Families of Sheffer Sequences Satisfying
Generalizations of Power and Alternating Power Sum Identities

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Abstract

In this paper, we will consider one family of Sheffer sequences satisfying a generalization of the classical power sum identity. Also, we will study another family of Sheffer sequences satisfying a generalization of the classical alternating power sum identity.
1. Introduction

Let

\[ \mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} a_k \bigg| a_k \in \mathbb{C} \right\} . \]  

(1.1)

For \( \mathbb{P} = \mathbb{C}[x] \), let us assume that \( \mathbb{P}^* \) is the vector space of all linear functionals on \( \mathbb{P} \). \( \langle L \mid p(x) \rangle \) denotes the action of the linear functional \( L \) on \( p(x) \) which satisfies \( \langle L + M \mid p(x) \rangle = \langle L \mid p(x) \rangle + \langle M \mid p(x) \rangle \), and \( \langle cL \mid p(x) \rangle = c \langle L \mid p(x) \rangle \), where \( c \) is a complex constant. The linear functional \( \langle f(t) \mid \cdot \rangle \) on \( \mathbb{P} \) is defined by \( \langle f(t) \mid x^n \rangle = a_n, \ (n \geq 0) \), for \( f(t) \in \mathcal{F} \).

Thus, we have

\[ \langle t^k \mid x^n \rangle = n! \delta_{n,k}, \quad (n, k \geq 0), \quad (\text{see [14, 17]}), \]  

(1.2)

where \( \delta_{n,k} \) is the Kronecker’s symbol.

The order \( o(f(t)) \) of a power series \( f(t) (\neq 0) \) is the smallest integer \( k \) for which the coefficient of \( t^k \) does not vanish. If \( o(f(t)) = 0 \), then \( f(t) \) is called an invertible series; if \( o(f(t)) = 1 \), then \( f(t) \) is called a delta series (see [10, 17]).

Let us assume that \( f_L(t) = \sum_{k=0}^{\infty} \langle L \mid x^k \rangle \frac{t^k}{k!} \). From (1.2), we note that

\[ \langle f_L(t) \mid x^n \rangle = \langle L \mid x^n \rangle. \]  

So, the map \( L \mapsto f_L(t) \) is a vector space isomorphism from \( \mathbb{P}^* \) onto \( \mathcal{F} \). Henceforth, \( \mathcal{F} \) denotes both the algebra of formal power series in \( t \) and the vector space of all linear functionals on \( \mathbb{P} \), and so an element \( f(t) \) of \( \mathcal{F} \) will be thought of as both a formal power series and a linear functional. We call \( \mathcal{F} \) the umbral algebra and the umbral calculus is the study of umbral algebra. Let \( f(t), g(t) \in \mathcal{F} \), with \( o(f(t)) = 1 \) and \( o(g(t)) = 0 \). Then there exists a unique sequence \( s_n(x) (\deg s_n(x) = n) \) such that \( \langle g(t) f(t)^k \mid s_n(x) \rangle = n! \delta_{n,k}, \ (n, k \geq 0) \). Such a sequence \( s_n(x) \) is called the Sheffer sequence for \( (g(t), f(t)) \) which is denoted by \( s_n(x) \sim (g(t), f(t)) \). The sequence \( s_n(x) \) is Sheffer for \( (g(t), f(t)) \) if and only if

\[ \frac{1}{g(f(t))} e^{x \bar{f}(t)} = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \quad (x \in \mathbb{C}), \quad (\text{see [15, 17]}), \]  

(1.3)

where \( \bar{f}(t) \) is the compositional inverse of \( f(t) \) with \( \bar{f}(f(t)) = f(\bar{f}(t)) = t \). Let \( f(t) \in \mathcal{F} \) and \( p(x) \in \mathbb{P} \). Then, by (1.2), we get

\[ f(t) = \sum_{k=0}^{\infty} \langle f(t) \mid x^k \rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \langle t^k \mid p(x) \rangle \frac{x^k}{k!}. \]  

(1.4)

From (1.4), we can derive the following equations:

\[ t^k p(x) = p^{(k)}(x), \quad e^xp(x) = p(x+y), \quad e^{yt}p(x) = p(y). \]  

(1.5)

In this paper, we will consider one family of Sheffer sequences satisfying a generalization of the classical power sum identity. Also, we will study another
family of Sheffer sequences satisfying a generalization of the classical alternating power sum identity. One family consists of those Sheffer sequences \( s_n(x) \) for the pair \( \left( g(t) = \frac{(e^{a_1 t} - 1) \cdots (e^{a_r t} - 1)}{f(t)}, f(t) \right) \), where \( f(t) \) is any delta series, \( r \in \mathbb{Z}_{>0} \), and \( a_1, a_2, \ldots, a_r \neq 0 \). Note that \( g(t) \) is an invertible series. That is, \( s_n(x) \sim \frac{(e^{a_1 t} - 1) \cdots (e^{a_r t} - 1)}{f(t)} f(t) \),

(1.6)

We will show later that this family contains many interesting Sheffer sequences. Another family is composed of those Sheffer sequences \( s_n(x) \) for the pair

\[
\left( g(t) = \prod_{i=1}^{r} \left( e^{a_i t} + \frac{1}{2} \right), f(t) \right),
\]

where \( a_1, a_2, \ldots, a_r \neq 0 \), and \( f(t) \) is any delta series. Again, we will see that this family also has many interesting members.

In the previous paper ([10]) “A generalization of power and alternating power sums to any Appell polynomials”, we introduced Barnes’ multiple Bernoulli and Appell mixed-type polynomials and Barnes’ multiple Euler and Appell mixed-type polynomials. Then we established one main identity for each of them connecting a sum for the Appell polynomial and that for the mixed-type polynomial. We note that the present result has overlaps with those ones only when \( f(t) = t \).

2. Families of Sheffer sequences satisfying generalizations of power and alternating power sum identities

Let \( a \neq 0 \). From (1.5), we note that

\[
\frac{e^{(m+1)at} - 1}{e^{at} - 1} p(x) = \sum_{i=0}^{m} p(x + ai), \quad \left\langle \frac{e^{(m+1)at} - 1}{e^{at} - 1} p(x) \right\rangle = \sum_{i=0}^{m} p(ai),
\]

(2.1)

and

\[
\frac{(-1)^m e^{(m+1)at} + 1}{e^{at} + 1} p(x) = \sum_{i=0}^{m} (-1)^i p(x + ai),
\]

\[
\left\langle \frac{(-1)^m e^{(m+1)at} + 1}{e^{at} + 1} p(x) \right\rangle = \sum_{i=0}^{m} (-1)^i p(ai),
\]

(2.2)

where \( p(x) \) is any polynomial.

**Lemma 1.** Let \( m_1, m_2, \ldots, m_r \in \mathbb{Z} \) with \( m_i \geq 0 \) (\( i = 1, 2, \ldots, r \)), \( a_1, \ldots, a_r \in \mathbb{C} \setminus \{0\} \). Then, for any polynomial \( p(x) \), we have

\[
(e^{(m_r+1)at} - 1) \cdots (e^{(m_1+1)at} - 1) p(x)
\]

(2.3)
\[\sum_{i=0}^{r} (-1)^{r-i} \sum_{J \subseteq [1, r], |J|=i} \left( x + \sum_{j \in J} (m_j + 1) a_j \right).\]

**Proof.** We prove Lemma 1 by induction on \( r \). It is easy to check that it holds for \( r = 1 \).

Assume that, for \( r > 1 \), the following holds:

\[ \left( e^{(m_{r-1}+1)a_{r-1}t} - 1 \right) \cdots \left( e^{(m_1+1)a_1t} - 1 \right) p(x) \]

\[ = \sum_{i=0}^{r-1} (-1)^{r+i} \sum_{J \subseteq [1, r-1], |J|=i} \left( x + \sum_{j \in J} (m_j + 1) a_j \right). \tag{2.4} \]

Thus, by (2.4), we see that the LHS of (2.3) is

\[ \sum_{i=0}^{r-1} (-1)^{r-1-i} \sum_{J \subseteq [1, r-1], |J|=i} \left( e^{(m_{r-1}+1)a_{r-1}t} - 1 \right) p \left( x + \sum_{j \in J} (m_j + 1) a_j \right) \]

\[ = \sum_{i=0}^{r-1} (-1)^{r-1-i} \sum_{J \subseteq [1, r-1], |J|=i} \left( x + \sum_{j \in J} (m_j + 1) a_j + (m_r + 1) a_r \right) \]

\[ - p \left( x + \sum_{j \in J} (m_j + 1) a_j \right) \]

\[ = \sum_{i=0}^{r-1} (-1)^{r-1-i} \sum_{J \subseteq [1, r-1], |J|=i} \left( x + \sum_{j \in J} (m_j + 1) a_j + (m_r + 1) a_r \right) \]

\[ + \sum_{i=0}^{r-1} (-1)^{r-i} \sum_{J \subseteq [1, r-1], |J|=i} \left( x + \sum_{j \in J} (m_j + 1) a_j \right) \]

\[ = p \left( x + \sum_{j \in [1, r]} (m_j + 1) a_j \right) \]

\[ + \sum_{i=1}^{r-1} (-1)^{r-i} \sum_{J \subseteq [1, r-1], |J|=i-1} \left( x + \sum_{j \in J} (m_j + 1) a_j + (m_r + 1) a_r \right) \]

\[ + \sum_{i=1}^{r-1} (-1)^{r-i} \sum_{J \subseteq [1, r-1], |J|=i} \left( x + \sum_{j \in J} (m_j + 1) a_j \right) + (-1)^r p(x). \]
Continuing in this fashion, we obtain the expression on the LHS of (2.5).

\[ p \left( x + \sum_{j \in [1,r]} (m_j + 1) a_j \right) \]

\[ + \sum_{i=1}^{r-1} (-1)^{r-i} \sum_{J \subseteq [1,r], |J| = i} p \left( x + \sum_{j \in J} (m_j + 1) a_j \right) + (-1)^r p(x) \]

\[ = \sum_{i=0}^{r} (-1)^{r-i} \sum_{J \subseteq [1,r], |J| = i} p \left( x + \sum_{j \in J} (m_j + 1) a_j \right). \]

\[ \square \]

**Theorem 1.** Let \( s_n (x) \sim \left( \frac{(e^{a_1 t} - 1) \cdots (e^{a_r t} - 1)}{f(t)} \right), w_n (x) \sim (1, f(t)) \), where \( a_1, a_2, \ldots, a_r \in \mathbb{C} \setminus \{0\}, r \in \mathbb{Z} \) with \( r > 0 \) and \( o(f(t)) = 1 \). Then, we have

\[ \sum_{i_1=0}^{m_1} \cdots \sum_{i_r=0}^{m_r} w_n (x + a_1 i_1 + \cdots + a_r i_r) \]

\[ = \frac{1}{(n+r)} \sum_{i=0}^{r} (-1)^{r-i} \sum_{J \subseteq [1,r], |J| = i} s_{n+r} \left( x + \sum_{j \in J} (m_j + 1) a_j \right), \]

where \((x)_n = x(x-1) \cdots (x-n+1) = \sum_{i=0}^{n} S_1(n,l) x^i\) and \( S_1(n,l) \) is the Stirling number of the first kind.

**Proof.** The result is obtained by computing the following in two different ways:

\[ \left( \frac{e^{(m_r+1)a_{r} t} - 1}{e^{a_{r} t} - 1} \right) \times \cdots \times \left( \frac{e^{(m_1+1)a_{1} t} - 1}{e^{a_{1} t} - 1} \right) w_n (x). \quad (2.6) \]

On one hand, it is

\[ \left( \frac{e^{(m_r+1)a_{r} t} - 1}{e^{a_{r} t} - 1} \right) \times \cdots \times \left( \frac{e^{(m_1+1)a_{1} t} - 1}{e^{a_{1} t} - 1} \right) w_n (x) \]

\[ = \frac{e^{(m_r+1)a_{r} t} - 1}{e^{a_{r} t} - 1} \cdots \frac{e^{(m_2+1)a_{2} t} - 1}{e^{a_{2} t} - 1} \left( \sum_{i_1=0}^{m_1} w_n (x + a_1 i_1) \right) \]

\[ = \sum_{i_1=0}^{m_1} \frac{e^{(m_r+1)a_{r} t} - 1}{e^{a_{r} t} - 1} \cdots \frac{e^{(m_2+1)a_{2} t} - 1}{e^{a_{2} t} - 1} \left( \sum_{i_2=0}^{m_2} w_n (x + a_1 i_1 + a_2 i_2) \right) \]

\[ = \sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} \frac{e^{(m_r+1)a_{r} t} - 1}{e^{a_{r} t} - 1} \cdots \frac{e^{(m_3+1)a_{3} t} - 1}{e^{a_{3} t} - 1} w_n (x + a_1 i_1 + a_2 i_2). \]

Continuing in this fashion, we obtain the expression on the LHS of (2.5).
On the other hand, we first observe that
\[
\frac{(e^{a_1t} - 1) \cdots (e^{a_rt} - 1)}{f(t)^r} \cdot s_n = w_n(x) \cdot f(t) \cdot w_n(x) = n w_{n-1}(x).
\] (2.8)
Thus, by (2.8), we get
\[
(e^{a_1t} - 1) \cdots (e^{a_rt} - 1) \cdot s_n(x) = f(t)^r \cdot w_n(x) = (n) \cdot w_{n-r}(x).
\] (2.9)
Replacing $n$ by $n + r$, we have
\[
(e^{a_1t} - 1) \cdots (e^{a_rt} - 1) \cdot s_{n+r}(x) = (n + r) \cdot w_n(x).
\] (2.10)
From (2.6) and (2.10), we can derive the following equation:
\[
\frac{e^{(m_r+1)a_1t} - 1}{e^{a_1t} - 1} \cdots \frac{e^{(m_1+1)a_rt} - 1}{e^{a_rt} - 1} \cdot w_n(x) = \frac{1}{(n + r)^r} \left( e^{(n+1)t} - 1 \right) s_{n+r}(x)
\] (2.11)
\[
= \frac{1}{(n + r)^r} \left( e^{(m_r+1)a_1t} - 1 \right) \cdots \left( e^{(m_1+1)a_rt} - 1 \right) \cdot s_{n+r}(x).
\]
Now, we get the expression on the RHS of (2.5) from Lemma 1.

**Corollary 1.**

(a) Let $s_n(x) \sim \left( \frac{(e^{a_1t} - 1) \cdots (e^{a_rt} - 1)}{f(t)} \right), w_n(x) \sim (1, f(t))$, where $a_1, a_2, \ldots, a_r \in \mathbb{C} \setminus \{0\}, r \in \mathbb{Z}$ with $r > 0$ and $o(f(t)) = 1$. Then we have
\[
\sum_{i_1=0}^{m_1} \cdots \sum_{i_r=0}^{m_r} w_n(a_1i_1 + \cdots + a_r i_r) = \frac{1}{(n + r)} \sum_{i=0}^{r} (-1)^{r-i} \sum_{j \in J \subset [1, r]} s_{n+r} \left( \sum_{j \in J} (m_j + 1) a_j \right).
\]

(b) Let $s_n(x) \sim \left( \left( \frac{e^{f^i - 1}}{f(t)} \right)^r, f(t) \right), w_n(x) \sim (1, f(t))$, where $o(f(t)) = 1$ and $r > 0$. Then
\[
\sum_{i_1=0}^{m_1} \cdots \sum_{i_r=0}^{m_r} w_n(x + i_1 + \cdots + i_r) = \frac{1}{(n + r)} \sum_{i=0}^{r} (-1)^{r-i} \sum_{J \subset [1, r]} s_{n+r} \left( x + \sum_{j \in J} (m_j + 1) \right).
\]

(c) With $s_n(x), w_n(x)$ as in (b), we have
\[
\sum_{i_1=0}^{m_1} \cdots \sum_{i_r=0}^{m_r} w_n(i_1 + \cdots + i_r) = \frac{1}{(n + r)} \sum_{i=0}^{r} (-1)^{r-i} \sum_{J \subset [1, r]} s_{n+r} \left( \sum_{j \in J} (m_j + 1) \right).
\]

(d) With $s_n(x), w_n(x)$ as in (b), we have
\[
\sum_{i_1, \cdots, i_r=0}^{m} w_n(x + i_1 + \cdots + i_r) = \frac{1}{(n + r)} \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} s_{n+r} \left( x + (m+1) i \right).
\]
(c) With \( s_n(x), w_n(x) \) as in (b), we have

\[
\sum_{i_1, \ldots, i_r = 0}^{m} w_n(i_1 + \cdots + i_r) = \frac{1}{(n+r)} \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} s_{n+r}((m+1)i).
\]

**Lemma 2.** Let \( m_1, m_2, \ldots, m_r \in \mathbb{Z} \) with \( m_i \geq 0 \) \((i = 1, 2, \ldots, r)\), \( a_1, \ldots, a_r \in \mathbb{C} \setminus \{0\} \). Then, for any polynomial \( p(x) \), we have

\[
((-1)^{m_r} e^{(m_r+1)a_r t} + 1) \cdots ((-1)^{m_1} e^{(m_1+1)a_1 t} + 1) p(x) = \sum_{i=0}^{r} \sum_{J \subseteq [1,r]} (-1)^{m_J} p(x + ((m+1)a)_J),
\]

where \( m_J = \sum_{j \in J} m_j, ((m+1)a)_J = \sum_{j \in J} (m_j + 1)a_j \).

**Proof.** We show this by induction on \( r \). It is easy to check that it holds for \( r = 1 \). Assume that, for \( r > 1 \), the following holds:

\[
((-1)^{m_{r-1}} e^{(m_{r-1}+1)a_{r-1} t} + 1) \cdots ((-1)^{m_1} e^{(m_1+1)a_1 t} + 1) p(x)
\]

\[
= \sum_{i=0}^{r-1} \sum_{J \subseteq [1,r-1]} (-1)^{m_J} p(x + ((m+1)a)_J),
\]

From (2.12) and (2.13), we note that the LHS of (2.12) is

\[
((-1)^{m_r} e^{(m_r+1)a_r t} + 1) \cdots ((-1)^{m_1} e^{(m_1+1)a_1 t} + 1) p(x)
\]

\[
= \sum_{i=0}^{r-1} ((-1)^{m_r} e^{(m_r+1)a_r t} + 1) \sum_{J \subseteq [1,r-1]} (-1)^{m_J} p(x + ((m+1)a)_J)
\]

\[
= \sum_{i=0}^{r-1} \left\{ \sum_{J \subseteq [1,r-1]} (-1)^{m_J + m_r} p(x + ((m+1)a)_J + (m_r + 1)a_r) \right\}
\]

\[
= \sum_{J \subseteq [1,r-1]} (-1)^{m_J} p(x + ((m+1)a)_J)
\]

\[
+ \sum_{i=1}^{r-1} \sum_{J \subseteq [1,r-1]} (-1)^{m_J + m_r} p(x + ((m+1)a)_J + (m_r + 1)a_r)
\]

with \( s_n(x), w_n(x) \) as in (b), we have

\[
\sum_{i_1, \ldots, i_r = 0}^{m} w_n(i_1 + \cdots + i_r) = \frac{1}{(n+r)} \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} s_{n+r}((m+1)i).
\]
\begin{equation}
+ \sum_{i=1}^{r-1} \sum_{J \subseteq [1, r-1]} (-1)^{m_J} p(x + ((m + 1) a)_J) + p(x)
\end{equation}

\begin{equation}
= (-1)^{m_{[1,r]}} p(x + ((m + 1) a)_{[1,r]}) + \sum_{i=1}^{r-1} \sum_{J \subseteq [1, r]} (-1)^{m_J} p(x + ((m + 1) a)_J) + p(x)
\end{equation}

\begin{equation}
= \sum_{i=0}^{r} \sum_{J \subseteq [1, r]} (-1)^{m_J} p(x + ((m + 1) a)_J).
\end{equation}

\textbf{Theorem 2.} Let \( s_n(x) \sim \left( \prod_{i=1}^{r} \left( \frac{e^{a_i t} + 1}{2} \right), f(t) \right) \), \( w_n(x) \sim (1, f(t)) \), where \( a_1, a_2, \ldots, a_r \in \mathbb{C} \setminus \{0\} \), \( r \in \mathbb{N} \) with \( r > 0 \) and \( o(f(t)) = 1 \). Then we have

\begin{equation}
\sum_{i_1=0}^{m_1} \cdots \sum_{i_r=0}^{m_r} (-1)^{i_1+\cdots+i_r} w_n(x + a_1 i_1 + \cdots + a_r i_r) = \frac{1}{2^r} \sum_{i=0}^{r} \sum_{J \subseteq [1, r]} (-1)^{m_J} s_n \left( x + \sum_{j \in J} (m_j + 1) a_j \right),
\end{equation}

where \( m_J = \sum_{j \in J} m_j \).

\textbf{Proof.} The result is obtained by computing the following in two different ways:

\begin{equation}
\frac{(-1)^{m_r} e^{(m_r+1)a_r t} + 1}{e^{a_r t} + 1} \cdots \frac{(-1)^{m_1} e^{(m_1+1)a_1 t} + 1}{e^{a_1 t} + 1} w_n(x).
\end{equation}

On one hand, it is

\begin{equation}
\frac{(-1)^{m_r} e^{(m_r+1)a_r t} + 1}{e^{a_r t} + 1} \cdots \frac{(-1)^{m_2} e^{(m_2+1)a_2 t} + 1}{e^{a_2 t} + 1} \sum_{i_1=0}^{m_1} (-1)^{i_1} w_n(x + a_1 i_1)
\end{equation}

\begin{equation}
= \sum_{i_1=0}^{m_1} (-1)^{i_1} \frac{(-1)^{m_r} e^{(m_r+1)a_r t} + 1}{e^{a_r t} + 1} \cdots \frac{(-1)^{m_3} e^{(m_3+1)a_3 t} + 1}{e^{a_3 t} + 1} \sum_{i_2=0}^{m_2} (-1)^{i_2} w_n(x + a_1 i_1 + a_2 i_2)
\end{equation}

Continuing in this fashion, we get the expression on the LHS of (2.15).

On the other hand, (2.16) is

\begin{equation}
\frac{1}{2^r} \left( (-1)^{m_r} e^{(m_r+1)a_r t} + 1 \right) \cdots \left( (-1)^{m_1} e^{(m_1+1)a_1 t} + 1 \right) \prod_{i=1}^{r} \left( \frac{2}{e^{a_i t} + 1} \right) w_n(x)
\end{equation}

\end{document}
\[ \frac{1}{2^r} \left( (-1)^{m_r} e^{(m_r+1)a_r t} + 1 \right) \cdots \left( (-1)^{m_1} e^{(m_1+1)a_1 t} + 1 \right) s_n(x). \]

Here we observe that

\[ s_n(x) = \prod_{i=1}^{r} \left( \frac{2}{e^{a_i t} + 1} \right) w_n(x), \]

which follows from

\[ \prod_{i=1}^{r} \left( \frac{e^{a_i t} + 1}{2} \right) s_n(x) = w_n(x) \sim (1, f(t)). \]

Now, we obtain the expression on the RHS of (2.15) from Lemma 2.

**Corollary 2.**

(a) Let \( s_n(x) \sim \left( \frac{e^t - 1}{2} \right)^r, f(t) \), \( w_n(x) \sim (1, f(t)) \), where \( f(t) \) is a delta series. Then we have

\[ \sum_{i_1=0}^{m_1} \cdots \sum_{i_r=0}^{m_r} (-1)^{i_1 + \cdots + i_r} w_n(x + i_1 + \cdots + i_r) \]

\[ = \frac{1}{2^r} \sum_{i=0}^{r} \sum_{J \subseteq [1,r]} (-1)^{m_j} s_n \left( \sum_{j \in J} (m_j + 1) \right). \]

(b) With \( s_n(x), w_n(x) \) as in (a), we have

\[ \sum_{i_1=0}^{m_1} \cdots \sum_{i_r=0}^{m_r} (-1)^{i_1 + \cdots + i_r} w_n(i_1 + \cdots + i_r) = \frac{1}{2^r} \sum_{i=0}^{r} \sum_{J \subseteq [1,r]} (-1)^{m_j} s_n \left( \sum_{j \in J} (m_j + 1) \right). \]

(c) With \( s_n(x), w_n(x) \) as in (a), we have

\[ \sum_{i_1,\ldots,i_r=0}^{m} (-1)^{i_1 + \cdots + i_r} w_n(x + i_1 + \cdots + i_r) = \frac{1}{2^r} \sum_{i=0}^{r} (-1)^{m_i} \binom{r}{i} s_n(x + (m + 1)i). \]

(d) With \( s_n(x), w_n(x) \) as in (a), we have

\[ \sum_{i_1,\ldots,i_r=0}^{m} (-1)^{i_1 + \cdots + i_r} w_n(i_1 + \cdots + i_r) = \frac{1}{2^r} \sum_{i=0}^{r} (-1)^{m_i} \binom{r}{i} s_n((m + 1)i). \]

3. **Examples on Theorem 1**

(A) Let \( s_n(x) = B_n(x \mid a_1, \ldots, a_r) \sim \frac{e^{x(t-1)} \cdots e^{x(t-1)}}{t^r}, f(t), w_n(x) \sim (1, t). \)

Here \( B_n(x \mid a_1, \ldots, a_r) \) are the Barnes' multiple Bernoulli polynomials whose generating function is given by

\[ \frac{t^r}{(e^{a_1 t} - 1) \cdots (e^{a_r t} - 1)} e^{x t} = \sum_{n=0}^{\infty} B_n(x \mid a_1, \ldots, a_r) \frac{t^n}{n!}. \]
From Theorem 1, we have

\[
\sum_{i_1=0}^{m_1} \cdots \sum_{i_r=0}^{m_r} (x + a_1 i_1 + \cdots + a_r i_r)^n
= \frac{1}{(n+r)^r} \sum_{i=0}^{r} (-1)^{r-i} \sum_{J \subseteq [1,r]} B_{n+r} \left( x + \sum_{j \in J} (m_j + 1) a_j \right) a_1, \ldots, a_r .
\]

Letting \( x = 0 \), we also get

\[
\sum_{i_1=0}^{m_1} \cdots \sum_{i_r=0}^{m_r} (a_1 i_1 + \cdots + a_r i_r)^n
= \frac{1}{(n+r)^r} \sum_{i=0}^{r} (-1)^{r-i} \sum_{J \subseteq [1,r]} B_{n+r} \left( \sum_{j \in J} (m_j + 1) a_j \right) a_1, \ldots, a_r .
\]

For \( r = 1 \), \( a_1 = 1 \), \( m_1 = m \), we have

\[
\sum_{i=0}^{m} (x + i)^n = \frac{1}{n+1} \left( B_{n+1} (x + m) - B_{n+1} (x) \right).
\]

Thus, for \( x = 0 \), we get the classical power sum identity:

\[
\sum_{i=0}^{m} i^n = \frac{1}{n+1} \left( B_{n+1} (m) - B_{n+1} \right).
\]

(B) Let \( s_n (x) = \beta_n^{(r)} (\lambda, x) \sim \left( \frac{e^x - 1}{x} \right)^r, \frac{1}{\lambda} (e^{\lambda t} - 1) \), \( w_n (x) = (x \mid \lambda)_n = x (x-\lambda) \cdots (x-\lambda (n-1)) \sim (1, \frac{1}{\lambda} (e^{\lambda t} - 1)) \). Here, \( \beta_n^{(r)} (\lambda, x) \) are the degenerate Bernoulli polynomials of order \( r \) whose generating function is given by

\[
\left( \frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} \right)^r = \sum_{n=0}^{\infty} \beta_n^{(r)} (\lambda, x) \frac{t^n}{n!}.
\]

They are called the degenerate Bernoulli polynomials of order \( r \), since

\[
\lim_{\lambda \to 0} \beta_n^{(r)} (\lambda, x) = B_n^{(r)} (x), \quad \lim_{\lambda \to \infty} \lambda^{-n} \beta_n^{(r)} (\lambda, \lambda x) = b_n^{(r)} (x).
\]

Here \( B_n^{(r)} (x) \) are the Bernoulli polynomials of order \( r \), with

\[
\left( \frac{t}{e^t - 1} \right)^r e^{tx} = \sum_{n=0}^{\infty} B_n^{(r)} (x) \frac{t^n}{n!}, \quad (\text{see [1–20]}),
\]

and \( b_n^{(r)} (x) \) are the Bernoulli polynomials of the second kind of order \( r \), with

\[
\left( \frac{t}{\log (1 + t)} \right)^r (1 + t)^x = \sum_{n=0}^{\infty} b_n^{(r)} (x) \frac{t^n}{n!}.
\]
From Corollary 1 (b), we have

\[
\sum_{i_1=0}^{m_1} \cdots \sum_{i_r=0}^{m_r} (x + i_1 + \cdots + i_r \mid \lambda) = \frac{1}{(n+r)} \sum_{i=0}^{r} (-1)^{r-i} \sum_{J \subseteq [1,r] \mid |J|=i} \beta_{n+r}^{(r)} \left( \lambda, x + \sum_{j \in J} (m_j + 1) \right).
\]

For \(x = 0\) we obtain

\[
\sum_{i_1=0}^{m_1} \cdots \sum_{i_r=0}^{m_r} (i_1 + \cdots + i_r \mid \lambda) = m \frac{1}{(n+r)} \sum_{i=0}^{r} (-1)^{r-i} \sum_{J \subseteq [1,r] \mid |J|=i} \beta_{n+r}^{(r)} \left( \lambda, \sum_{j \in J} (m_j + 1) \right),
\]

which reduces, in the simplest possible case, to

\[
\sum_{i=0}^{m} (i \mid \lambda) = \frac{1}{n+1} (\beta_{n+1} (\lambda, m + 1) - \beta_{n+1} (\lambda)). \tag{\star}
\]

Here, \(\beta_n (\lambda, x) = \beta_n^{(1)} (\lambda, x)\) were introduced by Carlitz in [3, 4] and the identity (\star) was found also by Carlitz in [4]. Also, the higher-order degenerate Bernoulli polynomials \(\beta_n^{(r)} (\lambda, x)\) were studied in [11] by using umbral calculus and in [13] by exploiting \(p\)-adic integrals.

(C) Let \(s_n (x) = \beta_n (\lambda, x \mid a_1, \ldots, a_r) \sim \left( \frac{(e^{a_1 x} - 1) \cdots (e^{a_r x} - 1)}{(e^{x} - 1)^r}, \frac{1}{\lambda} (e^{\lambda x} - 1) \right)\), and \(w_n (x) = (x \mid \lambda) \sim (1, \frac{1}{\lambda} (e^{\lambda x} - 1))\). We recall that \(\beta_n (\lambda, x \mid a_1, \ldots, a_r)\) are called Barnes-type degenerate Bernoulli polynomials and studied with umbral calculus viewpoint in [14]. Here one shows easily that

\[
\lim_{\lambda \to 0} \beta_n (\lambda, x \mid a_1, \ldots, a_r) = B_n (x \mid a_1, \ldots, a_r),
\]

\[
\lim_{\lambda \to \infty} \lambda^{-n} \beta_n (\lambda, x a_1, \ldots, a_r) = \left( \prod_{i=1}^{r} a_i \right)^{-1} b_n^{(r)} (x).
\]

From Theorem 1, we note that

\[
\sum_{i_1=0}^{m_1} \cdots \sum_{i_r=0}^{m_r} (x + a_1 i_1 + \cdots + a_r i_r \mid \lambda) = \frac{1}{(n+r)} \sum_{i=0}^{r} (-1)^{r-i} \sum_{J \subseteq [1,r] \mid |J|=i} \beta_{n+r} \left( \lambda, x + \sum_{j \in J} (m_j + 1) \mid a_1, \ldots, a_r \right).
\]
(D) Let \( s_n (x) \sim \left( \frac{e^{2t} - 1}{e^t - 1} \right)^r = (e^t + 1)^r - e^t - 1 \), \( w_n (x) = (x)_n \sim (1, e^t - 1) \). Note that the generating function for \( s_n (x) \) is

\[
\left( \frac{1}{2 + t} \right)^r (1 + t)^x = \sum_{n=0}^{\infty} s_n (x) \frac{t^n}{n!}.
\]

Thus, we have \( s_n (x) = 2^{-r} \text{Ch}_n^{(r)} (x) \) where \( \text{Ch}_n^{(r)} (x) \) are the Changhee polynomials of the first kind of order \( r \) given by the generating function

\[
\left( \frac{2}{2 + t} \right)^r (1 + t)^x = \sum_{n=0}^{\infty} \text{Ch}_n^{(r)} (x) \frac{t^n}{n!}, \quad \text{(cf. [15])}.
\]

Now, from Theorem 1, we obtain

\[
\sum_{i_1=0}^{m_1} \cdots \sum_{i_r=0}^{m_r} (x + 2 (i_1 + \cdots + i_r))_n
\]

\[
= \frac{1}{2^r (n + r)} \sum_{i=0}^{r} (-1)^{r-i} \sum_{J \subseteq [1,r], \left| J \right| = i} \text{Ch}_n^{(r)} \left( x + 2 \sum_{j \in J} (m_j + 1) \right).
\]

(E) Let \( s_n (x) = (x)_n \sim \left( \frac{e^{2t} - 1}{e^t - 1} \right)^r = 1, e^t - 1 \), \( w_n (x) = (x)_n \sim (1, e^t - 1) \). In this simple case, from Theorem 1, we have

\[
\sum_{i_1=0}^{m_1} \cdots \sum_{i_r=0}^{m_r} (x + i_1 + \cdots + i_r)_n
\]

\[
= \frac{1}{(n + r)} \sum_{i=0}^{r} (-1)^{r-i} \sum_{J \subseteq [1,r], \left| J \right| = i} \left( x + \sum_{j \in J} (m_j + 1) \right)_{n+r}.
\]

4. Examples on Theorem 2

(A) Let \( s_n (x) = E_n (x \mid a_1, \ldots, a_r) \sim \left( \prod_{i=1}^{r} \left( \frac{e^{a_i t} + 1}{2} \right), t \right), w_n (x) = x^n \sim (1, t) \). Here, \( E_n (x \mid a_1, \ldots, a_r) \) are the Barnes-type Euler polynomials whose generating function is given by

\[
\prod_{i=1}^{r} \left( \frac{2}{e^{a_i t} + 1} \right) e^{tx} = \sum_{n=0}^{\infty} E_n (x \mid a_1, \ldots, a_r) \frac{t^n}{n!}.
\]

From Theorem 2, we obtain

\[
\sum_{i_1=0}^{m_1} \cdots \sum_{i_r=0}^{m_r} (-1)^{i_1 + \cdots + i_r} (x + a_1 i_1 + \cdots + a_r i_r)^n
\]
\[ \frac{1}{2r} \sum_{i=0}^{r} \sum_{J \subseteq [1,r]} (-1)^{m_J} E_n \left( x + \sum_{j \in J} (m_j + 1) a_j \mid a_1, \ldots, a_r \right). \]

For \( r = 1, a_1 = 1, m_1 = m \), we have
\[ \sum_{i=0}^{m} (-1)^i (x + i)^m = \frac{1}{2} \left( (-1)^m E_n(x + m + 1) + E_n(x) \right). \]

When \( x = 0 \), we obtain the classical alternating power sum identity
\[ \sum_{i=0}^{m} (-1)^i i^n = \frac{1}{2} \left( (-1)^m E_n(m + 1) + E_n \right). \] (**)

(B) Let \( s_n(x) = E_n^{(r)}(\lambda, x) \sim \left( \left( \frac{e^{\lambda t} + 1}{2} \right)^r, \frac{1}{\lambda} \left( e^\lambda - 1 \right) \right), \ w_n(x) = (x \mid \lambda)_n \sim (1, \frac{1}{\lambda} \left( e^\lambda - 1 \right)) \). Here \( E_n^{(r)}(\lambda, x) \) are the degenerate Euler polynomials of order \( r \) whose generating function is given by
\[ \left( \frac{2}{(1 + \lambda t)^{\lambda} + 1} \right)^r (1 + \lambda t)^{\lambda} \sum_{n=0}^{\infty} E_n^{(r)}(\lambda, x) \frac{t^n}{n!}. \]

These polynomials were studied in [12]. We observe that
\[ \lim_{\lambda \to 0} E_n^{(r)}(\lambda, x) = E_n^{(r)}(x), \quad \lim_{\lambda \to \infty} \Lambda^{-n} E_n^{(r)}(\lambda, \lambda x) = (x \mid \lambda)_n \quad \text{for any } r \in \mathbb{Z}_{>0}. \]

Here \( E_n^{(r)}(x) \) are the Euler polynomials of order \( r \) given by
\[ \left( \frac{2}{e^t + 1} \right)^r e^{tx} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}. \]

From Corollary 2 (a), we have
\[ \sum_{i_1=0}^{m_1} \cdots \sum_{i_r=0}^{m_r} (-1)^{i_1+\cdots+i_r} (x + i_1 + \cdots + i_r \mid \lambda) = \frac{1}{2r} \sum_{i=0}^{r} \sum_{J \subseteq [1,r]} (-1)^{m_J} E_n^{(r)}(\lambda, x + \sum_{j \in J} (m_j + 1)) \].

In the simplest possible case, this reduces to
\[ \sum_{i=0}^{m} (-1)^i (i \mid \lambda) = \frac{1}{2} \left( (-1)^m E_n(\lambda, m + 1) + E_n(\lambda) \right), \]

which becomes, by letting \( \lambda \to 0 \), the classical alternating power sum identity (**). Higher order degenerate Euler polynomials \( E_n^{(r)}(\lambda, x) \) were introduced in [19] and studied in [12] by using umbral calculus.

(C) Let \( s_n(x) = E_n(\lambda, x \mid a_1, \ldots, a_r) \sim \left( \prod_{i=1}^{r} \left( \frac{e^{\lambda i} + 1}{2} \right), \frac{1}{\lambda} \left( e^{\lambda} - 1 \right) \right), \ w_n(x) =
\( (x \mid \lambda) \sim (1, \frac{1}{\xi} (e^{\xi t} - 1)) \). Here \( \mathcal{E}_n (\lambda, x \mid a_1, \ldots, a_r) \) are the Barnes-type degenerate Euler polynomials given by

\[
\prod_{i=1}^{r} \left( \frac{2}{(1 + \lambda t)^{2^i \lambda}} \right) (1 + \lambda t)^{\frac{x}{\xi}} = \sum_{n=0}^{\infty} \mathcal{E}_n (\lambda, x \mid a_1, \ldots, a_r) \frac{t^n}{n!},
\]

which were studied in [9]. Here we note that

\[
\lim_{\lambda \to 0} \mathcal{E}_n (\lambda, x \mid a_1, \ldots, a_r) = E_n (x \mid a_1, \ldots, a_r),
\]

\[
\lim_{\lambda \to \infty} \lambda^{-n} \mathcal{E}_n (\lambda, \lambda x \mid a_1, \ldots, a_r) = (x)_n.
\]

From Theorem 2, we can derive the following equation:

\[
\sum_{i_1=0}^{m_1} \cdots \sum_{i_r=0}^{m_r} (-1)^{i_1 + \cdots + i_r} (x + a_1 i_1 + \cdots + a_r i_r \mid \lambda)_n
\]

\[
= \frac{1}{2^r} \sum_{i=0}^{r} \sum_{|J| \subseteq [1, r]} (-1)^{m_J} \mathcal{E}_n \left( \lambda, x + \sum_{j \in J} (m_j + 1) a_j \mid a_1, \ldots, a_r \right).
\]

(D) Let \( s_n (x) = \text{Ch}_n^{(r)} (x) \sim \left( \left( e^{t^2} + 1 \right)^{\frac{x}{2}} \right), w_n (x) = (x)_n \sim (1, e^t - 1) \).

From Theorem 2, we note that

\[
\sum_{i_1=0}^{m_1} \cdots \sum_{i_r=0}^{m_r} (-1)^{i_1 + \cdots + i_r} (x + a_1 i_1 + \cdots + a_r i_r)_n
\]

\[
= \frac{1}{2^r} \sum_{i=0}^{r} \sum_{|J| \subseteq [1, r]} (-1)^{m_J} \text{Ch}_n^{(r)} \left( x + \sum_{j \in J} (m_j + 1) a_j \right).
\]

(E) Let \( s_n (x) \sim \left( \left( e^{t^2} + 1 \right)^{\frac{x}{2}}, \log (1 + t) \right), w_n (x) \sim (1, \log (1 + t)) \). Here \( s_n (x) \) are the polynomials whose generating function is

\[
\left( \frac{2}{e^{e^t-1} + 1} \right)^{\frac{x}{e^{e^t-1}}} e^{x(e^t-1)}
\]

\[
= \sum_{n=0}^{\infty} E_i^{(r)} (x) \frac{1}{l!} (e^t - 1)^l
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} S_2 (n, l) E_i^{(r)} (x) \right) \frac{t^n}{n!},
\]

where \( S_2 (n, l) \) is the Stirling number of the second kind.

Thus, \( s_n (x) = \sum_{l=0}^{n} S_2 (n, l) E_i^{(r)} (x) \).
Also, \( w_n(x) = \phi_n(x) = \sum_{l=0}^{n} S_2(n,l) x^l \) are the exponential polynomials given by
\[
e^{x(e^t-1)} = \sum_{n=0}^{\infty} \phi_n(x) \frac{t^n}{n!}.
\]

From Corollary 2 (c), we have
\[
\sum_{i_1=0}^{m_1} \cdots \sum_{i_r=0}^{m_r} (-1)^{i_1+\cdots+i_r} \phi_n(x + i_1 + \cdots + i_r)
\]
\[
= \frac{1}{2^r} \sum_{i=0}^{r} \sum_{J \subset [1,r]} (-1)^{m_J} \sum_{l=0}^{n} S_2(n,l) E_l^{(r)} \left( x + \sum_{j \in J} (m_j + 1) \right).
\]

**Acknowledgements**

This work was supported by Research Grant of Pukyong National University (2014 year). The corresponding author of this paper is Jong Jin Seo.

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Received: January 20, 2015; Published: March 5, 2015