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Symmetric Identities for the Generalized Higher-order q-Bernoulli Polynomials

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Abstract

In this paper, we give identities of symmetry for the generalized higher-order q-Bernoulli polynomials attached to χ which are derived from the symmetric properties of multivariate p-adic invariant integrals on $\mathbb{Z}_p$.

1. Introduction

As is well known, the higher order Bernoulli polynomials are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^r e^{xt}=\sum_{n=0}^{\infty}B_n^{(r)}(x)\frac{t^n}{n!},\ (n\in\mathbb{N}),\ (\text{see [1,4,11]}). \tag{1}$$

When $x=0$, $B_n^{(r)}=B_n^{(r)}(0)$ is called the n-th Bernoulli number of order r. Let χ be a primitive Dirichlet character with conductor $d \in \mathbb{N}$. Then the generalized Bernoulli polynomials attached to χ are defined by the generating function to be

$$\left(\frac{t}{e^{dt}-1}\sum_{a=0}^{d-1}\chi(a)e^{at}\right)e^{xt}=\sum_{n=0}^{\infty}B_{n,\chi}(x)\frac{t^n}{n!},\ \text{(see [5,6])}. \tag{2}$$

For $r \in \mathbb{N}$, in view of (1), we may also consider the generalized higher-order Bernoulli polynomials attached to χ as follows:

$$\left(\frac{t\sum_{a=0}^{d-1}\chi(a)e^{at}}{e^{dt}-1}\right)^r e^{xt}=\sum_{n=0}^{\infty}B_{n,\chi}^{(r)}(x)\frac{t^n}{n!}. \tag{3}$$

When $x=0$, $B_{n,\chi}^{(r)}=B_{n,\chi}^{(r)}(0)$ is called the n-th generalized Bernoulli numbers attached to χ with order r. Let p be a fixed prime number. Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p=p^{-\nu_p(p)}=\frac{1}{p}$. When one talks about q-extensions, q is variously considered as an indeterminate, a complex number $q\in\mathbb{C}$, or a p-adic number $q\in\mathbb{C}_p$. If $q\in\mathbb{C}$, one usually assumes $|q|<1$;if $q\in\mathbb{C}_p$, one usually assumes $|1-q|_p<p^{-\frac{1}{p-1}}$ so that $q^x=exp(x\log q)$ for $|x|_p\le 1$. The q-number of x is defined by $[x]_q=\frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q=x$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f\in UD(\mathbb{Z}_p)$, the p-adic invariant integral on $\mathbb{Z}_p$ is defined by

$$I_0(f)=\int_{\mathbb{Z}_p}f(x)d\mu_0(x)=\lim_{N\to\infty}\frac{1}{p^N}\sum_{x=0}^{p^N-1}f(x),\ \text{(see [1-11])}. \tag{4}$$

From (4), we note that

$$I_0(f_1)=I_0(f)+f'(0),\ \text{where}\ f_1(x)=f(x+1). \tag{5}$$

Thus, by (5), we get

$$I_0(f_n)=I_0(f)+\sum_{l=0}^{n-1}f'(l), \tag{6}$$

where $n\in\mathbb{N}$ and $f_n(x)=f(x+n)$.
Let d be a fixed positive integer and let p be a fixed prime number. For $N\in\mathbb{N}$, we set

$$X = \lim_{\overleftarrow{N}} \left(\mathbb{Z} / dp^N \mathbb{Z} \right), \quad X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$

$$a + dp^N \mathbb{Z}_p = \{x \in X | x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$.

It is not difficult to show that

$$\int_X f(x) d\mu_0(x) = \int_{\mathbb{Z}_p} f(x) d\mu_0(x), \tag{7}$$

where $f(x) \in UD(\mathbb{Z}_p)$, (see [5,6,7]).
Let us take $f(x) = e^{xt}\chi(x)$. Then, by (6), we get

$$\int_X \chi(x) e^{xt} d\mu_0(x) = \frac{t}{e^{dt}-1} \sum_{a=0}^{d-1} \chi(a) e^{at} = \sum_{n=0}^{\infty} B_{x,\chi} \frac{t^n}{n!}. \tag{8}$$

By (8), we see that

$$\begin{aligned} \int_X \chi(y) e^{(x+y)t} d\mu_0(y) &= \left(\frac{t}{e^{dt}-1} \sum_{a=0}^{d-1} \chi(a) e^{at} \right) e^{xt} \\ &= \sum_{n=0}^{\infty} B_{n,\chi}(x) \frac{t^n}{n!}. \end{aligned} \tag{9}$$

From (9), we can derive the following equation (10):

$$\begin{aligned} &\int_X \cdots \int_X \left(\Pi_{l=1}^r \chi(y_l) \right) e^{(x+y_1+\cdots+y_r)t} d\mu_0(y_1) \ldots d\mu_0(y_r) \\ &= \left(\frac{t \sum_{a=0}^{d-1} \chi(a) e^{at}}{e^{dt}-1} \right)^r e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi}^{(r)}(x) \frac{t^n}{n!}. \end{aligned} \tag{10}$$

In the next section, we consider q-extensions of (10). The purpose of this paper is to give identities of symmetry for the generalized q-Bernoulli polynomials of order r which are derived from the symmetric properties of multivariate p-adic invariant integrals on $\mathbb{Z}_p$.

2. Symmetric identities of the generalized higher-order q-Bernoulli polynomials

Now, we consider the generalized higher-order q-Bernoulli polynomials attached to χ with the viewpoint of (10) as follows:

$$\begin{aligned}&\int_X \cdots \int_X \left(\Pi_{l=1}^r \chi(y_l)\right) e^{[x+y_1+\cdots+y_r]_q t} d\mu_0(y_1)\ldots d\mu_0(y_r)\\ &=\sum_{n=0}^{\infty} B_{n,\chi,q}^{(r)}(x)\frac{t^n}{n!}.\end{aligned} \tag{11}$$

Thus, by (11), we get

$$\begin{aligned}B_{n,\chi,q}^{(r)}(x) &= \int_X \cdots \int_X \left(\Pi_{l=1}^r \chi(y_l)\right) [x+y_1+\cdots+y_r]_q^n d\mu_0(y_1)\ldots d\mu_0(y_r)\\ &= \frac{[d]_q^n}{d^r}\sum_{a_1,\cdots,a_r=0}^{d-1}\left(\Pi_{l=1}^r \chi(a_l)\right)\\ &\times \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left[\frac{x+a_1+\cdots+a_r}{d}+y_1+\cdots+y_r\right]_{q^d}^n d\mu_0(y_1)\ldots d\mu_0(y_r)\\ &= \frac{[d]_q^n}{d^r}\sum_{a_1,\cdots,a_r=0}^{d-1}\left(\Pi_{l=1}^r \chi(a_l)\right) B_{n,q^d}^{(r)}\left(\frac{x+a_1+\cdots+a_r}{d}\right),\end{aligned} \tag{12}$$

where $B_{n,q}^{(r)}(x)$ are higher-order q-Bernoulli polynomials which are defined by

$$B_{n,q}^{(r)}(x) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+y_1+\cdots+y_r]_q^n d\mu_0(y_1)\ldots d\mu_0(y_r).$$

From (12), we note that

$$B_{n,\chi,q}^{(r)}(x) = \sum_{l=0}^{n}\binom{n}{l}[x]_q^{n-l}q^{lx}B_{l,\chi,q}^{(r)}, \tag{13}$$

where $B_{l,\chi,q}^{(r)} = B_{l,\chi,q}^{(r)}(0)$ is the l-th generalized higher-order q-Bernoulli number attached to χ.

Let w_1, w_2 be natural numbers. Then, by (11), we get

$$\begin{aligned}&\frac{1}{w_1^r}\int_X\cdots\int_X\left(\Pi_{l=1}^r \chi(y_l)\right) e^{[w_1w_2x+w_2\sum_{l=1}^r j_l+w_1\sum_{l=1}^r y_l]_q t}d\mu_0(y_1)\ldots d\mu_0(y_r)\\ &=\lim_{N\to\infty}\left(\frac{1}{dw_1w_2p^N}\right)^r\sum_{i_1,\cdots,i_r=0}^{dw_2-1}\left(\Pi_{l=1}^r \chi(i_l)\right)\\ &\qquad\times\sum_{y_1,\cdots,y_r=0}^{p^N-1} e^{[w_1w_2x+w_2\sum_{l=1}^r j_l+w_1\sum_{l=1}^r (i_l+dw_2y_l)]_q t}\end{aligned} \tag{14}$$

Thus, by (14), we get

$$\begin{aligned}&\frac{1}{w_1^r}\sum_{j_1,\cdots,j_r=0}^{dw_1-1}(\Pi_{l=1}^r\chi(j_l))\int_X\cdots\int_X(\Pi_{l=1}^r\chi(y_l))\\&\qquad\times e^{[w_1w_2x+w_2\sum_{l=1}^r j_l+w_1\sum_{l=1}^r y_l]_q t}d\mu_0(y_1)\ldots d\mu_0(y_r)\\&=\lim_{N\to\infty}\left(\frac{1}{dw_1w_2p^N}\right)^r\sum_{y_1,\cdots,y_r=0}^{p^N-1}\sum_{j_1,\cdots,j_r=0}^{dw_1-1}\sum_{i_1,\cdots,i_r=0}^{dw_2-1}(\Pi_{l=1}^r\chi(i_l)\chi(j_l))\\&\qquad\times e^{[w_1w_2x+w_2\sum_{l=1}^r j_l+w_1\sum_{l=1}^r(i_l+dw_2y_l)]_q t}\end{aligned}\tag{15}$$

By the same method as (15), we get

$$\begin{aligned}&\frac{1}{w_2^r}\sum_{j_1,\cdots,j_r=0}^{dw_2-1}(\Pi_{l=1}^r\chi(j_l))\int_X\cdots\int_X(\Pi_{l=1}^r\chi(y_l))\\&\qquad\times e^{[w_1w_2x+w_1\sum_{l=1}^r j_l+w_2\sum_{l=1}^r y_l]_q t}d\mu_0(y_1)\ldots d\mu_0(y_r)\\&=\lim_{N\to\infty}\left(\frac{1}{dw_1w_2p^N}\right)^r\sum_{y_1,\cdots,y_r=0}^{p^N-1}\sum_{j_1,\cdots,j_r=0}^{dw_2-1}\sum_{i_1,\cdots,i_r=0}^{dw_1-1}(\Pi_{l=1}^r\chi(i_l)\chi(j_l))\\&\qquad\times e^{[w_1w_2x+w_1\sum_{l=1}^r j_l+w_2\sum_{l=1}^r(i_l+dw_2y_l)]_q t}\end{aligned}\tag{16}$$

Therefore, by (15) and (16), we obtain the following theorem.

Theorem 2.1. *For $w_1, w_2, d\in\mathbb{N}$, we have*

$$\begin{aligned}&\frac{1}{w_1^r}\sum_{j_1,\cdots,j_r=0}^{dw_1-1}(\Pi_{l=1}^r\chi(j_l))\int_X\cdots\int_X(\Pi_{l=1}^r\chi(y_l))\\&\qquad\times e^{[w_1w_2x+w_2\sum_{l=1}^r j_l+w_1\sum_{l=1}^r y_l]_q t}d\mu_0(y_1)\ldots d\mu_0(y_r)\\&=\frac{1}{w_2^r}\sum_{j_1,\cdots,j_r=0}^{dw_2-1}(\Pi_{l=1}^r\chi(j_l))\int_X\cdots\int_X(\Pi_{l=1}^r\chi(y_l))\\&\qquad\times e^{[w_1w_2x+w_1\sum_{l=1}^r j_l+w_2\sum_{l=1}^r y_l]_q t}d\mu_0(y_1)\ldots d\mu_0(y_r)\end{aligned}$$

Note that

$$\left[w_1w_2x+w_2\sum_{l=1}^r j_l+w_1\sum_{l=1}^r y_l\right]_q=[w_1]_q\left[w_2x+\frac{w_2}{w_1}\sum_{l=1}^r j_l+\sum_{l=1}^r y_l\right]_{q^{w_1}},\tag{17}$$

and

$$\left[w_1w_2x+w_1\sum_{l=1}^r j_l+w_2\sum_{l=1}^r y_l\right]_q=[w_2]_q\left[w_1x+\frac{w_1}{w_2}\sum_{l=1}^r j_l+\sum_{l=1}^r y_l\right]_{q^{w_2}}.\tag{18}$$

Therefore, by Theorem 2.1, (17) and (18), we obtain the following Corollary.

Corollary 2.2. *For $n \geq 0$, we have*

$$\frac{[w_1]_q^n}{w_1^r} \sum_{j_1,\cdots,j_r=0}^{dw_1-1} (\Pi_{l=1}^r \chi(j_l)) \int_X \cdots \int_X (\Pi_{l=1}^r \chi(y_l)) \times \left[w_2 x + \frac{w_2}{w_1} \sum_{l=1}^r j_l + \sum_{l=1}^r y_l \right]_{q^{w_1}}^n d\mu_0(y_1) \ldots d\mu_0(y_r)$$

$$= \frac{[w_2]_q^n}{w_2^r} \sum_{j_1,\cdots,j_r=0}^{dw_2-1} (\Pi_{l=1}^r \chi(j_l)) \int_X \cdots \int_X (\Pi_{l=1}^r \chi(y_l)) \times \left[w_1 x + \frac{w_1}{w_2} \sum_{l=1}^r j_l + \sum_{l=1}^r y_l \right]_{q^{w_2}}^n d\mu_0(y_1) \ldots d\mu_0(y_r)$$

Therefore, by (12) and Corollary 2.2, we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, $w_1, w_2 \in \mathbb{N}$, we have*

$$\frac{[w_1]_q^n}{w_1^r} \sum_{j_1,\cdots,j_r=0}^{dw_1-1} (\Pi_{l=1}^r \chi(j_l))\, B_{n,\chi,q^{w_1}}^{(r)} \left(w_2 x + \frac{w_2}{w_1}(j_1 + \cdots + j_r) \right)$$

$$= \frac{[w_2]_q^n}{w_2^r} \sum_{j_1,\cdots,j_r=0}^{dw_2-1} (\Pi_{l=1}^r \chi(j_l))\, B_{n,\chi,q^{w_2}}^{(r)} \left(w_1 x + \frac{w_1}{w_2}(j_1 + \cdots + j_r) \right).$$

From (12), we can derive the following equation (19):

$$\begin{aligned}
&\int_X \cdots \int_X (\Pi_{l=1}^r \chi(y_l)) \left[w_2 x + \frac{w_2}{w_1} \sum_{l=1}^r j_l + \sum_{l=1}^r y_l \right]_{q^{w_1}}^n d\mu_0(y_1) \ldots d\mu_0(y_r) \\
&= \sum_{i=0}^n \binom{n}{i} \left(\frac{[w_2]_q}{[w_1]_q} \right)^i [j_1 + \cdots + j_r]_{q^{w_2}}^i q^{w_2(n-i)\sum_{l=1}^r j_l} \\
&\quad \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[w_2 x + \sum_{l=1}^r y_l \right]_{q^{w_1}}^{n-i} (\Pi_{l=1}^r \chi(y_l))\, d\mu_0(y_1) \ldots d\mu_0(y_r) \\
&= \sum_{i=0}^n \binom{n}{i} \left(\frac{[w_2]_q}{[w_1]_q} \right)^i [j_1 + \cdots + j_r]_{q^{w_2}}^i q^{w_2(n-i)\sum_{l=1}^r j_l} B_{n-i,\chi,q^{w_1}}^{(r)}(w_2 x).
\end{aligned} \tag{19}$$

Thus, by (19), we get

$$\begin{aligned}
&\frac{[w_1]_q^n}{w_1^r}\sum_{j_1,\cdots,j_r=0}^{dw_1-1}\left(\Pi_{l=1}^r\chi(j_l)\right)\int_X\cdots\int_X\left(\Pi_{l=1}^r\chi(y_l)\right)\\
&\qquad\qquad\times\left[w_2x+\frac{w_2}{w_1}\sum_{l=1}^r j_l+\sum_{l=1}^r y_l\right]_{q^{w_1}}^n d\mu_0(y_1)\ldots d\mu_0(y_r)\\
&=\sum_{i=0}^n\binom{n}{i}\frac{[w_1]_q^{n-i}}{w_1^r}[w_2]_q^i\left(\sum_{j_1,\cdots,j_r=0}^{dw_1-1}[j_1+\cdots+j_r]_{q^{w_2}}^i\left(\Pi_{l=1}^r\chi(j_l)\right)q^{w_2(n-i)\sum_{l=1}^r j_l}\right)\\
&\qquad\qquad\qquad\qquad\times B_{n-i,\chi,q^{w_1}}^{(r)}(w_2x)\\
&=\sum_{i=0}^n\binom{n}{i}\frac{[w_1]_q^{i}}{w_1^r}[w_2]_q^{n-i}\left(\sum_{j_1,\cdots,j_r=0}^{dw_1-1}[j_1+\cdots+j_r]_{q^{w_2}}^{n-i}q^{w_2i\sum_{l=1}^r j_l}\left(\Pi_{l=1}^r\chi(j_l)\right)\right)\\
&\qquad\qquad\qquad\qquad\times B_{i,\chi,q^{w_1}}^{(r)}(w_2x)\\
&=\sum_{i=0}^n\binom{n}{i}\frac{[w_1]_q^{i}}{w_1^r}[w_2]_q^{n-i}T_{n,i}^{(r)}(dw_1,q^{w_2}|\chi)B_{i,\chi,q^{w_1}}^{(r)}(w_2x),
\end{aligned}\tag{20}$$

where

$$T_{n,i}^{(r)}(w,q|\chi)=\sum_{j_1,\cdots,j_r=0}^{w-1}[j_1+\cdots+j_r]_q^{n-i}q^{i\sum_{l=1}^r j_l}\left(\Pi_{l=1}^r\chi(j_l)\right).\tag{21}$$

By the same method as (20), we get

$$\begin{aligned}
&\frac{[w_2]_q^n}{w_2^r}\sum_{j_1,\cdots,j_r=0}^{dw_2-1}\left(\Pi_{l=1}^r\chi(j_l)\right)\int_X\cdots\int_X\left(\Pi_{l=1}^r\chi(y_l)\right)\\
&\qquad\times\left[w_1x+\frac{w_1}{w_2}\sum_{l=1}^r j_l+\sum_{l=1}^r y_l\right]_{q^{w_2}}^n d\mu_0(y_1)\ldots d\mu_0(y_r)\\
&=\sum_{i=0}^n\binom{n}{i}\frac{[w_2]_q^{i}}{w_2^r}[w_1]_q^{n-i}T_{n,i}^{(r)}(dw_2,q^{w_1}|\chi)B_{i,\chi,q^{w_2}}^{(r)}(w_1x).
\end{aligned}\tag{22}$$

Therefore, by (20) and (22), we obtain the following theorem.

Theorem 2.4. *For $n\geq 0$, $w_1,w_2\in\mathbb{N}$, we have*

$$\begin{aligned}
&\sum_{i=0}^n\binom{n}{i}\frac{[w_1]_q^{i}}{w_1^r}[w_2]_q^{n-i}T_{n,i}^{(r)}(dw_1,q^{w_2}|\chi)B_{i,\chi,q^{w_1}}^{(r)}(w_2x)\\
&=\sum_{i=0}^n\binom{n}{i}\frac{[w_2]_q^{i}}{w_2^r}[w_1]_q^{n-i}T_{n,i}^{(r)}(dw_2,q^{w_1}|\chi)B_{i,\chi,q^{w_2}}^{(r)}(w_1x),
\end{aligned}$$

where

$$T_{n,i}^{(r)}(w,q|\chi)=\sum_{j_1,\cdots,j_r=0}^{w-1}[j_1+\cdots+j_r]_q^{n-i}q^{i\sum_{l=1}^{r}j_l}\left(\Pi_{l=1}^{r}\chi(j_l)\right).$$

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