Symmetric Properties for the \((h, q)\)-Tangent Polynomials

C. S. Ryoo

Department of Mathematics
Hannam University, Daejeon 306-791, Korea

Copyright © 2014 C. S. Ryoo. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In [4], we studied the \((h, q)\)-tangent numbers and polynomials. By using these numbers and polynomials, we give some interesting symmetric properties for the \((h, q)\)-tangent polynomials.

Mathematics Subject Classification: 11B68, 11S40, 11S80

Keywords: tangent numbers and polynomials, \((h, q)\)-tangent polynomials, alternating sums, symmetric properties

1 Introduction

Throughout this paper, we always make use of the following notations: \(\mathbb{N}\) denotes the set of natural numbers and \(\mathbb{Z}_+ = \mathbb{N} \cup \{0\}\), \(\mathbb{C}\) denotes the set of complex numbers, \(\mathbb{Z}_p\) denotes the ring of \(p\)-adic rational integers, \(\mathbb{Q}_p\) denotes the field of \(p\)-adic rational numbers, and \(\mathbb{C}_p\) denotes the completion of algebraic closure of \(\mathbb{Q}_p\). Let \(\nu_p\) be the normalized exponential valuation of \(\mathbb{C}_p\) with \(|p|_p = p^{-\nu_p(p)} = p^{-1}\). When one talks of \(q\)-extension, \(q\) is considered in many ways such as an indeterminate, a complex number \(q \in \mathbb{C}\), or \(p\)-adic number \(q \in \mathbb{C}_p\). If \(q \in \mathbb{C}\) one normally assume that \(|q| < 1\). If \(q \in \mathbb{C}_p\), we normally assume that \(|q - 1|_p < p^{-\nu_p(1)}\) so that \(q^x = \exp(x \log q)\) for \(|x|_p \leq 1\). For

\[ g \in UD(\mathbb{Z}_p) = \{g|g: \mathbb{Z}_p \rightarrow \mathbb{C}_p \text{ is uniformly differentiable function}\}, \]
the fermionic $p$-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} g(x)(-1)^x,$$

(see [2]).

(1.1)

If we take $g_1(x) = g(x+1)$ in (1.1), then we see that

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0),$$

(see [2]).

(1.2)

In [4], we introduced the $(h, q)$-tangent numbers $T_{n,q}^{(h)}$ and polynomials $T_{n,q}^{(h)}(x)$ and investigate their properties. Let us define the $(h, q)$-tangent numbers $T_{n,q}^{(h)}$ and polynomials $T_{n,q}^{(h)}(x)$ as follows:

$$I_{-1}(q^h y e^{2yt}) = \int_{\mathbb{Z}_p} q^h y e^{2yt} d\mu_{-1}(y) = \sum_{n=0}^{\infty} T_{n,q}^{(h)} \frac{t^n}{n!},$$

(1.3)

$$I_{-1}(q^h y e^{(2y+x)t}) = \int_{\mathbb{Z}_p} q^h y e^{(2y+x)t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} T_{n,q}^{(h)}(x) \frac{t^n}{n!}.$$

(1.4)

The following elementary properties of the $(h, q)$- tangent numbers $E_{n,q}^{(h)}$ and polynomials $T_{n,q}^{(h)}(x)$ are readily derived form (1.1), (1.2), (1.3) and (1.4)(see, for details, [4]). We, therefore, choose to omit details involved.

**Theorem 1.1** For $h \in \mathbb{Z}$, we have

$$\int_{\mathbb{Z}_p} q^h (2x)^n d\mu_{-1}(x) = T_{n,q}^{(h)}, \quad \int_{\mathbb{Z}_p} q^h y (2y + x)^n d\mu_{-1}(y) = T_{n,q}^{(h)}(x).$$

**Theorem 1.2** For any positive integer $n$, we have

$$T_{n,q}^{(h)}(x) = \sum_{k=0}^{n} \binom{n}{k} T_{k,q}^{(h)} x^{n-k}.$$

2 The alternating sums of powers of consecutive $(h, q)$-even integers

In this section, we assume that $q \in \mathbb{C}$, with $|q| < 1$ and $h \in \mathbb{Z}$. By using (1.4), we give the alternating sums of powers of consecutive $(h, q)$-even integers as follows:

$$\sum_{n=0}^{\infty} T_{n,q}^{(h)} \frac{t^n}{n!} = \frac{2}{q^h e^{2t} + 1} = 2 \sum_{n=0}^{\infty} (-1)^n q^n h e^{2nt}.$$
From the above, we obtain
\[- \sum_{n=0}^{\infty}(-1)^n q^n e^{(2n+2k)t} + \sum_{n=0}^{\infty}(-1)^{n-k} q^{(n-k)h} e^{2nt} = \sum_{n=0}^{k-1}(-1)^n q^{(n-k)h} e^{2nt}.\]

By using (1.3) and (1.4), we obtain
\[- \frac{1}{2} \sum_{j=0}^{\infty} T_{j,q}^{(h)} (2k) \frac{t^j}{j!} + \frac{1}{2} (-1)^{-k} q^{-kh} \sum_{j=0}^{\infty} T_{j,q}^{(h)} \frac{t^j}{j!} = \sum_{j=0}^{\infty} \left((-1)^{-k} q^{-kh} \sum_{n=0}^{k-1}(-1)^n q^n (2n)^j\right) \frac{t^j}{j!}.\]

By comparing coefficients of \( \frac{t^j}{j!} \) in the above equation, we obtain
\[\sum_{n=0}^{k-1} (-1)^n q^n (2n)^j = \left((-1)^{k+1} q^{kh} T_{j,q}^{(h)} (2k) + T_{j,q}^{(h)}\right) \frac{t^j}{2}.\]

By using the above equation we arrive at the following theorem:

**Theorem 2.1** Let \( k \) be a positive integer and \( q \in \mathbb{C} \) with \( |q| < 1 \). Then we obtain
\[T_{j,q}^{(h)} (k - 1) = \sum_{n=0}^{k-1} (-1)^n 2^j q^n n^j = \frac{(-1)^{k+1} q^{kh} T_{j,q}^{(h)} (2k) + T_{j,q}^{(h)}}{2},\] \hspace{1cm} (2.1)

**Remark 2.2** For the alternating sums of powers of consecutive even integers, we have
\[\lim_{q \to 1} T_{j,q}^{(h)} (k - 1) = \sum_{n=0}^{k-1} (-1)^n (2n)^j = \frac{(-1)^{k+1} T_j (2k) + T_j}{2},\]
where \( T_j(x) \) and \( T_j \) denote the tangent polynomials and the tangent numbers, respectively.

### 3 Symmetric properties for the \((h, q)\)-tangent polynomials

In this section, we assume that \( q \in \mathbb{C}_p \) and \( \in \mathbb{T}_p \). In [3], Kim investigated interesting properties of symmetry \( p \)-adic invariant integral on \( \mathbb{Z}_p \) for Bernoulli polynomials. By using same method of [3], expect for obvious modifications, we investigate interesting properties of symmetry \( p \)-adic invariant integral on \( \mathbb{Z}_p \) for \((h, q)\)-tangent polynomials. By using (1.1), we have
\[I_{-1}(g_n) + (-1)^{n-1} I_{-1}(g) = 2 \sum_{k=0}^{n-1} (-1)^{n-1-k} g(k),\]
where \( n \in \mathbb{N}, g_n(x) = g(x + n) \). If \( n \) is odd from the above, we obtain

\[
\int_{\mathbb{Z}_p} g(x + n) d\mu_{-1}(x) + \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = 2 \sum_{k=0}^{n-1} (-1)^{n-1-k} g(k). \tag{3.1}
\]

Substituting \( g(x) = q^{hx}e^{2xt} \) into the above, we obtain

\[
\int_{\mathbb{Z}_p} q^{h(x+n)}e^{(2x+2nt)t} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} q^{hx}e^{2xt} d\mu_{-1}(x) = 2 \sum_{j=0}^{n-1} (-1)^j q^{hj} e^{(2j)t}. \tag{3.2}
\]

After some elementary calculations, we have

\[
\int_{\mathbb{Z}_p} q^{h(x+n)}e^{(2x+2nt)t} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} q^{hx}e^{2xt} d\mu_{-1}(x) = \frac{2(1 + q^{hn}e^{2nt})}{q^{he^{2t}} + 1}.
\]

By substituting Taylor series of \( e^{2xt} \) into (3.2) and the above, we arrive at the following theorem:

**Theorem 3.1** Let \( n \) be odd positive integer. Then we obtain

\[
\frac{2 \int_{\mathbb{Z}_p} q^{hx}e^{2xt} d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} q^{hn}e^{2nt} d\mu_{-1}(x)} = \sum_{m=0}^{\infty} \left( 2T_{m,q}^{(h)}(n - 1) \right) \frac{t^m}{m!}. \tag{3.3}
\]

Let \( w_1 \) and \( w_2 \) be odd positive integers. By using (3.3), we have

\[
\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} q^{h(w_1x_1+w_2x_2)}e^{(w_12x_1+w_22x_2+w_1w_2x_2)t} d\mu_{-1}(x_1)d\mu_{-1}(x_2) = \frac{2 \int_{\mathbb{Z}_p} q^{hw_1x_1}e^{2w_1xt} d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} q^{hw_1x_1}e^{2w_1w_2xt} d\mu_{-1}(x)} = \frac{2e^{w_1w_2xt}(q^{hw_1}e^{2w_1w_2t} + 1)}{(q^{hw_1}e^{2w_1t} + 1)(q^{hw_2}e^{2w_2t} + 1)}. \tag{3.4}
\]

By using (3.3) and (3.4), after elementary calculations, we obtain

\[
a = \left( \frac{1}{2} \int_{\mathbb{Z}_p} q^{hw_1x_1}e^{(w_12x_1+w_1w_2x_2)t} d\mu_{-1}(x_1) \right) \left( \frac{2 \int_{\mathbb{Z}_p} q^{hw_2x_2}e^{2x_2w_2t} d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} q^{hw_1w_2x_1}e^{2w_1w_2xt} d\mu_{-1}(x)} \right) = \left( \frac{1}{2} \sum_{m=0}^{\infty} T_{m,q}^{(h)}(w_2x_1)w_1^m \frac{t^m}{m!} \right) \left( 2 \sum_{m=0}^{\infty} T_{m,q}^{(h)}(w_1 - 1)w_2^m \frac{t^m}{m!} \right) \tag{3.5}
\]

By using Cauchy product in the above, we have

\[
a = \sum_{m=0}^{\infty} \left( \sum_{j=0}^{m} \binom{m}{j} T_{j,q}^{(h)}(w_2x_1)w_1^j T_{m-j,q}^{(h)}(w_1 - 1)w_2^{m-j} \right) \frac{t^m}{m!}. \tag{3.6}
\]
By using the symmetry in (3.5), we have

\[
a = \left( \frac{1}{2} \int_{\mathbb{Z}_p} q^{hw_2x_2} e^{(w_2x_2+w_1w_2)t} d\mu_{-1}(x_2) \right) \left( \frac{2 \int_{\mathbb{Z}_p} q^{hw_1x_1} e^{2x_1w_1t} d\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p} q^{hw_1w_2x} e^{2w_1w_2x} d\mu_{-1}(x)} \right).
\]

Thus we obtain

\[
a = \left( \frac{1}{2} \sum_{m=0}^{\infty} T_{m,q}^{(h)}(w_1x)w_2^m \frac{t^m}{m!} \right) \left( \sum_{m=0}^{\infty} T_{m,q}^{(h)}(w_2 - 1)w_1^{m-j} \frac{t^m}{m!} \right).
\]

By comparing coefficients \(\frac{t^m}{m!}\) in the both sides of (3.6) and (3.7), we arrive at the following theorem:

**Theorem 3.2** Let \(w_1\) and \(w_2\) be odd positive integers. Then we have

\[
\sum_{j=0}^{m} \binom{m}{j} T_{j,q}^{(h)}(w_1x)T_{m-j,q}^{(h)}(w_2 - 1)w_1^{m-j}w_2^j = \sum_{j=0}^{m} \binom{m}{j} T_{j,q}^{(h)}(w_2x)T_{m-j,q}^{(h)}(w_1 - 1)w_1^jw_2^{m-j},
\]

where \(T_{k,q}^{(h)}(x)\) and \(T_{m,q}^{(h)}(k)\) denote the \((h, q)\)-tangent polynomials and the alternating sums of powers of consecutive \((h, q)\)-even integers, respectively.

By using Theorem 1.2, we have the following corollary:

**Corollary 3.3** Let \(w_1\) and \(w_2\) be odd positive integers. Then we obtain

\[
\sum_{j=0}^{m} \sum_{k=0}^{j} \binom{j}{k} \binom{m}{j} w_1^{m-k}w_2^j T_{k,q}^{(h)} T_{m-j,q}^{(h)}(w_2 - 1) = \sum_{j=0}^{m} \sum_{k=0}^{j} \binom{j}{k} \binom{m}{j} w_1^jw_2^{m-k} T_{k,q}^{(h)} T_{m-j,q}^{(h)}(w_1 - 1).
\]

By using (3.5), we have

\[
a = \left( \frac{1}{2} e^{w_1w_2x} \int_{\mathbb{Z}_p} q^{hw_1x_1} e^{2x_1w_1t} d\mu_{-1}(x_1) \right) \left( \sum_{j=0}^{w_1-1} (-1)^j q^{w_2hj} e^{2wj}\right)\left( \frac{2jw_2}{w_1} \right)T_{n,q}^{(h)}(w_2x + \frac{2jw_2}{w_1} w_1^n) \frac{t^n}{n!}.
\]
By using the symmetry property in (3.8), we also have
\[
\begin{align*}
a &= \left( \frac{1}{2} e^{w_1 w_2 t} \int_{Z_p} q^{h w_2 x_2} e^{2 x_2 w_2 t} d\mu_{-1}(x_2) \right) \left( 2 \sum_{j=0}^{w_2-1} (-1)^j q^{w_1 h_j} e^{2 j w_1 t} \right) \\
&= \sum_{j=0}^{w_2-1} (-1)^j q^{w_1 h_j} \int_{Z_p} q^{h w_2 x_2} e^{2 x_2 w_1 + 2 j w_1} \mu_{-1}(x_1) \\
&= \sum_{n=0}^{\infty} \left( \sum_{j=0}^{w_2-1} (-1)^j q^{w_1 h_j} T_{n,q}^{(h)} w_2 \right) \left( w_1 x + 2 j w_1 \right) w_2^n.
\end{align*}
\]  
(3.9)

By comparing coefficients \( \frac{t^n}{n!} \) in the both sides of (3.8) and (3.9), we have the following theorem.

**Theorem 3.4** Let \( w_1 \) and \( w_2 \) be odd positive integers. Then we obtain
\[
\sum_{j=0}^{w_1-1} (-1)^j q^{w_2 h_j} T_{n,q}^{(h)} \left( w_2 x + \frac{2 j w_2}{w_1} \right) \frac{w_1^n}{w_1}
\]  
(3.10)

Observe that if \( q \to 1 \), then (3.10) reduces to Theorem 3.4 in [5]. Substituting \( w_1 = 1 \) into (3.10), we arrive at the following corollary.

**Corollary 3.5** Let \( w_2 \) be odd positive integer. Then we obtain
\[
T_{n,q}^{(h)}(x) = w_2^n \sum_{j=0}^{w_2-1} (-1)^j q^{h_j} T_{n,q}^{(h)} \left( x + \frac{2 j}{w_2} \right).
\]

**References**


**Received: January 9, 2014**