A Note on the Modified
\((r, q)\)-Bernoulli Polynomials

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Abstract

In this paper, we defined a new \(q\)-extension of Bernoulli polynomials by bosonic \(q\)-integral, and investigate several properties of those polynomials. These results are generalizations for several well known identities on \(q\)-Bernoulli numbers and polynomials and Carlitz’s \(q\)-Bernoulli numbers.

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1 Introduction

Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will respectively denote the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = 1/p$.

When one speaks of $q$-extension, $q$ is considered in many ways such as indeterminate, a complex number $q \in \mathbb{C}$ or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we assume that $|1 - q|_p < 1$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. The $q$-number of $x$ is denoted by

$$[x]_q = \frac{1 - q^x}{1 - q}.$$  

Note that $\lim_{q \to 1} [x]_q = x$. Let $d$ be a fixed positive integer and let $p$ be a fixed prime number. We set

$$X_d = \lim_{N \to \infty} \mathbb{Z}/d^N \mathbb{Z}, \quad X^* = \bigcup_{0 < a < dp \atop (a, p) = 1} \{a + dp \mathbb{Z}_p\},$$

$$a + dp^N \mathbb{Z}_p = \{x \in X | x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$.

The distribution on $X$ is given by $\mu_q (a + dp^N \mathbb{Z}_p) = q^a/\lbrack dp^N \rbrack_q$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x \text{ (see [9, 10]).} \quad (1)$$

As is well known, Bernoulli polynomials are defined by the generating function to be

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n \text{ for } |t| < 2\pi \text{ with } t \in \mathbb{C}. \quad (2)$$

In the special case, $x = 0$, $B_n = B_n(0)$ are called the $n$-th Bernoulli numbers. From (2), we note that

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k. \quad (3)$$
In [2, 3], L. Carlitz defined the $q$-extension of Bernoulli numbers as follows:

$$\beta_{0,q} = 1, q(\beta + 1)^k - \beta_{k,q} = \delta_{k,1} \quad (4)$$

with the usual convention of replacing $\beta^l_q$ by $\beta^l_{k,q}$ where $\delta_{k,1}$ is the Kronecker’s symbol.

Note that, from (4), we note that

$$\beta_{n,q} = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1 + l}{[l + 1]_q}, \quad (n \geq 0). \quad (5)$$

L. Carlitz have defined $q$-extension of Bernoulli polynomials as follows:

$$\beta_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^l x^l \beta_{l,q}, \quad (\text{see [3,4]}). \quad (6)$$

In [9], Kim showed that Carlitz’s $q$-Bernoulli numbers and polynomials can be expressed as an $p$-adic $q$-integral on $\mathbb{Z}_p$ as follows:

$$\beta_{n,q} = \int_{\mathbb{Z}_p} [x]^n d\mu_q(x), \quad \beta_{n,q}(x) = \int_{\mathbb{Z}_p} [x + y]^n d\mu_q(x).$$

$q$-Bernoulli numbers and polynomials have been studied by many mathematicians, and possess many interesting properties (see [1-17]). As new extension of Bernoulli polynomials, T. Kim et. al., Seo et. al., Y. Simsek gave new $q$-extensions of Bernoulli polynomials (see [13, 16, 17]). In [16], authors defined modified $q$-Bernoulli polynomials $\tilde{\beta}_{n,q}(x)$ by generating function, and represent $\tilde{\beta}_{n,q}(x)$ as a $p$-adic $q$-integral on $\mathbb{Z}_p$.

In this paper, we construct new $q$-extension of Bernoulli polynomials, and Bernoulli polynomials of order $k$. Finally, we investigate several properties of those polynomials.

2 A new extension of $q$-Bernoulli polynomials

As a new $q$-extension of Bernoulli polynomials, we define the modified $(r, q)$-Bernoulli polynomials with weight $\alpha$ and weak weight $\beta$ which are defined by the generation function to be

$$\sum_{n=0}^{\infty} \bar{B}_n^{(\alpha, \beta)}(x|r, q) (1 - r^\alpha)^n \frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{(x+[y],_{\alpha})(1-r^\alpha)} d\mu_{q^\beta}(y). \quad (7)$$

In the special case, $x = 0$, $\bar{B}_n^{(\alpha, \beta)}(0|r, q) = \bar{B}_n^{(\alpha, \beta)}(r, q)$ is called modified $(r, q)$-Bernoulli numbers with weight $\alpha$ and weak weight $\beta$. 
Note that, by (1), the following equations are obtained easily:

\[
\bar{B}_n^{(\alpha,\beta)}(r, q) = \int_{\mathbb{Z}_p} [x]_p^n d\mu_{q^\beta}(x)
\]

\[
= \lim_{N \to \infty} \frac{1}{[p^N]_{q^\beta}} \sum_{x=0}^{p^N-1} [x]_p^n \left( q^\beta \right)^x
\]

\[
= \frac{1}{(1 - r^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{\alpha l \log r + \beta \log q}{(1 - r^\alpha q^\beta) \beta \log q}. \tag{8}
\]

In particular, if we put \( r = q \), then \( \bar{B}_n^{(\alpha,\beta)}(q, q) = \beta_n^{(\alpha,\beta)} \) is the \( q \)-Bernoulli numbers with weight \( \alpha \) and weak weight \( \beta \) (see [15]).

From (8), we obtain the following equation:

\[
\int_{\mathbb{Z}_p} e^{(x + y)\alpha (1 - r^\alpha)t} d\mu_{q^\beta}(y)
\]

\[
= e^{x(1 - r^\alpha)t} \sum_{n=0}^{\infty} (1 - r^\alpha)^n \int_{\mathbb{Z}_p} [y]_p^n d\mu_{q^\beta}(y) \frac{t^n}{n!}
\]

\[
= e^{x(1 - r^\alpha)t} \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{\alpha l \log r + \beta \log q}{(1 - r^\alpha q^\beta) \beta \log q} \frac{t^n}{n!}
\]

\[
= e^{x(1 - r^\alpha)t+1} \sum_{l=0}^{\infty} (-1)^l \frac{\alpha l \log r + \beta \log q}{(1 - r^\alpha q^\beta) \beta \log q} \frac{t^l}{l!}
\]

\[
= \left( \sum_{m=0}^{\infty} ((1 - r^\alpha)x + 1)^m \frac{t^m}{m!} \right) \left( \sum_{l=0}^{\infty} (-1)^l \frac{\alpha l \log r + \beta \log q}{(1 - r^\alpha q^\beta) \beta \log q} \frac{t^l}{l!} \right)
\]

\[
= \sum_{n=0}^{\infty} \sum_{l=0}^{n} (-1)^l \binom{n}{l} \frac{\alpha l \log r + \beta \log q}{(1 - r^\alpha q^\beta) \beta \log q} ((1 - r^\alpha)x + 1)^{n-l} \frac{t^n}{n!}.
\]

Thus, by (7) and (9), we get

\[
\bar{B}_n^{(\alpha,\beta)}(x|r, q) = \frac{1}{(1 - r^\alpha)^n} \sum_{l=0}^{n} (-1)^l \binom{n}{l} \frac{\alpha l \log r + \beta \log q}{(1 - r^\alpha q^\beta) \beta \log q} ((1 - r^\alpha)x + 1)^{n-l}
\]

\[
= \frac{1}{(1 - r^\alpha)^n} \sum_{l=0}^{n} \sum_{j=0}^{n-l} (-1)^l \binom{n}{l} \binom{n-l}{j} \frac{\alpha l \log r + \beta \log q}{(1 - r^\alpha q^\beta) \beta \log q} (1 - r^\alpha)^j x^j. \tag{10}
\]

Therefore, by (10), we obtain the following theorem.
Remark 2.4. In the special case of the Theorem 2.1, if we put \( \alpha = \beta = 1 \) and \( r = q \), then

\[
\bar{B}^{(1,1)}_n(x|q,q) = \frac{1}{(1-q)^n} \sum_{l=0}^{n-1} \binom{n}{l} (-1)^l \frac{l+1}{l+1}_q ((1-q)x + 1)^{n-l} \\
= \sum_{l=0}^{n-1} \sum_{j=0}^{n-l} \binom{n}{l} \binom{n-l}{j} (-1)^j \frac{(l+1)}{(l+1)_q} (1-q)^j x^j ,
\]
and is the modified $q$-Bernoulli polynomials (see [16]).

\section{A new approach to modified $q$-Bernoulli polynomials of order $k$}

From now on, we consider the modified $(r, q)$-Bernoulli numbers of order $k$ with weight $\alpha$ and weak weight $\beta_1, \ldots, \beta_k$ as follows:

$$
\tilde{B}^{(\alpha, \beta_1, \ldots, \beta_k)}_n(r, q) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k]^n d\mu_{q^\alpha_1}(x_1) \cdots d\mu_{q^\beta_k}(x_k).
$$

Note that, by (1),

$$
\tilde{B}^{(\alpha, \beta_1, \ldots, \beta_k)}_n(r, q) = \frac{1}{(1 - r^\alpha)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l k \prod_{j=1}^k \frac{(1 - q^{\beta_j}) (\alpha l \log r + \beta_j \log q) \beta_j \log q (1 - r^\alpha q^{\beta_j})}{\beta_j \log q (1 - r^\alpha q^{\beta_j})}.
$$

Let us consider the modified $(r, q)$-Bernoulli polynomials of order $k$ with weight $\alpha$ and weak weight $\beta_1, \ldots, \beta_k$ as follows:

$$
\tilde{B}^{(\alpha, \beta_1, \ldots, \beta_k)}_n(x | r, q) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x + [x_1 + \cdots + x_k]^n d\mu_{q^\alpha_1}(x_1) \cdots d\mu_{q^\beta_k}(x_k)
$$

$$
= \sum_{l=0}^n \binom{n}{l} \tilde{B}^{(\alpha, \beta_1, \ldots, \beta_k)}_l(r, q) x^{n-l} = \sum_{l=0}^n \binom{n}{l} \tilde{B}^{(\alpha, \beta_1, \ldots, \beta_k)}_{n-l}(r, q) x^l
$$

where $\tilde{B}^{(\alpha, \beta_1, \ldots, \beta_k)}_l(0 | r, q) = \tilde{B}^{(\alpha, \beta_1, \ldots, \beta_k)}_l(r, q)$ is the modified $(r, q)$-Bernoulli numbers of order $k$ with weight $\alpha$ and weak weight $\beta_1, \ldots, \beta_k$.

By (14),

$$
(1 - r^\alpha)^n \tilde{B}^{(\alpha, \beta_1, \ldots, \beta_k)}_n(r, q) = \sum_{l=0}^n \binom{n}{l} (-1)^l k \prod_{j=1}^k \frac{(1 - q^{\beta_j}) (\alpha l \log r + \beta_j \log q) \beta_j \log q (1 - r^\alpha q^{\beta_j})}{\beta_j \log q (1 - r^\alpha q^{\beta_j})}.
$$
Consider the equation

\[
\sum_{n=0}^{\infty} (1 - r^\alpha)^n \tilde{B}_n^{(\alpha, \beta_1, \ldots, \beta_k)} \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \prod_{j=1}^{k} \left(1 - q^\beta_j\right) \frac{(\alpha l \log r + \beta_j \log q) \ t^n}{\beta_j \log q \ (1 - r^\alpha q^\beta_j)} \frac{1}{n!}
\]

\[
= \left( \sum_{m=0}^{\infty} \frac{t^m}{m!} \right) \left( \sum_{l=0}^{\infty} (-1)^l \prod_{j=1}^{k} \left(1 - q^\beta_j\right) \frac{(\alpha l \log r + \beta_j \log q) \ t^l}{\beta_j \log q \ (1 - r^\alpha q^\beta_j)} \frac{1}{l!} \right)
\]

\[
= e^t \left( \sum_{l=0}^{\infty} (-1)^l \prod_{j=1}^{k} \left(1 - q^\beta_j\right) \frac{(\alpha l \log r + \beta_j \log q) \ t^l}{\beta_j \log q \ (1 - r^\alpha q^\beta_j)} \frac{1}{l!} \right)
\]

Since

\[
e^{(1-r^\alpha)xt} \sum_{n=0}^{\infty} (1 - r^\alpha)^n \tilde{B}_n^{(\alpha, \beta_1, \ldots, \beta_k)} (r, q) \frac{t^n}{n!}
\]

\[
= \left( \sum_{l=0}^{\infty} (1 - r^\alpha)^l x^l \frac{t^l}{l!} \right) \left( \sum_{n=0}^{\infty} (1 - r^\alpha)^n \tilde{B}_n^{(\alpha, \beta_1, \ldots, \beta_k)} (r, q) \frac{t^n}{n!} \right)
\]

\[
= \sum_{m=0}^{\infty} (1 - r^\alpha)^m \sum_{n=0}^{m} \binom{m}{n} \tilde{B}_n^{(\alpha, \beta_1, \ldots, \beta_k)} (r, q) x^{m-n} \frac{t^m}{m!}
\]

\[
= \sum_{m=0}^{\infty} (1 - r^\alpha)^m \tilde{B}_m^{(\alpha, \beta_1, \ldots, \beta_k)} (x\ r, q) \frac{t^m}{m!}
\]

and

\[
e^{(1-r^\alpha)xt} \left( e^t \sum_{l=0}^{\infty} (-1)^l \prod_{j=1}^{k} \left(1 - q^\beta_j\right) \frac{(\alpha l \log r + \beta_j \log q) \ t^l}{\beta_j \log q \ (1 - r^\alpha q^\beta_j)} \frac{1}{l!} \right)
\]

\[
= e^{(1-r^\alpha)x+1)t} \left( \sum_{l=0}^{\infty} (-1)^l \prod_{j=1}^{k} \left(1 - q^\beta_j\right) \frac{(\alpha l \log r + \beta_j \log q) \ t^l}{\beta_j \log q \ (1 - r^\alpha q^\beta_j)} \frac{1}{l!} \right)
\]

\[
= \left( \sum_{m=0}^{\infty} ((1 - r^\alpha)x + 1)^m \frac{t^m}{m!} \right) \left( \sum_{l=0}^{\infty} (-1)^l \prod_{j=1}^{k} \left(1 - q^\beta_j\right) \frac{(\alpha l \log r + \beta_j \log q) \ t^l}{\beta_j \log q \ (1 - r^\alpha q^\beta_j)} \frac{1}{l!} \right)
\]

\[
= \sum_{n=0}^{\infty} \sum_{l=0}^{n} (-1)^l \binom{n}{l} \prod_{j=1}^{k} \left(1 - q^\beta_j\right) \frac{(\alpha l \log r + \beta_j \log q) \ t^l}{\beta_j \log q \ (1 - r^\alpha q^\beta_j)} \frac{1}{l!} ((1 - r^\alpha)x + 1)^{n-l} \frac{t^n}{n!}.
\]
By (16) and (17),
\[(1 - r^\alpha)^n \tilde{B}_n^{(\alpha, \beta_1, \ldots, \beta_k)}(x | r, q)\]
\[= \sum_{l=0}^{n-l} \sum_{j=0}^{n-l} \left( \begin{array}{c} n-l \\ j \end{array} \right) (-1)^l \prod_{j=1}^{k} \frac{(1 - q^{\beta_j})}{\beta_j \log q (1 - r^{\alpha} q^{\beta_j})} (1 - r^\alpha)^{j-1} x^j.\]

Thus, we have the following result.

**Theorem 3.1.** For \( n \geq 1 \),
\[\tilde{B}_n^{(\alpha, \beta_1, \ldots, \beta_k)}(x | r, q)\]
\[= \frac{1}{(1 - r^\alpha)^n} \sum_{l=0}^{n-l} \sum_{j=0}^{n-l} \left( \begin{array}{c} n-l \\ j \end{array} \right) (-1)^l \prod_{j=1}^{k} \frac{(1 - q^{\beta_j})}{\beta_j \log q (1 - r^{\alpha} q^{\beta_j})} (1 - r^\alpha)^{j-1} x^j.\]

Note that, by (15) and (16), the generating function of \( \tilde{b}_{n,q} \) is represented as follows:
\[\sum_{m=0}^{\infty} (1 - r^\alpha)^m \tilde{B}_n^{(\alpha, \beta_1, \ldots, \beta_k)}(x | r, q) \frac{t^m}{m!}\]
\[= e^{(1-r^\alpha)xt} \sum_{n=0}^{\infty} (1 - r^\alpha)^n \tilde{B}_n^{(\alpha, \beta_1, \ldots, \beta_k)}(r, q) \frac{t^n}{n!}\]
\[= e^{(1-r^\alpha)(x+1)t} \sum_{n=0}^{\infty} (-1)^l \prod_{j=1}^{k} \frac{(1 - q^{\beta_j})}{\beta_j \log q (1 - r^{\alpha} q^{\beta_j})} \frac{t^l}{l!}.\]

and
\[\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(1-r^\alpha)(x+[x_1+\cdots+x_k])} d\mu_{q^{\beta_1}}(x_1) \cdots d\mu_{q^{\beta_k}}(x_k)\]
\[= \sum_{n=0}^{\infty} (1 - r^\alpha)^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x + [x_1 + \cdots + x_k])^n d\mu_{q^{\beta_1}}(x_1) \cdots d\mu_{q^{\beta_k}}(x_k) \frac{t^n}{n!}\]
\[= \sum_{n=0}^{\infty} (1 - r^\alpha)^n \tilde{B}_n^{(\alpha, \beta_1, \ldots, \beta_k)}(x | r, q) \frac{t^n}{n!}.\]
So, by (18) and (19), we have the following corollary.

**Corollary 3.2.** For $n \geq 0$,

$$
\sum_{n=0}^{\infty} (1 - r^n) \tilde{B}_n^{(\alpha_1, \ldots, \alpha_k)}(x | r, q) \frac{t^n}{n!} =
$$

$$
e^{(1 - r^n)x + 1} \sum_{l=0}^{\infty} (-1)^l \prod_{j=1}^{k} \frac{(1 - q^\alpha)}{\beta_j \log q (1 - r^\alpha q^\beta_j)} t^l l!
$$

$$=
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(1 - r^n)t(x + [x_1 + \cdots + x_k]_r, \alpha)} d\mu_{q^\alpha}(x_1) \cdots d\mu_{q^\alpha}(x_k)
$$

where $\tilde{B}_n^{(\alpha_1, \ldots, \alpha_k)}(r, q)$ are the $n$-th modified $(r, q)$-Bernoulli numbers of order $k$.

In the Theorem 3.1, if we put $r = q$, then

$$
\tilde{B}_n^{(\alpha_1, \ldots, \alpha_k)}(r | r, q) = \tilde{b}_{n,q}^{(\alpha_1, \ldots, \alpha_k)}(x)
$$

$$=
\frac{1}{(1 - q^n)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \prod_{j=1}^{k} \frac{(\alpha l + \beta_j)[\beta_j]_q}{\beta_j(\alpha l + \beta_j)_q} ((1 - q^\alpha)x + 1)^{n-l}
$$

(see [15]).

**References**


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