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On the Twisted Modified q-Daehee Numbers and Polynomials

Dongkyu Lim

Department of Mathematics
Kyungpook National University, Daegu 702-701, S. Korea

Abstract

The p-adic q-integral(or q-Volkenborn integration) was defined by Kim(see [9,10]). From p-adic q-integrals' equations, we can derive various q-extension of Bernoulli numbers and polynomials(see [1-21]). In [4], D.S.Kim and T.Kim have studied Daehee numbers and polynomials and their applications. For the twisted Daehee numbers and polynomials are investigate in [17]. In [11], Kim-Lee-Mansour-Seo introduced the q-analogue of Daehee numbers and polynomials which are called q-Daehee numbers and polynomials. In [16], Park investigated twisted version of Daehee polynomials as numbers with q-parameter, which related with usual Bernoulli numbers and polynomials. Lim considered in [13], the modified q-Daehee numbers and polynomials which are different from the q-Daehee numbers and polynomials of Kim-Lee-Mansour-Seo. For the twisted version of Daehee polynomials, In this paper, we give some useful properties and identities of twisted modified q-Daehee numbers and polynomials related with twisted q-Bernoulli numbers and polynomials.

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1 Introduction

Let p be a fixed prime number. Throuhout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will repectively denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion s of algebraic closure of $\mathbb{Q}_p$. The p-adic norm is defined $|p|_p = \frac{1}{p}$.

When one talks of q-extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. Throughout this paper, we use the notation:

$$[x]_q = \frac{1-q^x}{1-q}.$$

Note that $\lim_{q\to 1}[x]_q = x$ for each $x \in \mathbb{Z}_p$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x \quad \text{(see [9, 10])}. \tag{1}$$

As is well known, the *Stirling number of the first kind* is defined by

$$x_{(n)} = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l) x^l, \tag{2}$$

and the *Stirling number of the first kind* is given by the generating function to be

$$(e^t-1)^m = m! \sum_{l=m}^{\infty} S_2(l,m) \frac{t^l}{l!} \quad \text{(see [8])}. \tag{3}$$

Unsigned Stirling numbers of the first kind are given by

$$x^{\underline{n}} = x(x+1)\cdots(x+n-1) = \sum_{l=0}^{n} |S_1(n,l)| x^l. \tag{4}$$

Note that if we place x to $-x$ in (2), then

$$\begin{aligned}(-x)_{(n)} = (-1)^n x^{\underline{n}} &= \sum_{l=0}^{n} S_1(n,l)(-1)^l x^l \\ &= (-1)^n \sum_{l=0}^{n} |S_1(n,l)| x^l.\end{aligned} \tag{5}$$

Hence, $S_1(n,l) = |S_1(n,l)|(-1)^{n-l}$.

Using integration (1), the q-Daehee polynomials $D_{n,q}(x)$ are defined and studied by Kim et al.(see [11]), the generating function to be

$$\frac{1-q+\frac{1-q}{\log q}\log(1+t)}{1-q-qt}(1+t)^x = \sum_{n=0}^{\infty} D_{n,q}(x)\frac{t^n}{n!}. \tag{6}$$

And the modified q-Daehee polynomials are defined and studied by the author. The generating function to be

$$\frac{q-1}{\log q}\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x|q)\frac{t^n}{n!} \quad \text{(see [13])}. \tag{7}$$

From (1), we have the following integral identity.

$$qI_q(f_1) - I_q(f) = \frac{q-1}{\log q}f'(0) + (q-1)f(0), \tag{8}$$

where $f_1(x) = f(x+1)$ and $f'(x) = \frac{d}{dx}f(x)$.

In special case, we apply $f(x) = e^{tx}$ on (8), we have the modified q-Bernoulli number $B_n(q)$ as follows:

$$\int_{\mathbb{Z}_p} q^{-x}e^{xt}d\mu_q(x) = \frac{q-1}{\log q}\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n(q)\frac{t^n}{n!} \quad \text{(see [13])}. \tag{9}$$

Indeed if $q \to 1$, we have $\lim_{q\to 1} B_n(q) = B_n$. The nth modified q-Bernoulli polynomials and the generating function to be

$$\frac{q-1}{\log q}\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x|q)\frac{t^n}{n!}. \tag{10}$$

When $x=0$, $B_n(0|q) = B_n(q)$ are the nth q-Bernoulli numbers(see [13]).

From (8) and (10), we have

$$B_n(x|q) = \int_{\mathbb{Z}_p} q^{-y}(x+y)^n d\mu_q(y).$$

and

$$B_n(x|q) = \sum_{l=0}^{n}\binom{n}{l}B_l(q)x^{n-l}.$$

We define the twisted modified q-Bernoulli numbers by the generating function as follows:

$$\sum_{n=0}^{\infty} B_{n,\xi}(x)\frac{t^n}{n!} = \frac{q-1}{\log q}\frac{\xi t}{e^{\xi t}-1}, \tag{11}$$

where $|t|_p \le p^{-\frac{1}{p-1}}$.

If we apply $f(x) = q^{-x} e^{\xi t x}$ in (8), we have

$$\int_{\mathbb{Z}_p} q^{-x} e^{\xi t x} d\mu_q(x) = \sum_{n=0}^{\infty} B_{n,\xi}(q) \frac{t^n}{n!}. \tag{12}$$

The nth twisted modified q-Bernoulli polynomials $B_{n,\xi}(x|q)$ are given by,

$$B_{n,\xi}(x|q) = \int_{\mathbb{Z}_p} q^{-x} \xi^n x^n d\mu_q(x) = \frac{q-1}{\log q} \frac{\xi t}{e^{\xi t} - 1} e^{xt}. \tag{13}$$

The generating function of Daehee polynomials are introduced by Kim as follows:

$$\sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!} = \frac{\log(1+t)}{t} (1+t)^x = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x+y)_n d\mu_0(x) \quad \text{(see [11])}. \tag{14}$$

When $x = 0$, $D_n(0) = D_n$ are called the Daehee numbers.

For $n \in \mathbb{N}$, let T_p be the p-adic locally constant space defined by

$$T_p = \bigcup_{n \ge 1} C_{p^n} = \lim_{n \to \infty} C_{p^n},$$

where $C_{p^n} = \{\omega | \omega^{p^n} = 1\}$ is the cyclic group of order p^n.

We assume that q is an indeterminate in $\mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$. Then we define the q-analog of a falling fractorial sequence as follows:

$$(x)_{n,q} = x(x-q)(x-2q)\cdots(x-(n-1)q) \quad (n \ge 1), \qquad (x)_{0,1} = 1.$$

Note that

$$\lim_{q \to 1} (x)_{n,q} = (x)_n = \sum_{l=0}^{n} S_1(n,l) x^l.$$

From the view point of a generalization of the midified q-Daehee polynomials, we consider the twisted modified q-Daehee polynomials defined to be

$$\sum_{n=0}^{\infty} D_{n,\xi}(x|q) \frac{t^n}{n!} = (1+q\xi t)^{\frac{x}{q}} \frac{q-1}{q \log q} \frac{\log(1+q\xi t)}{(1+q\xi t)^{\frac{1}{q}} - 1},$$

where $t, q \in \mathbb{C}_p$ with $|t|_p \le |q|_p p^{-\frac{1}{p-1}}$ and $\xi \in T_p$

The p-adic q-integral(or q-Volkenborn integration) was defined by Kim(see [9,10]). From p-adic q-integrals' equations, we can derive various q-extension of Bernoulli numbers and polynomials(see [1-21]). In [4], D.S.Kim and T.Kim have studied Daehee numbers and polynomials and their applications. For the twisted

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2 Witt-type formula for the nth twisted modified q-Daehee polynomials

Let us now consider the p-adic q-integral representation as follows:

$$\xi^n \int_{\mathbb{Z}_p} q^{-y}(x+y)_{n,q} d\mu_q(y) \quad (n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}, \quad \xi \in T_p). \tag{15}$$

From (15), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\xi^n \int_{\mathbb{Z}_p} q^{-y}(x+y)_{n,q} d\mu_q(y)\right)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} \xi^n q^n \int_{\mathbb{Z}_p} q^{-y}\left(\frac{x+y}{q}\right)_n d\mu_q(y)\frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} q^{-y}(1+q\xi t)^{\frac{x+y}{q}} d\mu_q(y),\end{aligned} \tag{16}$$

where $t \in \mathbb{C}_p$ with $|t|_p < |q|_p p^{-\frac{1}{p-1}}$.

For $|t|_p < |q|_p p^{-\frac{1}{p-1}}$, we apply $f(y) = q^{-y}(1+q\xi t)^{\frac{x+y}{q}}$ in (1).

By (8), we have

$$\begin{aligned}\int_{\mathbb{Z}_p} q^{-y}(1+q\xi t)^{\frac{x+y}{q}} d\mu_q(y) &= (1+q\xi t)^{\frac{x}{q}} \frac{q-1}{q\log q}\frac{\log(1+q\xi t)}{(1+q\xi t)^{\frac{1}{q}}-1} \\ &= \sum_{n=0}^{\infty} D_{n,\xi}(x|q)\frac{t^n}{n!}.\end{aligned} \tag{17}$$

By (16) and (17), we obtain the following theorem, which may be called Witt-type formula for the twisted modified q-Daehee polynomials.

Theorem 2.1 *For $n \geq 0$, we have*

$$D_{n,\xi}(x|q) = \xi^n \int_{\mathbb{Z}_p} q^{-y}(x+y)_{n,q} d\mu_q(y).$$

In (17), by replacing t by $\frac{1}{\xi q}(e^{\xi q t}-1)$, we have

$$\sum_{n=0}^{\infty} D_{n,\xi}(x|q)\frac{1}{\xi^n q^n}\frac{(e^{\xi q t}-1)^n}{n!} = \frac{q-1}{\log q}\frac{\xi t}{e^{\xi t}-1}e^{\xi x t} = \sum_{n=0}^{\infty} B_{n,\xi}(x|q)\frac{t^n}{n!}. \tag{18}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{D_{n,\xi}(x|q)}{\xi^n q^n}\frac{1}{n!}(e^{\xi q t}-1)^n &= \sum_{n=0}^{\infty} \frac{D_{n,\xi}(x|q)}{\xi^n q^n}\sum_{m=n}^{\infty} \xi^m q^m S_2(m,n)\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{D_{n,\xi}(x|q)}{\xi^n q^n}\xi^m q^m S_2(m,n)\frac{t^m}{m!}.\end{aligned} \tag{19}$$

By (18) and (19), we obtain the following corollary.

Corollary 2.2 *For $n \geq 0$, we have*

$$B_{n,\xi}(x|q) = \sum_{m=0}^{n} D_{m,\xi}(x|q)\xi^{n-m}q^{n-m}S_2(n,m).$$

By Theorem 2.1,

$$\begin{aligned}D_{n,\xi}(x|q) &= \xi^n \int_{\mathbb{Z}_p} q^{-y}(x+y)_{n,q}\,d\mu_q(y) \\ &= \xi^n q^n \sum_{l=0}^{n} \frac{1}{q^l} S_1(n,l) \int_{\mathbb{Z}_p} q^{-y}(x+y)^l\,d\mu_q(y).\end{aligned} \tag{20}$$

By (20), we obtain the following corollary.

Corollary 2.3 *For $n \geq 0$, we have*

$$D_{n,\xi}(x|q) = \sum_{l=0}^{n} \xi^{n-l}q^{n-l}S_1(n,l)B_{l,\xi}(x|q) = \sum_{l=0}^{n} \xi^{n-l}|S_1(n,l)|(-q)^{n-l}B_{l,\xi}(x|q).$$

From now on, we consider twisted modified q-Daehee polynomials of order $k \in \mathbb{N}$. Twisted modified q-Daehee polynomials of order $k \in \mathbb{N}$ are defined by the multivariant p-adic q-integral on $\mathbb{Z}_p$:

$$D_{n,\xi}^{(k)}(x|q) = \xi^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-(x_1+\cdots+x_k)}(x_1+\cdots+x_k+x)_{n,q}\,d\mu_q(x_1)\cdots d\mu_q(x_k), \tag{21}$$

where n is a nonnegative integer and $k \in \mathbb{N}$. In the special case, $x = 0$, $D_{n,\xi}^{(k)}(q) = D_{n,\xi}^{(k)}(0|q)$ are called the twisted modified q-Daehee numbers of order k.

From (21), we can derive the generating function of $D_{n,\xi}^{(k)}(x|q)$ as follows:

$$\begin{aligned}
&\sum_{n=0}^{\infty} D_{n,\xi}^{(k)}(x|q)\frac{t^n}{n!} \\
&\quad= \sum_{n=0}^{\infty} \xi^n q^n \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+\cdots+x_k)} \binom{\frac{x_1+\cdots+x_k+x}{q}}{n} d\mu_q(x_1)\cdots d\mu_q(x_k) t^n \\
&\quad= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+\cdots+x_k)} (1+q\xi t)^{\frac{x_1+\cdots+x_k+x}{q}} d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&\quad= (1+q\xi t)^{\frac{x}{q}} \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+\cdots+x_k)} (1+q\xi t)^{\frac{x_1+\cdots+x_k}{q}} d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&\quad= (1+q\xi t)^{\frac{x}{q}} \big(\frac{q-1}{q\log q}\frac{\log(1+q\xi t)}{(1+q\xi t)^{\frac{1}{q}}-1}\big)^k.
\end{aligned} \tag{22}$$

Note that, by (22),

$$\begin{aligned}
D_{n,\xi}^{(k)}(x|q) = \xi^n q^n \sum_{m=0}^{n} \frac{S_1(n,m)}{q^m} \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+\cdots+x_k)} (x_1+\cdots+x_k+x)^m \\
\times d\mu_q(x_1)\cdots d\mu_q(x_k).
\end{aligned} \tag{23}$$

Since

$$\begin{aligned}
\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} \xi^n q^{-(x_1+\cdots+x_k)} e^{(x_1+\cdots+x_k+x)t} d\mu_q(x_1)\cdots d\mu_q(x_k) \\
= \left(\frac{q-1}{\log q}\frac{\xi t}{e^{\xi t}-1}\right)^k e^{xt} \\
= \sum_{n=0}^{\infty} B_{n,\xi}^{(k)}(x|q)\frac{t^n}{n!},
\end{aligned}$$

we can derive easily

$$B_{n,\xi}^{(k)}(x|q) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} \xi^n q^{-(x_1+\cdots+x_k)} (x_1+\cdots+x_k+x)^n d\mu_q(x_1)\cdots d\mu_q(x_k). \tag{24}$$

Thus, by (23) and (24), we have

$$\begin{aligned}
D_{n,\xi}^{(k)}(x|q) &= q^n \sum_{m=0}^{n} \xi^{n-m} \frac{S_1(n,m)}{q^m} B_{m,\xi}^{(k)}(x|q) \\
&= \sum_{m=0}^{n} q^{n-m} \xi^{n-m} S_1(n,m) B_m^{(k)}(x|q) \\
&= \sum_{m=0}^{n} \xi^{n-m} |S_1(n,m)| (-q)^{n-m} B_m^{(k)}(x|q).
\end{aligned} \tag{25}$$

In (22), by replacing t by $\frac{1}{q\xi}(e^{\xi qt}-1)$, we get

$$\sum_{n=0}^{\infty} D_{n,\xi}^{(k)}(x|q)\frac{(e^{\xi qt}-1)^n}{\xi^n q^n n!} = e^{\xi tx}\left(\frac{q-1}{\log q}\frac{\xi t}{e^{\xi t}-1}\right)^k = \sum_{n=0}^{\infty} B_{n,\xi}^{(k)}(x|q)\frac{t^n}{n!}. \tag{26}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{D_{n,\xi}^{(k)}(x|q)}{\xi^n q^n}\frac{1}{n!}(e^{\xi qt}-1)^n &= \sum_{n=0}^{\infty}\frac{D_{n,\xi}^{(k)}(x|q)}{\xi^n q^n}\sum_{m=n}^{\infty}\xi^m q^m S_2(m,n)\frac{t^m}{m!}\\ &= \sum_{m=0}^{\infty}\left(\xi^m q^m\sum_{n=0}^{m}\frac{D_{n,\xi}^{(k)}(x|q)}{\xi^n q^n}S_2(m,n)\right)\frac{t^m}{m!}.\end{aligned} \tag{27}$$

By (25),(26) and (27), we obtain the following theorem.

Theorem 2.4 *For $n \geq 0$ and $k \in \mathbb{N}$, we have*

$$\begin{aligned}D_{n,\xi}^{(k)}(x|q) &= \sum_{m=0}^{n} q^{n-m}\xi^{n-m}S_1(n,m)B_m^{(k)}(x|q)\\ &= \sum_{m=0}^{n}\xi^{n-m}|S_1(n,m)|(-q)^{n-m}B_m^{(k)}(x|q).\end{aligned}$$

Now, we consider the twisted modified q-Daehee polynomials of the second kind as follows:

$$\widehat{D}_{n,\xi}(x|q) = \xi^n\int_{\mathbb{Z}_p} q^{-y}(-y+x)_{n,q}d\mu_q(y) \quad (n \geq 0). \tag{28}$$

In the special case $x = 0$, $\widehat{D}_{n,\xi}(q) = \widehat{D}_{n,\xi}(0|q)$ are the called the twisted modified q-Daehee numbers of the second kind.

By (28), we have

$$\widehat{D}_{n,\xi}(x|q) = \xi^n q^n\int_{\mathbb{Z}_p} q^{-y}\big(\frac{-y+x}{q}\big)_n d\mu_q(y), \tag{29}$$

and so we can derive the generating function of $\widehat{D}_{n,\xi}(x|q)$ by (8) as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{D}_{n,\xi}(x|q)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\xi^n q^n\int_{\mathbb{Z}_p} q^{-y}\big(\frac{-y+x}{q}\big)_n d\mu_q(y)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\xi^n q^n\int_{\mathbb{Z}_p} q^{-y}\binom{\frac{-y+x}{q}}{n}d\mu_q(y)t^n\\ &= \int_{\mathbb{Z}_p} q^{-y}(1+q\xi t)^{\frac{-y+x}{q}}d\mu_q(y)\\ &= (1+q\xi t)^{\frac{x}{q}}\frac{1-q}{q\log q}\frac{\log(1+q\xi t)}{(1+q\xi t)^{-\frac{1}{q}}-1}.\end{aligned} \tag{30}$$

From (29), we get

$$\begin{aligned}
\widehat{D}_{n,\xi}(x|q) &= \xi^n q^n \int_{\mathbb{Z}_p} q^{-y} \Big(\frac{-y+x}{q}\Big)_n d\mu_q(y) \\
&= \xi^n q^n \int_{\mathbb{Z}_p} q^{-y} \sum_{l=0}^{n} \frac{S_1(n,l)}{q^l}(-y+x)^l d\mu_q(y) \\
&= \sum_{l=0}^{n} S_1(n,l)(-1)^l \int_{\mathbb{Z}_p} \xi^l q^{-y}(y-x)^l d\mu_q(y) q^{n-l}\xi^{n-l} \\
&= \sum_{l=0}^{n} S_1(n,l)(-1)^l B_{l,\xi}(-x|q) q^{n-l}\xi^{n-l} \\
&= (-1)^n \sum_{l=0}^{n} |S_1(n,l)| B_{l,\xi}(-x|q) q^{n-l}\xi^{n-l}.
\end{aligned} \tag{31}$$

It is easy to show $B_{n,\xi}(-x|q) = (-1)^n B_{n,\xi}(x+1|q)$. Thus from (31), we have

$$\begin{aligned}
\widehat{D}_{n,\xi}(x|q) &= (-1)^n \sum_{l=0}^{n} |S_1(n,l)| B_{l,\xi}(-x|q) q^{n-l}\xi^{n-l} \\
&= \sum_{l=0}^{n} |S_1(n,l)| B_{l,\xi}(x+1|q)(-q)^{n-l}\xi^{n-l}.
\end{aligned} \tag{32}$$

From (31) and (32), we have

$$B_{n,\xi}(x+1|q) = \sum_{m=0}^{n} q^{n-m}\xi^{n-m} \widehat{D}_{m,\xi}(x|q)|S_1(n,m). \tag{33}$$

Thus, from (31), we have the following theorem.

Theorem 2.5 *For $n \geq 0$, we have*

$$\widehat{D}_{n,\xi}(x|q) = (-1)^n \sum_{l=0}^{n} |S_1(n,l)| B_{l,\xi}(-x|q) q^{n-l}\xi^{n-l}.$$

By replacing t by $\frac{1}{q\xi}(e^{\xi qt-1})$ in (30), we have

$$\sum_{n=0}^{\infty} \widehat{D}_{n,\xi}(x|q) \frac{1}{\xi^n q^n} \frac{(e^{\xi qt}-1)^n}{n!} = \frac{q-1}{\log q} \frac{-\xi t}{e^{-\xi t}-1} e^{\xi xt} = \sum_{n=0}^{\infty} B_{n,-\xi}(-x|q)\frac{t^n}{n!}. \tag{34}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\widehat{D}_{n,\xi}(x|q)}{\xi^n q^n} \frac{1}{n!}(e^{\xi qt}-1)^n &= \sum_{n=0}^{\infty} \frac{\widehat{D}_{n,\xi}(x|q)}{\xi^n q^n} \sum_{m=n}^{\infty} \xi^m q^m S_2(m,n)\frac{t^m}{m!} \\
&= \sum_{m=0}^{\infty} \Big(\sum_{n=0}^{m} \widehat{D}_{n,\xi}(x|q) S_2(m,n) q^{m-n}\xi^{m-n} \Big) \frac{t^m}{m!}.
\end{aligned} \tag{35}$$

By (34) and (35), we obtain the following theorem.

Theorem 2.6 *For $n \geq 0$, we have*

$$B_{n,-\xi}(-x|q) = \sum_{m=0}^{n} q^{n-m}\xi^{n-m}\widehat{D}_{m,\xi}(x|q)S_2(n,m).$$

Now we consider higher-order twisted modified q-Daehee polynomials of the second kind. Higher-order twisted modified q-Daehee polynomials of the second kind are defined by the multivariant p-adic q-integral on $\mathbb{Z}_p$:

$$\widehat{D}^{(k)}_{n,\xi}(x|q) = \xi^n \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+\cdots+x_k)}(-x_1-\cdots-x_k+x)_{n,q}d\mu_q(x_1)\cdots d\mu_q(x_k), \tag{36}$$

where $n \in \mathbb{Z}_+$ and $k \in \mathbb{N}$. In the special case, $x = 0$, $\widehat{D}^{(k)}_{n,\xi}(q) = \widehat{D}^{(k)}_{n,\xi}(0|q)$ are called the higher-order twisted modified q-Daehee numbers of the second kind.

From (36), we can derive the generating function of $\widehat{D}^{(k)}_{n,\xi}(x|q)$ as follows:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\widehat{D}^{(k)}_{n,\xi}(x|q)\frac{t^n}{n!} \\
&\quad= \sum_{n=0}^{\infty}\xi^n q^n \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+\cdots+x_k)}\binom{\frac{-x_1-\cdots-x_k+x}{q}}{n} d\mu_q(x_1)\cdots d\mu_q(x_k)t^n \\
&\quad= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+\cdots+x_k)}(1+q\xi t)^{\frac{-x_1-\cdots-x_k+x}{q}} d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&\quad= (1+q\xi t)^{\frac{x}{q}}\Big(\frac{1-q}{q\log q}\frac{\log(1+q\xi t)}{(1+q\xi t)^{-\frac{1}{q}}-1}\Big)^k.
\end{aligned} \tag{37}$$

By (37)

$$\begin{aligned}
\widehat{D}^{(k)}_{n,\xi}(x|q) &= \xi^n q^n \sum_{m=0}^{n}\frac{S_1(n,m)}{q^m}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+\cdots+x_k)}(-x_1-\cdots-x_k+x)^m \\
&\quad\times d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&= \xi^n q^n \sum_{m=0}^{n}\frac{S_1(n,m)}{(-q)^m}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+\cdots+x_k)}(x_1+\cdots+x_k+x)^m \\
&\quad\times d\mu_q(x_1)\cdots d\mu_q(x_k) \\
&= \xi^n q^n \sum_{m=0}^{n}\frac{S_1(n,m)}{q^m}B^{(k)}_m(-x|q) \\
&= \xi^n q^n \sum_{m=0}^{n} q^{n-m}|S_1(n,m)|B^{(k)}_m(-x|q).
\end{aligned} \tag{38}$$

It is easy to show $B_n^{(k)}(-x|q) = (-1)^n B_n^{(k)}(x+k|q)$. Hence, by (38),

Theorem 2.7 *For $n \geq 0$ and $k \in \mathbb{N}$, we have*

$$\begin{aligned}\widehat{D}_{n,\xi}^{(k)}(x|q) &= \xi^n q^n \sum_{m=0}^{n} \xi^{n-m} q^{n-m} |S_1(n,m)| B_{m,\xi}^{(k)}(-x|q)\\ &= \xi^n q^n \sum_{m=0}^{n} (-1)^m \xi^{n-m} q^{n-m} |S_1(n,m)| B_{m,\xi}^{(k)}(x+k|q).\end{aligned}$$

In (37), by replacing t by $\frac{1}{q\xi}(e^{\xi q t}-1)$, we get

$$\sum_{n=0}^{\infty} \widehat{D}_{n,\xi}^{(k)}(x|q) \frac{(e^{\xi q t}-1)^n}{\xi^n q^n n!} = e^{\xi t(x+k)} \left(\frac{q-1}{\log q} \frac{\xi t}{e^{\xi t}-1} \right)^k = \sum_{n=0}^{\infty} B_{n,\xi}^{(k)}(x+k|q) \frac{t^n}{n!}. \tag{39}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\widehat{D}_{n,\xi}^{(k)}(x|q)}{\xi^n q^n} \frac{1}{n!}(e^{\xi q t}-1)^n &= \sum_{n=0}^{\infty} \frac{\widehat{D}_{n,\xi}^{(k)}(x|q)}{\xi^n q^n} \sum_{m=n}^{\infty} \xi^m q^m S_2(m,n) \frac{t^m}{m!}\\ &= \sum_{m=0}^{\infty} \left(\xi^m q^m \sum_{n=0}^{m} \frac{\widehat{D}_{n,\xi}^{(k)}(x|q)}{\xi^n q^n} S_2(m,n) \right) \frac{t^m}{m!}.\end{aligned} \tag{40}$$

By (39) and (40), we obtain the following theorem.

Theorem 2.8 *For $n \geq 0$ and $k \in \mathbb{N}$, we have*

$$B_{n,\xi}^{(k)}(x+k|q) = \sum_{m=0}^{n} \widehat{D}_{m,\xi}^{(k)}(x) \xi^{n-m} q^{n-m} S_2(n,m).$$

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