Neutral Thin Shell Stability in Reissner-Nordstrom de-Sitter Space-Time

A. Eid

Department of Physics
Collage of Science, Al Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh, KSA Saudi Arabia
and
Department of Astronomy
Faculty of Science, Cairo University, Giza, Egypt

Copyright © 2014 A. Eid. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

Starting from Israel equations for spherically symmetric thin shell, the neutral thin shell immersed into different types of Reissner-Nordstrom de-Sitter space-time is constructed. The static and stable configuration of the neutral shell, using only the gravitational field of the charged source as a stabilizing mechanism, is deduced. In particular, two types of shells are studied: a string gas shell and a dust shell with cosmological constant. The dynamical possibilities are also analyzed, including the possibility of child universe creation.

Keywords: general relativity; thick shell; cosmology; gravitation

1 Introduction

The formalism of the model of thin gravitating shells, that was first proposed by Israel [1] occupies an important place among the exactly solvable problems in general relativity, was subsequently developed in details and used for a wide class of cosmological and astrophysical problems. One can mention more recent applications of the thin-shell formalism, namely, gravitational collapse of radiating shells, the evolution of bubbles and domain walls in cosmological settings and in in-
flationary models, the structure and dynamics of voids in the large-scale structure of the universe, shells around black holes and their respective stability, signature changes, matching of cosmological solutions and applications to the Randall–Sundrum braneworld scenario to wormhole physics.

Kuchar [2] was among the first authors who considered the problem on the dynamical evolution of a shell in the Reissner metric. In particular, he showed that the electric charge of the black hole could prevent the shell collapse, i.e., a bounce point could exist. Novikov [3] showed that the collapse of a charged sphere could stop and subsequently expand into another universe. The global geometry of a Reissner-Nordstrom (R-N) manifold depends on the relation between the electrical charge and the total mass (energy) of the system (the latter includes both the energy of Coulomb field and the binding gravitational energy).

In particular, when the phase transitions in the early universe are analyzed [4], the model of thin shells is a very convenient formalism that allows the dynamics of the phase transitions themselves and the formation and evolution of baby universes to be traced in sufficient detail [5, 6]. Spherical solutions of domain walls are usually unstable towards collapse. However, prior studies have shown that some modifications of the effective surface tension (e.g., radially dependent surface tension) [7], as well as considering charged bubbles [8], can yield a static and stable shell solution.

In this paper, I wish to explore a different way of achieving stabilization of a two-dimensional extended object using the gravitational effects of a massive charged source on an uncharged spherical shell. Hence, the stabilization in the case presented here is induced purely from gravitational effects. This paper is organized as follows: In Section 2 the Darmois–Israel thin shell formalism is briefly reviewed. In section III match an interior RN de-Sitter solution to an exterior RN de-Sitter solution and determines the dynamics of a bubble that carries an arbitrary matter content on its surface, then analyze this equations for two analytical cases: a bubble which carries dust on its surface (i.e. a dust shell with cosmological constant) and a bubble that carries a string gas like content. Finally, some concluding remarks are made in Section 5. Also adopt the units such that $c = G = 1$.

2 The Darmois – Israel Formalism

Consider two distinct spacetime manifolds $M_+$ and $M_-$ with metrics given by $g_{\mu\nu}(x^\mu_\pm)$ and $\mathcal{S}^y \vec{K}_{\sigma j} = \left[ -T_{\rho\sigma} n^{\rho} n^{\sigma} - \frac{\Lambda}{8\pi} \right]$, in terms of independently defined coordinate systems $x^\mu_\pm$. The manifolds are bounded by hypersurfaces $\Sigma_\pm$.
Neutral thin shell stability

and $\Sigma_+$, respectively, with induced metrics $g_{ij}^\pm$. The hypersurfaces are isometric, i.e. $g_{ij}^+(\xi) = g_{ij}^-(\xi) = g_{ij}(\xi)$, in terms of the intrinsic coordinates, invariant under the isometry. A single manifold $M$ is obtained by gluing together $M_+$ and $M_-$ at their boundaries, i.e. $M = M_+ \cup M_-$, with the natural identification of the boundaries $\Sigma = \Sigma_+ = \Sigma_-$. The second fundamental forms (extrinsic curvature) associated with the two sides of the shell are:

$$K_{ij}^\pm = -n^\pm \left( \frac{\partial^2 x^\gamma}{\partial \xi^i \partial \xi^j} + \Gamma^\gamma_{\alpha \beta} \frac{\partial x^\alpha}{\partial \xi^i} \frac{\partial x^\beta}{\partial \xi^j} \right)_{\Sigma}$$

(2.1)

where $n^\pm$ are the unit normal 4-vector to $\Sigma$ in $M$, with $n^\mu n_\mu = 1$ and $n^\mu e_\mu^\nu = 0$. The Israel formalism requires that the normal point from $M_-$ to $M_+$. For the case of a thin shell $K_{ij}$ is not continuous across $\Sigma$, so that, the discontinuity in the second fundamental form is defined as $[K_{ij}] = K_{ij}^+ - K_{ij}^-$. The Einstein equation determines the relations between the extrinsic curvature and the three dimensional intrinsic energy momentum tensor are given by The Lanczos equations,

$$S_{ij} = -\frac{1}{8\pi} \left( [K_{ij}] - [K] g_{ij} \right)$$

(2.2)

where $[K]$ is the trace of $[K_{ij}]$ and $S_{ij}$ is the surface stress-energy tensor on $\Sigma$. The first contracted Gauss-Kodazzi equation or the “Hamiltonian” constraint

$$G_{ij} n^\mu n^\nu = \frac{1}{2} (K^2 - K_{ij} K^{ij} - 3R) ,$$

(2.3)

with the Einstein equations provide the evolution identity

$$S_{ij} \bar{K}_{ij} = -T_{ij} n^\mu n^\nu - \frac{\Lambda}{8\pi}$$

The convention, $\{X\} = X^+ - X^-$, and $X = \frac{1}{2}(X^+ + X^-)$, is used. The second contracted Gauss-Kodazzi equation or the “ADM” constraint,

$$G_{ij} e^\mu_{\nu} n^\nu = K^i_{j,i} - K_{j}$$

(2.4)

With the Lanczos equations gives the conservation identity
\[ S^i_{\cdot j} = [T_{\mu\nu} e^\mu_i n^\nu_j]^+ . \] (2.5)

The surface stress-energy tensor may be written in terms of the surface energy density \( \sigma \), and surface pressure \( p \): \( S^i_{\cdot j} = \text{diag} \cdot (-\sigma, p, p) \). For spherically symmetric thin shell, the Lanczos equations reduce to

\[
\sigma = \frac{-1}{4\pi} [K^\rho_{\cdot \rho} ]
\] (2.6)

\[
p = \frac{1}{8\pi} \left( [K^r_{\cdot r}] + [K^\theta_{\cdot \theta}] \right).
\] (2.7)

If the surface stress-energy terms are zero, the junction is denoted as a boundary surface. If surface stress terms are present, the junction is called a thin shell.

### 3 Dynamical Analysis

The matching of two Rissner Nordstrom de-Sitter space-times of \( M^\pm \), given by the following line elements:

\[
d s^2 = -f_\pm dt^2 + f_\pm dr^2 + r^2 d\theta^2 + s d\phi^2 \] (3.1)

with

\[
f_\pm = 1 - \frac{2m_\pm}{r} + \frac{q_\pm^2}{r^2} - \frac{1}{3} \Lambda_\pm r^2
\] (3.2)

where \( m_\pm \) is the mass of the electric source, \( m_\pm \) is the mass-energy of the system as viewed by an external observer, \( q_\pm \) and \( \Lambda_\pm \) are the charge and the cosmological constant outside and inside the shell. The suffix ‘+’ denotes a quantity evaluated just outside the shell and ‘-’ just inside the shell. Let the equation of the shell be \( r_\pm = R_\pm(\tau) \), the history of the shell is described by the hypersurface \( x^\mu = x^\mu(\tau, \theta, \phi) \), in the regions \( M^\pm \), respectively; the function \( R(\tau) \) describes the time evolution of the shell.

From (2.6) the bubble equation of motion is

\[
\varepsilon_+ \sqrt{f_+ + R^2} - \varepsilon_- \sqrt{f_- + R^2} = \frac{M(R)}{R} = \kappa(R)
\] (3.3)

where \( M(R) = 4\pi R^2 \sigma(R) \) describes the energy-matter content which is located on
the surface of the bubble and the coefficients \( \varepsilon_{z} = \text{sgn}(n^{\mu} \partial_{\nu} r)|_{m} \) determine if the radial coordinate \( r \) is decreasing \( (\varepsilon_{z} = -1) \) or increasing \( (\varepsilon_{z} = 1) \) along the normal coordinate \( n^{\mu} \). Taking both coefficients, \( \varepsilon_{z} \), be positive. As the shell does not carry any charge, the charge parameter that appears in both metrics should be equal.

Rearranging equation (3.3) into the form:

\[
\dot{R}^2 = \frac{M^2}{4R^2} - \frac{1}{2} (f_+ + f_-) + \frac{R^2}{4M^2} (f_- - f_+)^2 \tag{3.4}
\]

It represents the energy equation of the shell and can be written in the form:

\[
\dot{R}^2 + V(R) = 0
\]

where,

\[
V(R) = -1 + \frac{M^2}{4R^2} + \left( \frac{m_+ - m_-}{M} \right)^2 + \left( \frac{m_+ + m_-}{M} \right) - \frac{q^2}{R^2} + \frac{1}{3} \Lambda R^2 \tag{3.5}
\]

is called the effective potential.

Taking into account the transparency condition, \([G_{\mu \nu} U^\mu n^\nu] = 0\), the conservation identity, equation (2.5), provides the simple relationship:

\[
\frac{d}{d \tau} \sigma A + P \frac{d}{d \tau} A = 0
\]

where \( A = 4\pi R^2 \) is the area of the spheres of symmetry at constant \( R \). In general case, the conservation identity provides the following relationship:

\[
\dot{\sigma} = -\frac{2\dot{R}}{R}(\sigma + P) \tag{3.6}
\]

Let the equation of state for the matter on the bubble surface is \( P = \omega \sigma \), then the conservation energy (3.6) will be

\[
\sigma = \sigma_{a} R^{2(-1+\kappa_0)} \tag{3.7}
\]

where \( \sigma_{a} \) is a constant. Substitute equation (3.7) into (3.3) to get

\[
\kappa(R) = \kappa_{a} R^{-1+\frac{a}{2}} \tag{3.8}
\]

where \( \kappa_{a} = 4\pi \sigma_{a} \). The effective equation, in terms of (3.8), will be
\[ R^2 + 1 - \frac{1}{4} \kappa_s^2 R^{-2(1+2\omega)} - \left( \frac{m_- - m_+}{\kappa_s} \right)^2 R^{4\omega} - \left( \frac{m_- + m_+}{R} \right) + \frac{q^2}{R^2} - \frac{1}{3} \Lambda R^2 = 0 \]  

(3.9)

with the effective potential,

\[ V_{\text{eff}}(R) = 1 - \frac{1}{4} \kappa_s^2 R^{-2(1+2\omega)} - \left( \frac{m_- - m_+}{\kappa_s} \right)^2 R^{4\omega} - \left( \frac{m_- + m_+}{R} \right) + \frac{q^2}{R^2} - \frac{1}{3} \Lambda R^2 \]  

(3.10)

Let us now concentrate on two specific cases that have a clear physical meaning:

(i) \( \omega = -\frac{1}{2} \) and (ii) \( \omega = 0 \). For these two values of the equation of state parameter the effective potential is tractable and can be solved in closed form. It is known that a gas of strings in \( n \) spatial dimensions satisfies the equation of state \( p = -\frac{n}{n-1} \) [9].

Therefore, the first case describes a gas of strings which is located on the surface of the bubble, while the second choice corresponds to dust with a cosmological constant living on the wall.

i) For \( \omega = -\frac{1}{2} \): stringy gas bubble

The effective potential for this kind of bubble is:

\[ V_{\text{eff}}^{\omega = -\frac{1}{2}}(R) = 1 - \frac{1}{4} \kappa_s^2 R^{-2(1+2\omega)} - \left( \frac{m_- - m_+}{\kappa_s} \right)^2 R^{4\omega} - \left( \frac{m_- + m_+}{R} \right) + \frac{q^2}{R^2} - \frac{1}{3} \Lambda R^2 \]  

(3.11)

Solve equation (3.11), \( V_{\text{eff}}^{\omega = -\frac{1}{2}}(R) = 0 \), as a Polynomial equation of rank 2:

\[ R_\pm = \frac{1}{2(1 - \frac{1}{4} \kappa_s^2 - \frac{1}{2} \lambda)} \left( m_- + m_ \pm \sqrt{(m_- + m_+)^2 - 4(1 - \frac{1}{4} \kappa_s^2 - \frac{1}{2} \lambda) \left( q^2 - \frac{(m_+ - m_-)^2}{\kappa_s^2} \right)} \right) \]

where \( \lambda = \Lambda R^2 \). The discriminant of the effective potential function is given by

\[ \Delta = (m_- + m_+)^2 - 4(1 - \frac{1}{4} \kappa_s^2 - \frac{1}{2} \lambda) \left( q^2 - \frac{(m_+ - m_-)^2}{\kappa_s^2} \right) \]  

(3.12)

Let \( \Delta = 0 \) to give a constraint condition on the charge:

\[ q^2 = \frac{(m_- + m_+)^2}{4(1 - \frac{1}{4} \kappa_s^2 - \frac{1}{2} \lambda)} + \frac{(m_+ - m_-)^2}{\kappa_s^2} \]  

(3.13)

The radius of curvature of the stable bubble is thus given by

\[ R_{\text{min}} = \frac{2(m_- + m_+)}{(4 - \kappa_s^2 - \frac{1}{2} \lambda)} \]  

(3.14)

with \( R_{\text{min}} \) being positive under the imposed conditions. Now, if \( R_{\text{min}} \) is located behind any horizons; in principle, there might be two different horizons in each
Neutral thin shell stability

region. But, if $q^2 \succ m_+^2$, there is no horizons in the system (when $m_+ \succ m_-$, and surface tension not negative). The comparison between $q^2$, in equation(3.13), and $m_+^2$ is equivalent to the comparison between the function $f_1(x)$ and the quantity $g_1$, where:

$$f_1(x) = (x^2 - 2x + 1)(1 - \frac{1}{3}) + x \kappa_0^2$$

$$g_1(x) = \frac{1}{4} \kappa_0^2 (4 - \kappa_0^2 - \frac{4}{3} \lambda)$$

with $x = m_- / m_+$ being the ratio between the masses (which is constrained $0 < x < 1$). So the condition $q^2 \ll m_+^2$ is now $f_1(x) < g_1$. The minimum value of $f_1(x)$ is located at $x_{\text{min}} = \frac{1}{2(1 - \lambda/3)} [2(1 - \lambda/3) - \kappa_0^2]$, when $\kappa_0^2 < 2(1 - \lambda/3)$, and at $x = 0$ when $\kappa_0^2 \succ 2(1 - \lambda/3)$.

The minimum value of $f_1(x)$ satisfies $f_1(x_{\text{min}}) = g_1$. This means that $f_1(x)$ will always be greater than $g_1$ (i.e. $q^2 \ll m_+^2$) except for the case when $x_{\text{min}} = m_- / m_+$ and the surface tension is small enough (i.e. $\kappa_0^2 < 2(1 - \lambda/3)$). This limiting case means that there would be two degenerate horizons (i.e. an extremal black hole like object). When $x \neq x_{\text{min}}$, the naked singularity at $r = 0$ is exist, and this corresponds to $q^2 \ll m_+^2$. When $x = x_{\text{min}}$ (i.e. $m_- = \frac{1}{2(1 - \lambda/3)} [2(1 - \lambda/3) - \kappa_0^2] m_+$), the bubble would sit exactly on the degenerate horizon (i.e. $R_{\text{min}} = m_+$) and therefore it would be a light-like brane.

This result is actually an example of the more general relation between the effective potential and the metric coefficients which states that $f_\pm (R) - V_{\text{eff}} (R) \geq 0$. The geometries $M_\pm$ can not contain any horizons, when the effective potential is definite positive [10].

For the string gas bubble, the trajectory constants signs read $\text{sgn}(\epsilon_+ \epsilon_-) = \text{sgn}(2(m_+ - m_-) R^{-1} - \kappa_0^2)$, $\text{sgn}(\epsilon_+) = 1$.

When $\epsilon_- < 0$, the normal coordinate to the brane is pointing along a direction for which the radial coordinate is actually decreasing rather than, as in the more familiar possibility of non-wormhole geometry, increasing. Therefore, there is a possibility for the bubble to expand to infinity disconnected from the original spacetime (i.e. a child universe solution [9]).

ii) For $\omega = 0$: dust shell with cosmological constant.

The effective potential for this case is
\[ V_{\text{eff}}^{\alpha=0} = 1 - \frac{1}{4} R^{-2} \kappa_s^2 - \left( \frac{m_+ - m_-}{\kappa_s} \right)^2 - \left( \frac{m_- + m_+}{R} \right) + q^2 \frac{R^2}{2} - \frac{1}{3} \Lambda R^2 \]  

(3.17)

Solve equation (3.17), \( V_{\text{eff}}^{\alpha=0} (R_{\text{min}}) = 0 \), as a Polynomial equation of rank 2:

\[ R_\pm = \frac{1}{2[1-(\frac{m_+ - m_-}{\kappa_s})^2 - \frac{1}{4} \lambda]} \{ m_- + m_+ \pm \sqrt{(m_- + m_+)^2 - 4[1-(\frac{m_+ - m_-}{\kappa_s})^2 - \frac{1}{4} \lambda](q^2 - \frac{\kappa_s^2}{4})} \} \]

where \( \lambda = \Lambda R^2 \). The discriminant of the effective potential function is given by

\[ \Delta = 4(\frac{1}{4} \kappa_s^2 - q^2) \left( 1 - \frac{1}{4} \lambda - \frac{(m_+ - m_-)^2}{\kappa_s^2} \right) + (m_- + m_+)^2 \]

(3.18)

Let \( \Delta = 0 \) to give a constraint condition on the charge:

\[ q^2 = \frac{\kappa_s^2}{4} \frac{4m_- m_+ + \kappa_s^2 - \frac{1}{4} \lambda \kappa_s^2}{\kappa_s^2 - \frac{1}{4} \lambda \kappa_s^2 - (m_+ - m_-)^2} \]

(3.19)

The radius of curvature for the stable bubble is now given by

\[ R_{\text{min}} = \frac{(m_- + m_+)\kappa_s^2}{2[\kappa_s^2 - \frac{1}{4} \lambda \kappa_s^2 - (m_+ - m_-)^2]} \]

(3.20)

which is, again, positive under the imposed conditions. The comparison between \( q^2 \), in equation(3.19), and \( m_+^2 \) is equivalent to the comparison between the function \( f_2(x) \) and the quantity \( g_2 \), where:

\[ f_2(x) = m_+^4 (1-x)^2 + \kappa_s^2 m_+^2 (x + \frac{1}{4} \lambda) \]

(3.21)

\[ g_2(x) = m_+^2 \kappa_s^2 - \frac{1}{4} \kappa_s^4 (1 - \frac{1}{4} \lambda) \]

(3.22)

The minimum value of \( f_2(x) \) is located at \( x_{\text{min}} = 1 - \frac{\kappa_s^2}{2m_+^2} \), when \( \kappa_s^2 < 2m_+^2 \), and at \( x = 0 \) when \( \kappa_s^2 > 2m_+^2 \). The minimum value of \( f_2(x) \) satisfies \( f_2(x_{\text{min}}) = g_2 \). Therefore, there will be no horizons unless \( \kappa_s^2 < 2m_+^2 \) and \( x = x_{\text{min}} \), where the latter means that the two horizons are degenerate and located at \( R = m_+ \), exactly where the bubble would sit. Again, the effective potential is definite positive [10].
For the dust shell case, the trajectory constants signs read
\[ \text{sgn}(\varepsilon) = \text{sgn}(2(m_+ - m_-)R^{-1} - R^{-2} \kappa^3), \quad \text{sgn}(\varepsilon) = 1. \]

In the case \( R \to \infty, \, \varepsilon, \geq 0 \) is positive, which indicating that the normal to the brane does not point to a direction in which the radial coordinate decreases. Hence, there is no child universe formation [9].

5 Conclusions

General solution by matching an interior RN de-Sitter space-time to a RN de-Sitter space-time exterior solution at a junction surface was constructed. The electric charge causes a gravitational repulsive effect which balances the natural tendency of two dimensional extended objects to collapse and thus yields a static and stable shell configuration, even though the latter carries zero charge and does not interact directly with electric fields. When \( q_+ = 0 \) and \( \Lambda = 0 \), the results coincide with those presented in [11]. When \( f^- = 1 \), the space time is flat (Minkowski region, as \( R \to \infty \); independent of the mass), then, there is the possibility of 'ionization' which could be responsible for a dynamical creation of a universe, this solution approach those studied in [13, 15], which represent child universe creation. In contrast, for the case of a dust shell, the ionization does not produce a child universe, but instead it is simply an 'expansion' where the dust shell achieves the critical (escape velocity) necessary to expand to infinity in the existing space, i.e. not creating a new space of its own(a child universe).

References

http://dx.doi.org/10.1007/bf02710419

http://dx.doi.org/10.1007/bf01698208


http://dx.doi.org/10.1016/0370-2693(72)90109-8

http://dx.doi.org/10.1103/physrevd.43.1129

arXiv:gr-qc/9901066. http://dx.doi.org/10.1088/0264-9381/16/10/320

http://dx.doi.org/10.1007/bf02634175


arXiv: gr-qc 0706.1233.

Received: October 10, 2014; Published: November 20, 2014