Higher-order Changhee Numbers and Polynomials

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Abstract

In this paper, we consider the higher-order Changhee numbers and polynomials which are derived from the fermionic $p$-adic integral on $\mathbb{Z}_p$ and give some relations between higher-order Changhee polynomials and special polynomials.
1. INTRODUCTION

As is well known, the Euler polynomials of order $\alpha (\in \mathbb{N})$ are defined by the generating function to be

$$(\frac{2}{e^t + 1})^\alpha e^{\alpha t} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{t^n}{n!}, \text{ (see [1-16])}.$$ (1.1)

When $x = 0$, $E_n^{(\alpha)} = E_n^{(\alpha)}(0)$ are called the Euler numbers of order $\alpha$.

The Stirling number of the first kind is defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n, l) x^l, \text{ (} n \in \mathbb{Z}_{\geq 0}, \text{ (see [5,6,7])}.$$ (1.2)

where $(x)_n = x(x-1) \cdots (x-n+1)$.

The Stirling number of the second kind is also defined by the generating function to be

$$(e^t - 1)^n = n! \sum_{l=n}^{\infty} S_2(l, n) \frac{t^l}{l!}, \text{ (} n \in \mathbb{Z}_{\geq 0} \text{).}$$ (1.3)

Let $p$ be an odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of $p$-adic integers, the field of $p$-adic numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The $p$-adic norm $| \cdot |_p$ is normalized as $|p|_p = \frac{1}{p}$. Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic $p$-adic integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_{-1}^{f} = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^{N-1}} f(x)(-1)^x, \text{ (see [9]).}$$ (1.4)

For $f_1(x) = f(x+1)$, we have

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0).$$ (1.5)

As is well-known, the Changhee polynomials are defined by the generating function to be

$$\frac{2}{t+2}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x) \frac{t^n}{n!}, \text{ (see [6,8])}.$$ (1.6)

When $x = 0$, $Ch_n = Ch_n(0)$ are called the Changhee numbers. In this paper, we consider the higher-order Changhee numbers and polynomials which are derived from the multivariate fermionic $p$-adic integral on $\mathbb{Z}_p$ and give some relations between higher-order Changhee polynomials and special polynomials.
2. Higher-order Changhee polynomials

For \( k \in \mathbb{N} \), let us define the Changhee numbers of the first kind with order \( k \) as follows:

\[
Ch^{(k)}_n = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k)_n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k), \tag{2.1}
\]

where \( n \) is a nonnegative integer.

From (2.1), we can derive the generating function of \( Ch^{(k)}_n \) as follows:

\[
\sum_{n=0}^{\infty} \frac{Ch^{(k)}_n}{n!} t^n = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \binom{x_1 + \cdots + x_k}{n} t^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k)
\]

\[
= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + t)^{x_1 + \cdots + x_k} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k). \tag{2.2}
\]

By (1.5), we easily see that

\[
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + t)^{x_1 + \cdots + x_k} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) = \left( \frac{2}{2 + t} \right)^k. \tag{2.3}
\]

From (2.2) and (2.3), we have

\[
\sum_{n=0}^{\infty} Ch^{(k)}_n \frac{t^n}{n!} = \left( \frac{2}{2 + t} \right)^k. \tag{2.4}
\]

It is easy to show that

\[
\left( \frac{2}{2 + t} \right)^k = \sum_{n=0}^{\infty} \left( \sum_{l_1 + \cdots + l_k = n} \binom{n}{l_1, \ldots, l_k} Ch_{l_1} \cdots Ch_{l_k} \right) \frac{t^n}{n!}. \tag{2.5}
\]

Thus, by (2.4) and (2.5), we get

\[
Ch^{(k)}_n = \sum_{l_1 + \cdots + l_k = n} \binom{n}{l_1, \ldots, l_k} Ch_{l_1} \cdots Ch_{l_k}. \tag{2.6}
\]

It is not difficult to show that

\[
\left( \frac{2}{2 + t} \right)^k = \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right)^n \frac{1}{n!} n! \binom{k}{n} \frac{t^n}{n!}. \tag{2.7}
\]
From (2.4) and (2.7), we have
\[
2^n Ch_n^{(k)} = (-1)^n n! \binom{n + k - 1}{n} = (-1)^n (k + n - 1)_n
\]
\[
= (-1)^n \sum_{l=0}^{n} S_1(n, l)(k + n - 1)^l. \tag{2.8}
\]

Therefore, by (2.8), we obtain the following theorem.

**Theorem 2.1.** For \( n \geq 0 \), we have
\[
Ch_n^{(k)} = \left( -\frac{1}{2} \right) \sum_{l=0}^{n} S_1(n, l)(k + n - 1)^l.
\]

By (2.1), we get
\[
Ch_n^{(k)} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k)n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k)
\]
\[
= \sum_{l=0}^{n} S_1(n, l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k)^l d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k). \tag{2.9}
\]

Now, we observe that
\[
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1 + \cdots + x_k)t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) = \left( \frac{2}{e^t + 1} \right)^k = \sum_{n=0}^{\infty} E_n^{(k)} \frac{t^n}{n!}. \tag{2.10}
\]

By (2.9) and (2.10), we get
\[
Ch_n^{(k)} = \sum_{l=0}^{n} S_1(n, l) E_l^{(k)}. \tag{2.11}
\]

Therefore, by (2.12), we obtain the following theorem.

**Theorem 2.2.** For \( n \geq 0 \), we have
\[
Ch_n^{(k)} = \sum_{l=0}^{n} S_1(n, l) E_l^{(k)}. \tag{2.12}
\]

Replacing \( t \) by \( e^t - 1 \) in (2.4), we get
\[
\sum_{n=0}^{\infty} Ch_n^{(k)} \frac{(e^t - 1)^n}{n!} = \left( \frac{2}{e^t + 1} \right)^k = \sum_{m=0}^{\infty} E_m^{(k)} \frac{t^m}{m!}, \tag{2.13}
\]
and
\[
\sum_{n=0}^{\infty} Ch_n^{(k)} \frac{(e^t - 1)^n}{n!} = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} Ch_n^{(k)} S_2(m, n) \right) \frac{t^m}{m!}. \tag{2.14}
\]
Therefore, by (2.13) and (2.14), we obtain the following theorem.

**Theorem 2.3.** For $n \geq 0$, we have

\[
E_m^{(k)} = \sum_{n=0}^{m} C_n^{(k)} S_2(m, n).
\]

Now, we consider the higher-order Changhee polynomials of the first kind as follows:

\[
C_n^{(k)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots x_k + x)_n \mu(x_1) \cdots \mu(x_k).
\]  

(2.15)

By (2.15), we get

\[
\sum_{n=0}^{\infty} C_n^{(k)}(x) \frac{x^n}{n!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + t)^{x_1 + \cdots + x_k + x} \mu(x_1) \cdots \mu(x_k) = \left( \frac{2}{2 + t} \right)^k (1 + t)^x.
\]  

(2.16)

From (2.4), we have

\[
\left( \frac{2}{2 + t} \right)^k (1 + t)^x = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \binom{n}{m} C_n^{(k)} \right) \frac{t^n}{n!}.
\]  

(2.17)

By (2.16) and (2.17), we get

\[
C_n^{(k)}(x) = \sum_{m=0}^{n} \binom{x}{m} \frac{n!}{(n - m)!} C_n^{(k)} = \sum_{m=0}^{n} \left( \frac{x}{n - m} \right) \frac{n!}{m!} C_n^{(k)}.
\]  

(2.18)

From (2.15), we have

\[
C_n^{(k)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots x_k + x)_n \mu(x_1) \cdots \mu(x_k)
\]

\[
= \sum_{l=0}^{n} S_1(n, l) E_l^{(k)}(x).
\]  

(2.19)

Therefore, by (2.19), we obtain the following corollary.

**Corollary 2.4.** For $n \geq 0$, we have

\[
C_n^{(k)}(x) = \sum_{l=0}^{n} S_1(n, l) E_l^{(k)}(x).
\]

In (2.16), by replacing $t$ by $e^t - 1$, we get

\[
\sum_{n=0}^{\infty} C_n^{(k)}(x) \frac{(e^t - 1)^n}{n!} = \left( \frac{2}{e^t + 1} \right)^k e^t x = \sum_{m=0}^{\infty} E_n^{(k)}(x) \frac{t^m}{m!},
\]  

(2.20)

and
Therefore, by (2.20) and (2.21), we obtain the following theorem.

**Theorem 2.5.** For \( m \geq 0 \), we have

\[
E_m^{(k)}(x) = \sum_{n=0}^{m} Ch_n^{(k)}(x) S_2(m, n).
\]

The rising factorial is defined by

\[
(x)^{(n)} = x(x + 1) \cdots (x + n - 1) = (-1)^{n}(-x)_n. \tag{2.22}
\]

Here, we define the Changhee numbers of the second kind with order \( k(\in \mathbb{N}) \) as follows:

\[
\hat{Ch}_n^{(k)} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 - \cdots - x_k)_n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k). \tag{2.23}
\]

Thus, by (2.23), we get

\[
\hat{Ch}_n^{(k)} = \sum_{l=0}^{n} (-1)^l S_1(n, l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots x_k)_l d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k)
\]

\[
= \sum_{l=0}^{n} (-1)^l S_1(n, l) E_l^{(k)}. \tag{2.24}
\]

The generating function of \( \hat{Ch}_n^{(k)} \) is given by

\[
\sum_{n=0}^{\infty} \frac{\hat{Ch}_n^{(k)} t^n}{n!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + t)^{-x_1 - \cdots - x_k} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k)
\]

\[
= \left( \frac{2}{2 + t} \right)^k (1 + t)^k. \tag{2.25}
\]

Now, we observe that

\[
\left( \frac{2}{2 + t} \right)^k (1 + t)^k = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \binom{k}{m} Ch_{n-m}^{(k)} \frac{n!}{(n-m)!} \right) \frac{t^n}{n!}. \tag{2.26}
\]

Thus, by (2.25) and (2.26), we get

\[
\hat{Ch}_n^{(k)} = \sum_{m=0}^{n} m! \binom{k}{m} \binom{n}{m} Ch_{n-m}^{(k)}. \tag{2.27}
\]

Therefore, by (2.27), we obtain the following theorem.
**Theorem 2.6.** For $n \geq 0$, we have

$$\hat{Ch}_n^{(k)} = \sum_{m=0}^{n} m! \binom{k}{m} \binom{n}{m} Ch_{n-m}^{(k)}. \quad (2.28)$$

In (2.25), by replacing $t$ by $e^t - 1$, we get

$$\sum_{n=0}^{\infty} \frac{\hat{Ch}_n^{(k)}(e^t - 1)^n}{n!} = \left( \frac{2}{e^t + 1} \right)^k = \sum_{m=0}^{\infty} E_m^{(k)}(k) \frac{t^m}{m!}, \quad (2.29)$$

and

$$\sum_{n=0}^{\infty} \frac{\hat{Ch}_n^{(k)}(e^t - 1)^n}{n!} = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} \hat{Ch}_n^{(k)} S_2(m, n) \right) \frac{t^m}{m!}. \quad (2.30)$$

Therefore, by (2.29) and (2.30), we obtain the following theorem.

**Theorem 2.7.** For $m \geq 0$, we have

$$E_m^{(k)}(k) = \sum_{n=0}^{m} \hat{Ch}_n^{(k)} S_2(m, n).$$

Now, we consider the Changhee polynomials of the second kind with order $k (\in \mathbb{N})$ as follows:

$$\hat{Ch}_n^{(k)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-x_1 - \cdots - x_k + x)_n d\mu_1(x_1) \cdots d\mu_1(x_k). \quad (2.31)$$

From (2.25) and (2.31), we have

$$\sum_{n=0}^{\infty} \frac{\hat{Ch}_n^{(k)}(x)^{tn}}{n!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + t)^{-x_1 - \cdots - x_k + x} d\mu_1(x_1) \cdots d\mu_1(x_k)$$

$$= (1 + t)^{x+k} \left( \frac{2}{2 + t} \right)^k. \quad (2.32)$$

We observe that

$$\left( \frac{2}{2 + t} \right)^k (1 + t)^{x+k} = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} m! \binom{x}{m} \binom{n}{m} Ch_{n-m}^{(k)} \right) \frac{t^n}{n!}. \quad (2.33)$$

Thus, by (2.32) and (2.33), we obtain the following theorem.

**Theorem 2.8.** For $m \geq 0$, we have

$$\hat{Ch}_n^{(k)}(x) = \sum_{m=0}^{n} m! \binom{x}{m} \binom{n}{m} Ch_{n-m}^{(k)}.$$
From (2.31), we have
\[ \hat{C}_n^{(k)}(x) = \sum_{l=0}^{n} S_1(n, l)(-1)^l \int_{\mathbb{R}^p} \cdots \int_{\mathbb{R}^p} (x_1 + \cdots + x_k - x)^l d\mu_1(x_1) \cdots d\mu_1(x_k) \]
\[ = \sum_{l=0}^{n} S_1(n, l)(-1)^l E_l^{(k)}(-x). \] (2.34)

In (2.32), by replacing \( t \) by \( e^t - 1 \), we get
\[ \sum_{n=0}^{\infty} \widehat{C}_n^{(k)}(x) \frac{(e^t - 1)^n}{n!} = e^{(x+k)t} \left( \frac{2}{e^t + 1} \right)^k = \sum_{m=0}^{\infty} E_m^{(k)}(x+k) \frac{t^m}{m!}, \] (2.35)
and
\[ \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{1} (e^t - 1)^n = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} \hat{C}_n^{(k)} S_m(n, n) \right) \frac{t^m}{m!}. \] (2.36)

Therefore, by (2.35) and (2.36), we obtain the following theorem.

**Theorem 2.9.** For \( m \geq 0 \), we have
\[ E_m^{(k)}(x+k) = \sum_{n=0}^{m} \hat{C}_n^{(k)} S_m(n, n). \]

Now, we observe that
\[ (-1)^n \frac{\hat{C}_n^{(k)}(x)}{n!} = (-1)^n \int_{\mathbb{R}^p} \cdots \int_{\mathbb{R}^p} \left( -\frac{(x_1 + \cdots + x_k + x)}{n} \right) d\mu_1(x_1) \cdots d\mu_1(x_k) \]
\[ = \int_{\mathbb{R}^p} \cdots \int_{\mathbb{R}^p} \left( \frac{x_1 + \cdots + x_k - x + n - 1}{n} \right) d\mu_1(x_1) \cdots d\mu_1(x_k) \]
\[ = \sum_{m=0}^{n} \binom{n-1}{n-m} \frac{1}{m!} C_l^{(k)}(-x) = \sum_{m=1}^{n} \frac{(n-m)}{m!} C_l^{(k)}(-x). \] (2.37)

Therefore, by (2.37), we obtain the following theorem.

**Theorem 2.10.** For \( n \in \mathbb{N} \), we have
\[ (-1)^n \frac{\hat{C}_n^{(k)}(x)}{n!} = \sum_{m=1}^{n} \frac{(n-m)}{m!} C_l^{(k)}(-x). \]
By (2.15), we get

\[
(-1)^n \frac{C_{m_n}^{(k)}(x)}{n!} = (-1)^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( \frac{x_1 + \cdots + x_k + x}{n} \right) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k)
\]

\[
= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( -x_1 - x_2 - \cdots - x_k - x + n - 1 \right) d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k)
\]

\[
= \sum_{m=0}^{n} \frac{(n-1)}{m!} \hat{h}_m^{(k)}(-x) = \sum_{m=1}^{n} \frac{(n-1)}{m!} \hat{h}_m^{(k)}(-x).
\]  

(2.38)

Therefore, by (2.38), we obtain the following theorem.

**Theorem 2.11.** For \( n \in \mathbb{N} \), we have

\[
(-1)^n \frac{\hat{C}_m^{(k)}(x)}{n!} = \sum_{m=1}^{n} \frac{(n-1)}{m!} \hat{h}_m^{(k)}(-x).
\]

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