + \xi t)^{\frac{1}{q}} - 1},
$$

where $t, q \in \mathbb{C}_p$ with $|t|_p \leq |q|_p^{-\frac{1}{p-1}}$ and $\xi \in T_p$.

The $p$-adic $q$-integral (or $q$-Volkenborn integration) was defined by Kim (see [9,10]). From $p$-adic $q$-integrals’ equations, we can derive various $q$-extension of Bernoulli numbers and polynomials (see [1-21]). In [4], D.S.Kim and T.Kim have studied Daehee numbers and polynomials and their applications. For the twisted
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2 Witt-type formula for the $n$th twisted modified $q$-Dahee polynomials

Let us now consider the $p$-adic $q$-integral representation as follows:

$$\xi^n \int_{\mathbb{Z}_p} q^{-y}(x + y)_{n,q} d\mu_q(y) \quad (n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}, \; \xi \in T_p). \tag{15}$$

From (15), we have

$$\sum_{n=0}^{\infty} \left( \xi^n \int_{\mathbb{Z}_p} q^{-y}(x + y)_{n,q} d\mu_q(y) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \xi^n q^n \int_{\mathbb{Z}_p} q^{-y} \left( \frac{x + y}{q} \right)_n d\mu_q(y) \frac{t^n}{n!}$$

$$= \int_{\mathbb{Z}_p} q^{-y}(1 + q^{\xi}t)^{\frac{x+y}{q}} d\mu_q(y), \tag{16}$$

where $t \in \mathbb{C}_p$ with $|t|_p < |q|_p p^{-1/p-1}$.

For $|t|_p < |q|_p p^{-1/p-1}$, we apply $f(y) = q^{-y} (1 + q^{\xi}t)^{\frac{x+y}{q}}$ in (1).

By (8), we have

$$\int_{\mathbb{Z}_p} q^{-y} (1 + q^{\xi}t)^{\frac{x+y}{q}} d\mu_q(y) = (1 + q^{\xi}t)^{\frac{x}{q}} \frac{q - 1}{q \log q} \frac{\log (1 + q^{\xi}t)}{(1 + q^{\xi}t)^{\frac{x}{q}}} - 1 \tag{17}$$

By (16) and (17), we obtain the following theorem, which may be called Witt-type formula for the twisted modified $q$-Dahee polynomials.

**Theorem 2.1** For $n \geq 0$, we have

$$D_{n,\xi}(x|q) = \xi^n \int_{\mathbb{Z}_p} q^{-y}(x + y)_{n,q} d\mu_q(y).$$
In (17), by replacing $t$ by $\frac{1}{\xi q}(e^{\xi qt} - 1)$, we have
\[
\sum_{n=0}^{\infty} D_{n,\xi}(x|q) \frac{1}{\xi^n q^n} \frac{(e^{\xi qt} - 1)^n}{n!} = \frac{q - 1}{\log q} \frac{\xi t}{e^{\xi qt} - 1} e^{\xi xt} = \sum_{n=0}^{\infty} B_{n,\xi}(x|q) \frac{t^n}{n!}.
\]
(18)

and
\[
\sum_{n=0}^{\infty} \frac{D_{n,\xi}(x|q)}{\xi^n q^n} \frac{1}{n!} (e^{\xi qt} - 1)^n = \sum_{n=0}^{\infty} \frac{D_{n,\xi}(x|q)}{\xi^n q^n} \sum_{m=n}^{\infty} \xi^m q^m S_2(m, n) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{D_{n,\xi}(x|q)}{\xi^n q^n} \xi^m q^m S_2(m, n) \frac{t^m}{m!}.
\]
(19)

By (18) and (19), we obtain the following corollary.

**Corollary 2.2** For $n \geq 0$, we have
\[
B_{n,\xi}(x|q) = \sum_{m=0}^{n} D_{m,\xi}(x|q) \xi^{n-m} q^{n-m} S_2(n, m).
\]

By Theorem 2.1,
\[
D_{n,\xi}(x|q) = \xi^n \int_{\mathbb{Z}_p} q^{-y} (x + y)_{n,q} d\mu_q(y) = \xi^n q^n \sum_{l=0}^{n} \frac{1}{q^l} S_1(n, l) \int_{\mathbb{Z}_p} q^{-y} (x + y)^l d\mu_q(y).
\]
(20)

By (20), we obtain the following corollary.

**Corollary 2.3** For $n \geq 0$, we have
\[
D_{n,\xi}(x|q) = \sum_{l=0}^{n} \xi^{n-l} q^{n-l} S_1(n, l) B_{l,\xi}(x|q) = \sum_{l=0}^{n} \xi^{n-l} |S_1(n, l)| (-q)^{n-l} B_{l,\xi}(x|q).
\]

From now on, we consider twisted modified $q$-Daehee polynomials of order $k \in \mathbb{N}$. Twisted modified $q$-Daehee polynomials of order $k \in \mathbb{N}$ are defined by the multivariant $p$-adic $q$-integral on $\mathbb{Z}_p$:
\[
D^{(k)}_{n,\xi}(x|q) = \xi^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-(x_1 + \cdots + x_k)} (x_1 + \cdots + x_k + x)_{n,q} d\mu_q(x_1) \cdots d\mu_q(x_k),
\]
(21)

where $n$ is a nonnegative integer and $k \in \mathbb{N}$. In the special case, $x = 0$, $D^{(k)}_{n,\xi}(0|q) = D^{(k)}_{n,\xi}(0|0)$ are called the twisted modified $q$-Daehee numbers of order $k$. 
From (21), we can derive the generating function of $D_{n,\xi}^{(k)}(x|q)$ as follows:

$$
\sum_{n=0}^{\infty} D_{n,\xi}^{(k)}(x|q) \frac{t^n}{n!}
$$

$$
= \sum_{n=0}^{\infty} \xi^n q^n \int_{z_p} \cdots \int_{z_p} q^{-(x_1+\cdots+x_k)} \left( \frac{x_1+\cdots+x_k+x}{q} \right) d\mu_q(x_1) \cdots d\mu_q(x_k) t^n
$$

$$
= \int_{z_p} \cdots \int_{z_p} q^{-(x_1+\cdots+x_k)} (1 + q\xi t)^{\frac{x_1+\cdots+x_k+x}{q}} d\mu_q(x_1) \cdots d\mu_q(x_k)
$$

$$
= (1 + q\xi t)^{\frac{x}{q}} \int_{z_p} \cdots \int_{z_p} q^{-(x_1+\cdots+x_k)} (1 + q\xi t)^{\frac{x_1+\cdots+x_k}{q}} d\mu_q(x_1) \cdots d\mu_q(x_k)
$$

$$
= (1 + q\xi t)^{\frac{x}{q}} \left( \frac{q - 1}{q \log (1 + q\xi t)} \right)^k.
$$

Note that, by (22),

$$
D_{n,\xi}^{(k)}(x|q) = \xi^n q^n \sum_{m=0}^{n} \frac{S_1(n, m)}{q^m} \int_{z_p} \cdots \int_{z_p} q^{-(x_1+\cdots+x_k)} (x_1 + \cdots + x_k + x)^m
$$

$$
\times d\mu_q(x_1) \cdots d\mu_q(x_k).
$$

Since

$$
\int_{z_p} \cdots \int_{z_p} \xi^n q^{-(x_1+\cdots+x_k)} e^{(x_1+\cdots+x_k+x)t} d\mu_q(x_1) \cdots d\mu_q(x_k)
$$

$$
= \left( \frac{q - 1}{q \log q e^{kt} - 1} \right)^k e^{xt}
$$

$$
= \sum_{n=0}^{\infty} B_{n,\xi}^{(k)}(x|q) \frac{t^n}{n!},
$$

we can derive easily

$$
B_{n,\xi}^{(k)}(x|q) = \int_{z_p} \cdots \int_{z_p} \xi^n q^{-(x_1+\cdots+x_k)} (x_1 + \cdots + x_k + x)^n d\mu_q(x_1) \cdots d\mu_q(x_k).
$$

Thus, by (23) and (24), we have

$$
D_{n,\xi}^{(k)}(x|q) = q^n \sum_{m=0}^{n} \xi^{n-m} \frac{S_1(n, m)}{q^m} B_{m,\xi}^{(k)}(x|q)
$$

$$
= \sum_{m=0}^{n} q^{n-m} \xi^{n-m} S_1(n, m) B_{m}^{(k)}(x|q)
$$

$$
= \sum_{m=0}^{n} \xi^{n-m} |S_1(n, m)| (-q)^{r-m} B_{m}^{(k)}(x|q).
$$
In (22), by replacing $t$ by $\frac{1}{q^t}(e^{\xi qt} - 1)$, we get
\[
\sum_{n=0}^{\infty} D^{(k)}_{n,\xi}(x|q) \left(\frac{q^t - 1}{\xi^n q^n n!}\right)^n = e^{\xi tx} \left(\frac{q - 1}{\log q^t} - 1\right) = \sum_{n=0}^{\infty} B^{(k)}_{n,\xi}(x|q) \frac{t^n}{n!}.
\] (26)

and
\[
\sum_{n=0}^{\infty} \frac{D^{(k)}_{n,\xi}(x|q)}{\xi^n q^n} \frac{1}{n!} \left(\frac{q^t - 1}{\xi^n q^n} - 1\right)^n = \sum_{n=0}^{\infty} \frac{D^{(k)}_{n,\xi}(x|q)}{\xi^n q^n} \sum_{m=n}^{\infty} \xi^m q^m S_2(m, n) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \frac{\xi^m q^m}{\xi^n q^n} \sum_{n=0}^{m} \frac{D^{(k)}_{n,\xi}(x|q)}{\xi^n q^n} S_2(m, n) \frac{t^m}{m!}.
\] (27)

By (25), (26) and (27), we obtain the following theorem.

**Theorem 2.4** For $n \geq 0$ and $k \in \mathbb{N}$, we have
\[
D^{(k)}_{n,\xi}(x|q) = \sum_{m=0}^{n} q^{n-m} \xi^{n-m} S_1(n, m) B^{(k)}_{m}(x|q)
\]
\[
= \sum_{m=0}^{n} \xi^{n-m} |S_1(n, m)| (-q)^{n-m} B^{(k)}_{m}(x|q).
\]

Now, we consider the twisted modified $q$-Daehee polynomials of the second kind as follows:
\[
\widehat{D}_{n,\xi}(x|q) = \xi^n \int_{\mathbb{Z}_p} q^{-y} (-y + x)_{n,q} d\mu_q(y) \quad (n \geq 0).
\] (28)

In the special case $x = 0$, $\widehat{D}_{n,\xi}(q) = \widehat{D}_{n,\xi}(0|q)$ are the called the twisted modified $q$-Daehee numbers of the second kind.

By (28), we have
\[
\widehat{D}_{n,\xi}(x|q) = \xi^n q^n \int_{\mathbb{Z}_p} q^{-y} \left(\frac{-y + x}{q}\right)_n d\mu_q(y),
\] (29)

and so we can derive the generating function of $\widehat{D}_{n,\xi}(x|q)$ by (8) as follows:
\[
\sum_{n=0}^{\infty} \frac{\widehat{D}_{n,\xi}(x|q)}{n!} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \xi^n q^n \int_{\mathbb{Z}_p} q^{-y} \left(\frac{-y + x}{q}\right)_n d\mu_q(y) \frac{t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \xi^n q^n \int_{\mathbb{Z}_p} q^{-y} \left(\frac{-y + x}{q^n}\right) d\mu_q(y) t^n
\]
\[
= \int_{\mathbb{Z}_p} q^{-y} (1 + q\xi t)^{-\frac{y + x}{q^n}} d\mu_q(y)
\]
\[
= (1 + q\xi t)^{\frac{y + x}{q^n}} \frac{1 - q}{q} \frac{\log (1 + q\xi t)}{q \log q (1 + q\xi t)^{-\frac{1}{q^n}} - 1}.
\] (30)
From (29), we get
\[ \hat{D}_{n,\xi}(x|q) = \xi^n q^n \int_{\mathbb{Z}_p} q^{-y} \left( \frac{-y + x}{q} \right)_n d\mu_q(y) \]
\[ = \xi^n q^n \int_{\mathbb{Z}_p} q^{-y} \sum_{l=0}^{\infty} S_1(n, l) \frac{(-y + x)^l}{q^l} d\mu_q(y) \]
\[ = \sum_{l=0}^{\infty} S_1(n, l) (-1)^l \int_{\mathbb{Z}_p} \xi^l q^{-y} (y - x)^l d\mu_q(y) q^{n-l} \xi^{n-l} \]
\[ = \sum_{l=0}^{\infty} S_1(n, l) (-1)^l B_l(\xi) (-x|q) q^{n-l} \xi^{n-l} \]
\[ = (-1)^n \sum_{l=0}^{\infty} |S_1(n, l)| B_l(\xi) (-x|q) q^{n-l} \xi^{n-l}. \]  

(31)

It is easy to show \( B_{n,\xi}(-x|q) = (-1)^n B_{n,\xi}(x + 1|q) \). Thus from (31), we have
\[ \hat{D}_{n,\xi}(x|q) = (-1)^n \sum_{l=0}^{\infty} |S_1(n, l)| B_l(\xi) (-x|q) q^{n-l} \xi^{n-l} \]
\[ = \sum_{l=0}^{\infty} |S_1(n, l)| B_l(\xi) (x + 1|q) (-q)^{n-l} \xi^{n-l}. \]  

(32)

From (31) and (32), we have
\[ B_{n,\xi}(x + 1|q) = \sum_{m=0}^{\infty} q^{n-m} \xi^{n-m} \hat{D}_{m,\xi}(x|q) S_1(n, m). \]  

(33)

Thus, from (31), we have the following theorem.

**Theorem 2.5** For \( n \geq 0 \), we have
\[ \hat{D}_{n,\xi}(x|q) = (-1)^n \sum_{l=0}^{\infty} |S_1(n, l)| B_l(\xi) (-x|q) q^{n-l} \xi^{n-l}. \]

By replacing \( t \) by \( \frac{1}{q^\xi} (e^{\xi t} - 1) \) in (30), we have
\[ \sum_{n=0}^{\infty} \hat{D}_{n,\xi}(x|q) \frac{1}{\xi^n q^n} \left( e^{\xi t} - 1 \right)^n \frac{\xi^q t^{-n}}{n!} = \frac{q - 1}{\log q} e^{-\xi t} - 1 e^{\xi t} = \sum_{n=0}^{\infty} B_{n, -\xi} (-x|q) t^n \frac{t^n}{n!} \]  

(34)

and
\[ \sum_{n=0}^{\infty} \hat{D}_{n,\xi}(x|q) \frac{1}{\xi^n q^n} \left( e^{\xi t} - 1 \right)^n = \sum_{n=0}^{\infty} \hat{D}_{n,\xi}(x|q) \frac{1}{\xi^n q^n} \sum_{m=0}^{\infty} \xi^m q^m S_2(m, n) \frac{t^m}{m!} \]
\[ = \sum_{n=0}^{\infty} \left( \sum_{n=0}^{m} \hat{D}_{n,\xi}(x|q) S_2(m, n) q^{m-n} \xi^{m-n} \right) \frac{t^m}{m!}. \]  

(35)
By (34) and (35), we obtain the following theorem.

**Theorem 2.6** For $n \geq 0$, we have

$$B_{n,-\xi}(-x|q) = \sum_{m=0}^{n} q^{n-m} \xi^{n-m} \hat{D}_{m,\xi}(x|q) S_2(n, m).$$

Now we consider higher-order twisted modified $q$-Dahee polynomials of the second kind. Higher-order twisted modified $q$-Dahee polynomials of the second kind are defined by the multivariant $p$-adic $q$-integral on $\mathbb{Z}_p$:

$$\hat{D}_{n,\xi}^{(k)}(x|q) = \xi^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-(x_1 + \cdots + x_k)} (-x_1 - \cdots - x_k + x)_{n,q} d\mu_q(x_1) \cdots d\mu_q(x_k),$$

where $n \in \mathbb{Z}_+$ and $k \in \mathbb{N}$. In the special case, $x = 0$, $\hat{D}_{n,\xi}^{(k)}(q) = \hat{D}_{n,\xi}^{(k)}(0|q)$ are called the higher-order twisted modified $q$-Dahee numbers of the second kind.

From (36), we can derive the generating function of $\hat{D}_{n,\xi}^{(k)}(x|q)$ as follows:

$$\sum_{n=0}^{\infty} \frac{\hat{D}_{n,\xi}^{(k)}(x|q) t^n}{n!} = \xi^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-(x_1 + \cdots + x_k)} \left( \frac{-x_1 - \cdots - x_k + x}{q} \right) d\mu_q(x_1) \cdots d\mu_q(x_k) t^n$$

$$= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-(x_1 + \cdots + x_k)} (1 + q\xi t)^{-x_1 - \cdots - x_k + x} q \, d\mu_q(x_1) \cdots d\mu_q(x_k)$$

$$= (1 + q\xi t)^{-\xi \left( \frac{1 - q}{q \log q} \log (1 + q\xi t) - \frac{1}{2} \right) - 1}.$$

(37)

By (37)

$$\hat{D}_{n,\xi}^{(k)}(x|q) = \xi^n q^n \sum_{m=0}^{n} \frac{S_1(n, m)}{q^m} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-(x_1 + \cdots + x_k)} (-x_1 - \cdots - x_k + x)^m$$

$$\times d\mu_q(x_1) \cdots d\mu_q(x_k)$$

$$= \xi^n q^n \sum_{m=0}^{n} \frac{S_1(n, m)}{(-q)^m} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-(x_1 + \cdots + x_k)} (x_1 + \cdots + x_k + x)^m$$

$$\times d\mu_q(x_1) \cdots d\mu_q(x_k)$$

$$= \xi^n q^n \sum_{m=0}^{n} \frac{S_1(n, m)}{q^m} B^{(k)}_m(-x|q)$$

$$= \xi^n q^n \sum_{m=0}^{n} q^{-m} |S_1(n, m)| B^{(k)}_m(-x|q).$$

(38)
It is easy to show $B_n^{(k)}(-x|q) = (-1)^n B_n^{(k)}(x+k|q)$. Hence, by (38),

**Theorem 2.7** For $n \geq 0$ and $k \in \mathbb{N}$, we have

$$
\hat{D}_{n,\xi}^{(k)}(x|q) = \xi^n q^n \sum_{m=0}^{n} \xi^{n-m} q^{n-m} |S_1(n,m)| B_m^{(k)}(-x|q)
$$

$$
= \xi^n q^n \sum_{m=0}^{n} (-1)^m \xi^{n-m} q^{n-m} |S_1(n,m)| B_m^{(k)}(x+k|q).
$$

In (37), by replacing $t$ by $\frac{1}{q\xi} (e^{\xi qt} - 1)$, we get

$$
\sum_{n=0}^{\infty} \hat{D}_{n,\xi}^{(k)}(x|q) \frac{(e^{\xi qt} - 1)^n}{\xi^n q^n n!} = e^{t(x+k)} \left( \frac{q - 1}{\log q} \frac{\xi t}{e^{\xi t} - 1} \right)^k = \sum_{n=0}^{\infty} B_{n,\xi}^{(k)}(x+k|q) \frac{t^n}{n!}.
$$

(39)

and

$$
\sum_{n=0}^{\infty} \frac{\hat{D}_{n,\xi}^{(k)}(x|q)}{\xi^n q^n} \frac{1}{n!} (e^{\xi qt} - 1)^n = \sum_{n=0}^{\infty} \frac{\hat{D}_{n,\xi}^{(k)}(x|q)}{\xi^n q^n} \sum_{m=n}^{\infty} \xi^m q^m S_2(m,n) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \left( \xi^m q^m \sum_{n=0}^{m} \frac{\hat{D}_{n,\xi}^{(k)}(x|q)}{\xi^n q^n} S_2(m,n) \right) \frac{t^m}{m!}.
$$

(40)

By (39) and (40), we obtain the following theorem.

**Theorem 2.8** For $n \geq 0$ and $k \in \mathbb{N}$, we have

$$
B_{n,\xi}^{(k)}(x+k|q) = \sum_{m=0}^{n} \frac{\hat{D}_{m,\xi}^{(k)}(x) \xi^{n-m} q^{n-m} S_2(n,m)}{\xi^n q^n}.
$$

References


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