Symmetric Properties for the (h, q) -Tangent Polynomials

C. S. Ryoo

Department of Mathematics
Hannam University, Daejeon 306-791, Korea

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Abstract

In [4], we studied the (h, q) -tangent numbers and polynomials. By using these numbers and polynomials, we give some interesting symmetric properties for the (h, q) -tangent polynomials.

Mathematics Subject Classification: 11B68, 11S40, 11S80

Keywords: tangent numbers and polynomials, (h, q) -tangent polynomials, alternating sums, symmetric properties

1 Introduction

Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}_p$ denotes the ring of p -adic rational integers, $\mathbb{Q}_p$ denotes the field of p -adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q -extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p -adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. For

$$g \in UD(\mathbb{Z}_p) = \{g|g : \mathbb{Z}_p \rightarrow \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the fermionic p -adic invariant integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} g(x) (-1)^x, \quad (\text{see [2]}). \quad (1.1)$$

If we take $g_1(x) = g(x+1)$ in (1.1), then we see that

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0), \quad (\text{see [2]}). \quad (1.2)$$

In [4], we introduced the (h, q) -tangent numbers $T_{n,q}^{(h)}$ and polynomials $T_{n,q}^{(h)}(x)$ and investigate their properties. Let us define the (h, q) -tangent numbers $T_{n,q}^{(h)}$ and polynomials $T_{n,q}^{(h)}(x)$ as follows:

$$I_{-1}(q^{hy} e^{2yt}) = \int_{\mathbb{Z}_p} q^{hy} e^{2yt} d\mu_{-1}(y) = \sum_{n=0}^{\infty} T_{n,q}^{(h)} \frac{t^n}{n!}, \quad (1.3)$$

$$I_{-1}(q^{hy} e^{(2y+x)t}) = \int_{\mathbb{Z}_p} q^{hy} e^{(2y+x)t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} T_{n,q}^{(h)}(x) \frac{t^n}{n!}. \quad (1.4)$$

The following elementary properties of the (h, q) -tangent numbers $E_{n,q}^{(h)}$ and polynomials $T_{n,q}^{(h)}(x)$ are readily derived from (1.1), (1.2), (1.3) and (1.4) (see, for details, [4]). We, therefore, choose to omit details involved.

Theorem 1.1 *For $h \in \mathbb{Z}$, we have*

$$\int_{\mathbb{Z}_p} q^{hx} (2x)^n d\mu_{-1}(x) = T_{n,q}^{(h)}, \quad \int_{\mathbb{Z}_p} q^{hy} (2y+x)^n d\mu_{-1}(y) = T_{n,q}^{(h)}(x).$$

Theorem 1.2 *For any positive integer n , we have*

$$T_{n,q}^{(h)}(x) = \sum_{k=0}^n \binom{n}{k} T_{k,q}^{(h)} x^{n-k}.$$

2 The alternating sums of powers of consecutive (h, q) -even integers

In this section, we assume that $q \in \mathbb{C}$, with $|q| < 1$ and $h \in \mathbb{Z}$. By using (1.4), we give the alternating sums of powers of consecutive (h, q) -even integers as follows:

$$\sum_{n=0}^{\infty} T_{n,q}^{(h)} \frac{t^n}{n!} = \frac{2}{q^h e^{2t} + 1} = 2 \sum_{n=0}^{\infty} (-1)^n q^{nh} e^{2nt}.$$

From the above, we obtain

$$-\sum_{n=0}^{\infty} (-1)^n q^{nh} e^{(2n+2k)t} + \sum_{n=0}^{\infty} (-1)^{n-k} q^{(n-k)h} e^{2nt} = \sum_{n=0}^{k-1} (-1)^{n-k} q^{(n-k)h} e^{2nt}.$$

By using (1.3) and (1.4), we obtain

$$-\frac{1}{2} \sum_{j=0}^{\infty} T_{j,q}^{(h)}(2k) \frac{t^j}{j!} + \frac{1}{2} (-1)^{-k} q^{-kh} \sum_{j=0}^{\infty} T_{j,q}^{(h)} \frac{t^j}{j!} = \sum_{j=0}^{\infty} \left((-1)^{-k} q^{-kh} \sum_{n=0}^{k-1} (-1)^n q^{nh} (2n)^j \right) \frac{t^j}{j!}.$$

By comparing coefficients of $\frac{t^j}{j!}$ in the above equation, we obtain

$$\sum_{n=0}^{k-1} (-1)^n q^{nh} (2n)^j = \frac{(-1)^{k+1} q^{kh} T_{j,q}^{(h)}(2k) + T_{j,q}^{(h)}}{2}.$$

By using the above equation we arrive at the following theorem:

Theorem 2.1 *Let k be a positive integer and $q \in \mathbb{C}$ with $|q| < 1$. Then we obtain*

$$\mathcal{T}_{j,q}^{(h)}(k-1) = \sum_{n=0}^{k-1} (-1)^n 2^j q^{nh} n^j = \frac{(-1)^{k+1} q^{kh} T_{j,q}^{(h)}(2k) + T_{j,q}^{(h)}}{2}. \quad (2.1)$$

Remark 2.2 *For the alternating sums of powers of consecutive even integers, we have*

$$\lim_{q \rightarrow 1} T_{j,q}^{(h)}(k-1) = \sum_{n=0}^{k-1} (-1)^n (2n)^j = \frac{(-1)^{k+1} T_j(2k) + T_j}{2},$$

where $T_j(x)$ and T_j denote the tangent polynomials and the tangent numbers, respectively.

3 Symmetric properties for the (h, q) -tangent polynomials

In this section, we assume that $q \in \mathbb{C}_p$ and $\in T_p$. In [3], Kim investigated interesting properties of symmetry p -adic invariant integral on $\mathbb{Z}_p$ for Bernoulli polynomials. By using same method of [3], expect for obvious modifications, we investigate interesting properties of symmetry p -adic invariant integral on $\mathbb{Z}_p$ for (h, q) -tangent polynomials. By using (1.1), we have

$$I_{-1}(g_n) + (-1)^{n-1} I_{-1}(g) = 2 \sum_{k=0}^{n-1} (-1)^{n-1-k} g(k),$$

where $n \in \mathbb{N}$, $g_n(x) = g(x+n)$. If n is odd from the above, we obtain

$$\int_{\mathbb{Z}_p} g(x+n) d\mu_{-1}(x) + \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = 2 \sum_{k=0}^{n-1} (-1)^{n-1-k} g(k). \quad (3.1)$$

Substituting $g(x) = q^{hx} e^{2xt}$ into the above, we obtain

$$\int_{\mathbb{Z}_p} q^{h(x+n)} e^{(2x+2n)t} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} q^{hx} e^{2xt} d\mu_{-1}(x) = 2 \sum_{j=0}^{n-1} (-1)^j q^{hj} e^{(2j)t}. \quad (3.2)$$

After some elementary calculations, we have

$$\int_{\mathbb{Z}_p} q^{h(x+n)} e^{(2x+2n)t} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} q^{hx} e^{2xt} d\mu_{-1}(x) = \frac{2(1 + q^{hn} e^{2nt})}{q^h e^{2t} + 1}.$$

By substituting Taylor series of e^{2xt} into (3.2) and the above, we arrive at the following theorem:

Theorem 3.1 *Let n be odd positive integer. Then we obtain*

$$\frac{2 \int_{\mathbb{Z}_p} q^{hx} e^{2xt} d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} q^{hnx} e^{2ntx} d\mu_{-1}(x)} = \sum_{m=0}^{\infty} (2\mathcal{T}_{m,q}^{(h)}(n-1)) \frac{t^m}{m!}. \quad (3.3)$$

Let w_1 and w_2 be odd positive integers. By using (3.3), we have

$$\begin{aligned} & \frac{\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} q^{h(w_1x_1+w_2x_2)} e^{(w_12x_1+w_22x_2+w_1w_2x)t} d\mu_{-1}(x_1) d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} q^{hw_1w_2x} e^{2w_1w_2xt} d\mu_{-1}(x)} \\ &= \frac{2e^{w_1w_2xt} (q^{hw_1w_2} e^{2w_1w_2t} + 1)}{(q^{hw_1} e^{2w_1t} + 1)(q^{hw_2} e^{2w_2t} + 1)} \end{aligned} \quad (3.4)$$

By using (3.3) and (3.4), after elementary calculations, we obtain

$$\begin{aligned} a &= \left(\frac{1}{2} \int_{\mathbb{Z}_p} q^{hw_1x_1} e^{(w_12x_1+w_1w_2x)t} d\mu_{-1}(x_1) \right) \left(\frac{2 \int_{\mathbb{Z}_p} q^{hw_2x_2} e^{2x_2w_2t} d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} q^{hw_1w_2x} e^{2w_1w_2tx} d\mu_{-1}(x)} \right) \\ &= \left(\frac{1}{2} \sum_{m=0}^{\infty} \mathcal{T}_{m,q^{w_1}}^{(h)}(w_2x) w_1^m \frac{t^m}{m!} \right) \left(2 \sum_{m=0}^{\infty} \mathcal{T}_{m,q^{w_2}}^{(h)}(w_1-1) w_2^m \frac{t^m}{m!} \right). \end{aligned} \quad (3.5)$$

By using Cauchy product in the above, we have

$$a = \sum_{m=0}^{\infty} \left(\sum_{j=0}^m \binom{m}{j} \mathcal{T}_{j,q^{w_1}}^{(h)}(w_2x) w_1^j \mathcal{T}_{m-j,q^{w_2}}^{(h)}(w_1-1) w_2^{m-j} \right) \frac{t^m}{m!} \quad (3.6)$$

By using the symmetry in (3.5), we have

$$\begin{aligned} a &= \left(\frac{1}{2} \int_{\mathbb{Z}_p} q^{hw_2x_2} e^{(w_2 2x_2 + w_1 w_2 x) t} d\mu_{-1}(x_2) \right) \left(\frac{2 \int_{\mathbb{Z}_p} q^{hw_1x_1} e^{2x_1 w_1 t} d\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p} q^{hw_1 w_2 x} e^{2w_1 w_2 t x} d\mu_{-1}(x)} \right) \\ &= \left(\frac{1}{2} \sum_{m=0}^{\infty} T_{m,q^{w_2}}^{(h)}(w_1 x) w_2^m \frac{t^m}{m!} \right) \left(2 \sum_{m=0}^{\infty} \mathcal{T}_{m,q^{w_1}}^{(h)}(w_2 - 1) w_1^m \frac{t^m}{m!} \right). \end{aligned}$$

Thus we obtain

$$a = \sum_{m=0}^{\infty} \left(\sum_{j=0}^m \binom{m}{j} T_{j,q^{w_2}}^{(h)}(w_1 x) w_2^j \mathcal{T}_{m-j,q^{w_1}}^{(h)}(w_2 - 1) w_1^{m-j} \right) \frac{t^m}{m!} \quad (3.7)$$

By comparing coefficients $\frac{t^m}{m!}$ in the both sides of (3.6) and (3.7), we arrive at the following theorem:

Theorem 3.2 *Let w_1 and w_2 be odd positive integers. Then we have*

$$\begin{aligned} & \sum_{j=0}^m \binom{m}{j} T_{j,q^{w_2}}^{(h)}(w_1 x) \mathcal{T}_{m-j,q^{w_1}}^{(h)}(w_2 - 1) w_1^{m-j} w_2^j \\ &= \sum_{j=0}^m \binom{m}{j} T_{j,q^{w_1}}^{(h)}(w_2 x) \mathcal{T}_{m-j,q^{w_2}}^{(h)}(w_1 - 1) w_1^j w_2^{m-j}, \end{aligned}$$

where $T_{k,q}^{(h)}(x)$ and $\mathcal{T}_{m,q}^{(h)}(k)$ denote the (h, q) -tangent polynomials and the alternating sums of powers of consecutive (h, q) -even integers, respectively.

By using Theorem 1.2, we have the following corollary:

Corollary 3.3 *Let w_1 and w_2 be odd positive integers. Then we obtain*

$$\begin{aligned} & \sum_{j=0}^m \sum_{k=0}^j \binom{m}{j} \binom{j}{k} w_1^{m-k} w_2^j x^{j-k} T_{k,q^{w_2}}^{(h)} \mathcal{T}_{m-j,q^{w_1}}^{(h)}(w_2 - 1) \\ &= \sum_{j=0}^m \sum_{k=0}^j \binom{m}{j} \binom{j}{k} w_1^j w_2^{m-k} x^{j-k} T_{k,q^{w_1}}^{(h)} \mathcal{T}_{m-j,q^{w_2}}^{(h)}(w_1 - 1). \end{aligned}$$

By using (3.5), we have

$$\begin{aligned} a &= \left(\frac{1}{2} e^{w_1 w_2 x t} \int_{\mathbb{Z}_p} q^{hw_1x_1} e^{2x_1 w_1 t} d\mu_{-1}(x_1) \right) \left(2 \sum_{j=0}^{w_1-1} (-1)^j q^{w_2 h j} e^{2j w_2 t} \right) \\ &= \sum_{j=0}^{w_1-1} (-1)^j q^{w_2 h j} \int_{\mathbb{Z}_p} q^{hw_1x_1} e^{\left(2x_1 + w_2 x + \frac{2j w_2}{w_1} \right) (w_1 t)} d\mu_{-1}(x_1) \\ &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{w_1-1} (-1)^j q^{w_2 h j} T_{n,q^{w_1}}^{(h)} \left(w_2 x + \frac{2j w_2}{w_1} \right) w_1^n \right) \frac{t^n}{n!}. \end{aligned} \quad (3.8)$$

By using the symmetry property in (3.8), we also have

$$\begin{aligned}
 a &= \left(\frac{1}{2} e^{w_1 w_2 x t} \int_{\mathbb{Z}_p} q^{h w_2 x_2} e^{2 x_2 w_2 t} d\mu_{-1}(x_2) \right) \left(2 \sum_{j=0}^{w_2-1} (-1)^j q^{w_1 h j} e^{2 j w_1 t} \right) \\
 &= \sum_{j=0}^{w_2-1} (-1)^j q^{w_1 h j} \int_{\mathbb{Z}_p} q^{h w_2 x_2} e^{\left(2 x_2 + w_1 x + \frac{2 j w_1}{w_2} \right) (w_2 t)} d\mu_{-1}(x_1) \\
 &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{w_2-1} (-1)^j q^{w_1 h j} T_{n, q^{w_2}}^{(h)} \left(w_1 x + \frac{2 j w_1}{w_2} \right) w_2^n \right) \frac{t^n}{n!}.
 \end{aligned} \tag{3.9}$$

By comparing coefficients $\frac{t^n}{n!}$ in the both sides of (3.8) and (3.9), we have the following theorem.

Theorem 3.4 *Let w_1 and w_2 be odd positive integers. Then we obtain*

$$\begin{aligned}
 &\sum_{j=0}^{w_1-1} (-1)^j q^{w_2 h j} T_{n, q^{w_1}}^{(h)} \left(w_2 x + \frac{2 j w_2}{w_1} \right) w_1^n \\
 &= \sum_{j=0}^{w_2-1} (-1)^j q^{w_1 h j} T_{n, q^{w_2}}^{(h)} \left(w_1 x + \frac{2 j w_1}{w_2} \right) w_2^n.
 \end{aligned} \tag{3.10}$$

Observe that if $q \rightarrow 1$, then (3.10) reduces to Theorem 3.4 in [5]. Substituting $w_1 = 1$ into (3.10), we arrive at the following corollary.

Corollary 3.5 *Let w_2 be odd positive integer. Then we obtain*

$$T_{n, q}^{(h)}(x) = w_2^n \sum_{j=0}^{w_2-1} (-1)^j q^{h j} T_{n, q^{w_2}}^{(h)} \left(\frac{x + 2 j}{w_2} \right).$$

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Received: January 9, 2014