On the Characterization Feedback of Positive LTI Continuous Singular Systems of Index 1

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Abstract

In this paper, we study the characterization of feedback for the positive LTI continuous singular systems of index 1. For this fundamental issue, we establish a sufficient condition for characterization the feedback matrix such that the closed loop system is regular, stable and positive.

Keywords: Positive LTI singular system, stabilization, Drazin inverse

1 Introduction

In the latest decades, the linear time invariant(LTI) singular systems have attracted attention seriously of many researchers, especially, in the control system field. This is caused the fact that these kinds of dynamical systems appear as a model of various applications, particularly, in physical science, electrical engineering, power systems, etc. An overview of state of the art in the singular systems is given in the monographs [2], and its application for the optimal control, has been reported by many authors, see [2] and [5], for some comprehensive literatures.

A singular system for which the state trajectory lies entirely in the non-negative orthant whenever the initial state and the input are nonnegative is called a positive singular system. Recently, Herrero et al. in [3] proposes an algorithm to check the nonnegativity of singular system for the case of discrete
time. Moreover, they also study the nonnegativity, stability and regularization of singular system for the case of discrete time \[4\].

To the best of author’s knowledge, little work has been done with positivity properties of LTI singular system for the case of continuous time. Recent development in positivity of continuous singular system and some new results on its stability are given in \[6\] and \[7\]. Anyway, these both papers do not study on stabilization problem of positive LTI singular systems of continuous time. In general, the stabilization problem of positive LTI singular systems is to seek a controller feedback such that the closed loop system is regular, stable and positive. In this paper, we are interested to study on characterization of the controller feedback such that the closed loop system is regular, stable and positive.

**Notations:** Let \( \mathbb{R} \) and \( \mathbb{C} \) be the sets of real and complex numbers, respectively, and \( \mathbb{R}_+ := [0, \infty) \). \( \mathbb{R}^n \) the set of \( n \)-tuples for which all of components belong to \( \mathbb{R} \) while \( \mathbb{R}^n_+ \) is the set of \( n \)-tuples for which each of components belong to \( \mathbb{R}_+ \). \( \mathcal{M}_{mn} \) denotes the set of \( m \times n \) real matrices, \( \mathcal{M}_{mn}^+ \) denotes the set of \( m \times n \) real nonnegative matrices, \( \mathcal{M}_{mn}^- \) denotes the set of \( m \times n \) real nonpositive matrices, \( O \) and \( I \) denote zero matrix and identity matrix of suitable size, respectively.

## 2 Materials and Problem Formulation

Given the continuous linear time invariant (LTI) singular system:

\[
E \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0
\]

where \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \) denote state and control vector, respectively, \( E, A \in \mathcal{M}_{nn} \) with \( \text{rank}(E) < n \), \( B \in \mathcal{M}_{nm} \), and \( t \in \mathbb{R}_+ \). The system (1) is called regular if there exists \( \lambda \in \mathbb{C} \) such that \( \det(\lambda E + A) \neq 0 \) [3]. It is called nonregular otherwise. The index of the system (1), denote by \( \text{ind}(E, A) \), is a smallest nonnegative integer \( k \) such that \( \text{rank}(E^{k+1}) = \text{rank}(E^k) \). Under the assumption regularity and \( \text{rank}(E) < n \), Campbel derived in [1] a result on the solving the system (1) using the Drazin invers. The solution of system (1) with index \( k \) is given in the following theorem.

**Theorem 1** [1] Let \( E, A \in \mathcal{M}_{nn} \) such that \( (E, A) \) be a regular matrix pair with \( \text{ind}(E, A) = k \). Let \( \hat{E} = (\lambda E - A)^{-1}E \), \( \hat{A} = (\lambda E - A)^{-1}A \) and \( \hat{B} = (\lambda E - A)^{-1}B \) for some scalar \( \hat{\lambda} \). Then the solution of (1) has the form

\[
x(t) = e^{\hat{E} \hat{A} t} \hat{E} \hat{D} \hat{v} + \int_0^t e^{\hat{E} \hat{A} (t - \tau)} \hat{E} \hat{D} \hat{B} u(\tau) d\tau - (I - \hat{E} \hat{D} \hat{E}) \sum_{i=0}^{k-1} (\hat{E} \hat{A}^D)^i \hat{A}^D \hat{B} u(t)
\]
for some \( \mathbf{v} \in \mathbb{R}^n \), where superscript \( D \) denotes the Drazin inverse of corresponding square matrix.

**Definition 1 [7]** The LTI singular system (1) with \( \text{ind}(E, A) = k \) is called positive if for all \( t \in \mathbb{R}_+ \) we have \( \mathbf{x}(t) \in \mathbb{R}_+^n \) for any control \( \mathbf{u}(t) \in \mathbb{R}_+^m \) such that \( \mathbf{u}(\tau) = \frac{d\mathbf{u}(t)}{dt} \in \mathbb{R}_+^m \), \( j = 1, 2, \ldots, k - 1 \) and \( 0 \leq \tau \leq t \) and for any consistent initial state \( \mathbf{x}_0 \in \mathbb{R}_+^n \).

Virnik in [7] construct the criterion of positivity of the singular system in term of \( Z \)-matrix. The square matrix \( A \) is said \( Z \)-matrix if its off-diagonal entries are nonnegative. The following theorem informs the criterion for positivity of the system (1) for the case of \( \text{ind}(E, A) = 1 \).

**Theorem 2 [7]** Let \( E, A \in \mathcal{M}_{nn} \) such that \((E, A)\) is a regular matrix pair with \( \text{ind}(E, A) = 1 \). Let \( \hat{E} = (\hat{\lambda}E - A)^{-1} \), \( \hat{A} = (\hat{\lambda}E - A)^{-1}A \) and \( \hat{B} = (\hat{\lambda}E - A)^{-1}B \) for some scalar \( \hat{\lambda} \). If \( \begin{pmatrix} I - \hat{E} \hat{D} \hat{E} \\ \hat{A} \hat{D} \hat{B} \end{pmatrix} \in \mathcal{M}_{-nm}^+ \) and \( \hat{E}^D \hat{E} \in \mathcal{M}_{nn}^+ \) then the system (1) is positive if and only if the following two conditions hold:

1. There exists a scalar \( \alpha \geq 0 \) such that the matrix \( \bar{M} = -\alpha I + (\hat{E}^D \hat{A} + \alpha \hat{E} \hat{D} \hat{E}) \) is a \( Z \)-matrix,

2. \( \hat{E}^D \hat{B} \in \mathcal{M}_{nm}^+ \).

**Definition 2 [2]** The regular singular system (1) is said to be stable if there exists scalars \( \alpha, \beta > 0 \) such that its state \( \mathbf{x}(t) \) satisfies

\[
\| \mathbf{x}(t) \|_2 \leq \alpha e^{-\beta t} \| \mathbf{x}_0 \|_2, \quad t \geq 0. \tag{2}
\]

It is clear from this definition that, when (1) is stable then \( \lim_{t \to \infty} \mathbf{x}(t) = \mathbf{0} \) for any consistent initial state \( \mathbf{x}_0 \in \mathbb{R}^n \).

**Theorem 3 [2]** The regular descriptor linear system (1) is stable if and only if

\[
\sigma(E, A) \subset \mathbb{C}^- = \{ \lambda \in \mathbb{C} : \text{Re}(\lambda) < 0 \}, \tag{3}
\]

where

\[
\sigma(E, A) = \{ \lambda \in \mathbb{C} : \det(\lambda E - A) = 0 \}. \tag{4}
\]

Furthermore, assume that the system (1) is regular, positive and \( \text{ind}(E, A) = 1 \). Under the state feedback controller

\[
\mathbf{u}(t) = K \mathbf{x}(t) + \mathbf{w}(t) \tag{5}
\]
for some $K \in \mathcal{M}_{mn}$ and $w(t) \in \mathbb{R}^m_+$, the closed loop system (1) can be transformed into
\[
E \dot{x}(t) = (A + BK)x(t) + Bw(t).
\] (6)

Note that, under the transformation (5) the closed loop system (6) does not need regular and positive. The problem of stabilization is to seek a controller feedback (5) such that the closed loop system (6) is regular, stable and positive.

In this paper, we will not solve this stabilization problem, but we are interested in finding the conditions of the matrix $K \in \mathcal{M}_{mn}$ such that the closed loop system (6) is regular, stable and positive.

### 3 Main Result

The following we provide the main result of this paper.

**Theorem 4** Given the positive regular continuous linear singular system (1), where $EA = AE$, $\text{ind}(E, A) = 1$, $(I - E^D E) A^D B \in \mathcal{M}_{nn}^-$ and $E^D E \in \mathcal{M}_{nn}^+$. If there exists a matrix $K \in \mathcal{M}_{mn}$ and a vector $w(t) \in \mathcal{M}_{mn}^+$ that satisfy

1. $E(BK) = (BK)E$,
2. there exists some scalar $\beta \in \mathbb{R}$ that satisfies $A + BK = I - \beta E$, where $\beta > \text{Re}(\gamma)$ for some $\gamma$ that satisfies $\frac{1}{\gamma} \in \sigma(I, E) \setminus \{0\}$,
3. $E^D BK$ is a Z-matrix,
4. $(I - E^D E) (A + BK)^D B \in \mathcal{M}_{nn}^-$,

such that the feedback (5) transforms (1) into (6), then the the closed-loop system (6) is regular, stabil and positive.

**Proof.** Let the hypothesis of the theorem hold. First, we will show that the system (6) is regular. Let $\alpha \in \mathbb{C}$, $\alpha \notin \sigma(I, E) \setminus \{0\}$, then
\[
\det(\alpha I - E) = \det(\alpha(I - \frac{1}{\alpha}E)) = \alpha^n \det(I - \frac{1}{\alpha}E) \neq 0. \tag{7}
\]

Setting $\frac{1}{\alpha} = \beta - \lambda'$ for some $\lambda' \in \mathbb{C}$, then using the hypothesis (2) we find
\[
\det(I - (\beta - \lambda') E) = \det(I - \beta E + \lambda' E) = \det(\lambda'E + (A + BK)) \neq 0. \tag{8}
\]

This fact shows that the matrix pair $(E, A + BK)$ is regular. Next, we show that the feedback control (5) is stabilizing the system (1). Let $\mu \in \mathbb{C}$ be a finite eigenvalue of matrix pair $(E, A + BK)$, then
\[
\det(\mu E - (A + BK)) = \det(\mu E - (I - \beta E)) = \det((\mu + \beta) E - I) = 0. \tag{9}
\]
It is clear that $\mu + \beta \neq 0$ due to if $\mu + \beta = 0$ then (9) does not hold. Setting $\tilde{\gamma} = \mu + \beta$, then (9) can be written as follows:

$$\det((\mu + \beta) E - I) = \det(\tilde{\gamma} E - I) = \det(-\tilde{\gamma}(-E + \frac{1}{\tilde{\gamma}} I)) = (-\tilde{\gamma})^n \det(\frac{1}{\tilde{\gamma}} I - E) = \det(\frac{1}{\tilde{\gamma}} I - E) = 0.$$ (10)

This shows that $\frac{1}{\tilde{\gamma}} \in \sigma(I,E) \setminus \{0\}$. Moreover, $\Re(\tilde{\gamma}) < \beta$ implies $\mu < 0$, i.e.

$$\sigma(E,A + BK) \subset \mathbb{C}_-.$$ (11)

This fact shows that the closed-loop system (6) is stable. Last, we have to show that the system (6) is positive. Since $EA = AE$ and using the hypothesis (1) we have $E(A + BK) = (A + BK)E$. It implies that the solution of the system (6) with $\ind(E,A) = 1$ is given by

$$x(t) = e^{E^D(A+BK)t}E^DEx_0 + \int_0^t e^{E^D(A+BK)(t-\tau)}E^DBw(\tau)d\tau$$ (12)

for any consistent initial state $x_0 \in \mathbb{R}^n_+$. We now show that the three summands in (12) are nonnegative. Since the system (1) is positive, Theorem 2 implies that there exists a scalar $\alpha \geq 0$ such that the matrix

$$M := -\alpha I + (E^D A + \alpha E^D E)$$ (13)

is a $Z$-matrix and $E^D B \in M_{nm}^+$. Summing the both side of (13) by $E^D BK$, we obtain

$$M^\ast = -\alpha I + E^D A + \alpha E^D E + E^D BK,$$ (14)

where $M^\ast := M + E^D BK$. It is obvious that $M^\ast$ is a $Z$-matrix due to that the both $M$ and $E^D BK$ are the $Z$-matrix. Furthermore, since $E^D E$ is a projector on $\text{im}(E^D E)$, we have $E^D E v = v, \forall v \in \mathbb{R}^n_+$. Postmultiplying (14) by $v$, we get

$$M^\ast v = E^D(A + BK)v,$$

that gives $M^\ast = E^D(A + BK)$. Since $M^\ast$ is a $Z$-matrix, then $e^{E^D(A+BK)} \in M_{nm}^+$. Therefore, the nonnegativity of the first term of (12), i.e.

$$e^{E^D(A+BK)t}E^DEx_0 \in \mathbb{R}^n_+,$$ (15)

is deduced from $E^DEx_0 \in \mathbb{R}^n_+$ for any $x_0 \in \mathbb{R}^n_+$. For the second term, since $E^D B \in M_{nm}^+$ and $M^\ast$ is a $Z$-matrix, we have $e^{E^D(A+BK)(t-\tau)}E^DBw(\tau) \in \mathbb{R}^n_+$ for all $0 \leq \tau \leq t$. Since integration is monoton, the second term of (12) is nonnegative. Ultimately, since $(I - E^D E)(A + BK)^D B \in M_{nm}^+$ we have

$$-(I - E^D E)(A + BK)^D Bw(t) \in \mathbb{R}^n_+.$$ (16)

Thus $x(t) \in \mathbb{R}^n_+$. ■
4 Conclusion

A sufficient condition for characterization of the feedback for positive LTI continuous singular systems of index 1 has been established. This condition can be used to identify the feedback matrix such that the closed loop system is regular, stable and positive.

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References


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