Quantum Mechanics of Kepler Orbits

Alexander Rauh and Jürgen Parisi

Department of Physics, University of Oldenburg, D-26111 Oldenburg, Germany

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Abstract

It is shown that the Kepler orbits both elliptic, hyperbolic, and parabolic ones, are solutions of the Schrödinger equation of the hydrogen atom, and can be inferred from a single wave function in the classical limit. The latter corresponds to large quantum numbers of the angular momentum. The mean initial values of position and velocity are implemented in the state. The orbits evolve by a curve parameter which in the elliptic case corresponds to the eccentric anomaly. The mean values, including mean square deviations, are first calculated for the hyperbolic case and, then, continued into the elliptic region by extending the curve parameter to purely imaginary values. Time dependence is introduced by the assumption that it enters through the curve parameter only. However, as it is shown, this assumption holds in the classical limit only and is violated to next leading order. The wave function is derived by a proper projection of a four-dimensional harmonic oscillator state onto the ordinary three-dimensional space, it appears in a compact form without the necessity of an additional constraint. The initial wave function is connected with a nearly minimum uncertainty product.

Subject Classification: 03.65.Sq; 02.70.Wz; 31.15.-p

Keywords: Kustaanheimo-Stiefel transformation; Classical limit; quantum fluctuations; celestial mechanics

1 Introduction

We propose a quantum state $\psi$ in three-dimensional configuration space and calculate the mean values of position and velocity with the result that, in
the classical limit, the Kepler orbits emerge: elliptic, hyperbolic, or parabolic ones, depending on the initial parameters implemented in $\psi$. With regard to the Hamiltonian of the hydrogen atom, most previous work was focused on the subspace with discrete energy spectrum which, in the classical limit, leads to elliptic orbits. In the elliptic domain, the wave function has been constructed basically by means of three methods and their combination: (i) The principle of minimum uncertainty coherent states [17, 16, 18, 9, 6]; (ii) The use of the Kustaanheimo-Stiefel (KS) transformation [13, 7, 5, 23, 12, 4, 10, 2]; (iii) The application of Lie groups [19, 15, 3, 6, 2]. The quantum mechanics in the hyperbolic region was elaborated in [20]. In the present work, we do not rely on spectral representations. In the classical limit, we derive from a single wave function the Kepler orbits except for rectilinear ones. Our method is close to using constrained four-dimensional harmonic oscillator states (COS). However, we do not rely on a constraint.

The state $\psi$ depends analytically on the configuration vector $\mathbf{r} \in \mathbb{R}^3$ and five parameters as follows:

$$\psi = \psi_w(\mathbf{r}; r_0, v_0; \kappa, \nu),$$

where $r_0$ denotes the mean initial distance of an apsis point $P_0$ from the force center, and $v_0$ the length of the mean initial velocity at $P_0$, which is normal to the apses line; $w$ is a real curve parameter for hyperbolic orbits; in the elliptic case, $w$ is continued to purely imaginary values; $\kappa$ is an order of magnitude parameter, essentially the angular momentum in units of $\hbar$, with $\kappa \to \infty$ providing us with the classical limit, and $\nu > 0$ is an arbitrary constant. With increasing distance from the initial point, where $w = 0$, $\kappa$ has to be replaced by the dynamical parameter $K = \kappa h(w)$, where in the hyperbolic case $h(w)$ monotonically decreases with increasing $w$ from the value $h(0) = 1$. The initial data $(r_0, v_0)$ are mean values to leading order in $\kappa$.

Keeping the initial distance $r_0$ fixed, let us discuss the dependence on $v_0$ in the classical limit. If $v_0$ is large, then the mean energy $E$ is positive and corresponds to a hyperbolic orbit with eccentricity $e > 1$; the initial point $P_0$ is the peri-center, and one has the relation $r_0 = (e - 1)a$ with $a$ denoting the semi-major axis. When $v_0$ decreases, the critical energy $E = 0$ is reached which corresponds to a parabolic orbit where $e = 1$. Decreasing $v_0$ further, one comes into the elliptic region with $E < 0$ and $0 \leq e < 1$; $P_0$ is still the peri-center with $r_0 = (1 - e)a$, until the circular orbit is reached at the speed $v_0 = v_c$ where $e = 0$. For speeds below $v_c$, the initial point $P_0$ becomes the apo-center with the connection $r_0 = (1 + e)a$, and the eccentricity increases from the value $e = 0$ to the second singular case $e = 1$, where $v_0 = 0$ which corresponds to a rectilinear orbit with angular momentum $L = 0$. As compared to [20], the present theory also applies to parabolic orbits, but it does not cover the case when $v_0$ is close to zero.
The magnitude parameter $\kappa$, at first, characterizes the width of $\psi_0$ in configuration space. With increasing $\kappa$, the probability density $|\psi_0|^2$ becomes more and more peaked with the consequence that the deviations of mean values $\bar{r}$ from the Kepler orbit are of order $1/\kappa$. The mean values of the velocity turn out being proportional to $\kappa$ in the classical limit. At time $t = 0$, or for $w = 0$, one can relate $\kappa$ to the initial parameters $r_0, v_0$; in Sec VI, it is found that $\kappa$, essentially, equals the angular momentum $L = mr_0v_0$ in units of $\hbar$, which implies that the parameter $\kappa$ and, thus, the half width of $|\psi_0|^2$ is fixed by the orbital angular momentum.

Through the explicit $w$ dependence, the mean values with respect to the state $\psi_w$ produce the full orbits in the classical limit, but still without time behavior. In the elliptic case, the parameter $2w$ turns out to be the eccentric anomaly. In the COS formalism, $w$ emerged, up to a dimensional factor, as the time parameter of the harmonic oscillators, and was termed pseudo or fictitious time, since it differs from the time of the Kepler problem, see e.g. [4], [10], [12], [3], [7], [20]. It was observed some time ago in [11] that pseudo time should be related to the eccentric anomaly rather than to a physical time variable. Here, we consider $w$ simply as a curve parameter. As it turns out, if one assumes that time dependence enters exclusively through $w = w(t)$, then this can be proved in the classical limit; in general, however, a one to one correspondence between $w$ and $t$ cannot be maintained.

The wave function $\psi_w$ determines both the mean orbit $\bar{r}(w)$ and the mean velocity $\bar{v}(w)$. In Sec. VII, we will infer the time dependence of $w$ from the vector relation $d/(dt)\bar{r}(w) = \bar{v}(w)$. As it turns out, only in the classical limit can one find the same scalar function $w(t)$ for all vector components; to next higher approximation with respect to $\kappa$, the vector equation is inconsistent with the assumption $w = w(t)$. In the elliptic case, the classical limit reproduces Kepler’s equation for $W(t) \equiv 2w(t)$. In Sec. VII. B, we attempt to solve the time dependent Schrödinger equation in the classical limit for the initial state $\psi_0$. But we are only partially successful, since we have to restrict our consideration to the neighborhood of the maximum of the probability density $|\psi_w|^2$ rather than taking into account the full configuration space.

In Sec. III, the initial wave function $\psi_0$ is derived from an un-constrained 4D oscillator state by using the fact that the KS space is the direct product of the original $\mathbb{R}^3$ and a circle space with the latter described by a fourth coordinate $0 \leq \Phi < 2\pi$, see Sec. II. For any constant value $\Phi_0$, a wave function in $\mathbb{R}^3$ appears; however, the function is not single-valued under rotations in 3D configuration space, a property which is reminiscent of the KS transformation being an outgrowth of spinor theory [13]. A proper state in $\mathbb{R}^3$ is obtained by integrating with respect to $\Phi$ with equal weight in the interval $(0, 2\pi)$, see Sec. III. The dependence on the curve parameter $w$ is borrowed from [20].

The calculation of mean values, in particular of mean square deviations of the position and velocity vectors, turned to be quite involved. It is hard to
imagine that the given problem could have been successfully attacked without the computer software for symbolic mathematical manipulations now available. We have used the version number 9 of Mathematica [22]. Instead of working out the matrix elements in 3D, we made a detour via four-dimensional KS space by keeping, at first, the phase integral with respect to Φ un-evaluated. The advantage is, that we obtain four Gaussian integrals and are left with the Φ integration which eventually leads to modified Bessel functions. As a check of the calculations, we verified the positivity of mean square deviations. This, of course, must be true by definition. However, positivity is not evident in the end formulas which are the result of several intermediate simplification steps. In any case, positivity is verified in the three-dimensional space of the parameters \((w,e,\nu)\) which appear explicitly in the results; the parameter \(\nu > 0\) shows up as a disposable constant. The mean values are calculated at first in the hyperbolic region, and then analytically continued to elliptic orbits. In the latter case, the positivity proof of the mean square deviations in subsections 1, 2, and 3 of Appendix E shows that the method of analytic continuation is consistent. As a further check, we verify for the hyperbolic case that the uncertainty products obey the quantum mechanical lower bound in the three-dimensional parameter space, for details see Appendix E.4.

The uncertainty products of the initial state \(\psi_0\) are close to the minimum, up to a factor \(\sqrt{1 + \nu^2}\). The value \(\nu = 0\) is a singular point of the theory; a good choice would be \(\nu = 1\). With increasing curve parameter \(w\), there is increasing quantum diffusion which, however, in macroscopic examples like artificial satellites is irrelevant within realistic time spans, see Sec. VIII.

The plan of the paper is as follows. In Sec. II, we sketch the KS transformation which serves to establish in Sec. III the wave function. In Sec. IV, we develop two integration methods, one is based on a mean value theorem which is suitable to determine mean values to leading order in the dynamical magnitude parameter \(K\); for higher order correction, integration via detour in KS space is more efficient and outlined in Sec. IV. B. At the end of Sec. IV, the reader will find a list of the main auxiliary functions used in Sec. V, where we present the mean values of orbit vector, velocity, their mean square deviations, uncertainty products at zero time, and of the Hamiltonian. In Sec. VI, the magnitude parameter \(\kappa\) is fixed from the initial data and the classical energy conservation law; furthermore, the classical limit is related to the orbital angular momentum. In Sec. VII, we discuss the time dependence of the wave function. In Sec. VIII, we present numerical examples. In Sec. IX, the results for hyperbolic orbits are analytically continued into the elliptic domain. The conclusions of the paper are in Sec. X followed by five appendices which contain the proof of the mean value lemma, details of the mean value calculations, and several checks.
2 Coordinate systems

Let us describe the cartesian position components in terms of polar coordinates
\[ x = r \sin(\theta) \cos(\varphi), \quad y = r \sin(\theta) \sin(\varphi), \quad z = r \cos(\theta), \]
\[ r > 0, \quad 0 < \theta < \pi, \quad 0 \leq \varphi < 2\pi. \] (1)

By adding the further coordinate \(0 \leq \Phi < 2\pi\), we extend to four dimensions and write the KS transformation \((r, \theta, \varphi, \Phi) \rightarrow \mathbf{u} \in \mathbb{R}^4\) as in [20]
\[ u_1 = \sqrt{r} \cos(\theta/2) \cos(\varphi - \Phi); \quad u_2 = \sqrt{r} \cos(\theta/2) \sin(\varphi - \Phi); \]
\[ u_3 = \sqrt{r} \sin(\theta/2) \cos(\Phi); \quad u_4 = \sqrt{r} \sin(\theta/2) \sin(\Phi). \] (2)

The transformation (2) has been known for a long time, see e.g. [3] and references therein. Equivalent forms, with \(\Phi\) replaced by \(\Phi' = \varphi - 2\Phi\), were used in [4], [12]. According to [20], the volume elements of the different coordinate systems are related as follows
\[ du_1 du_2 du_3 du_4 = 1/(8r) dx dy dz d\Phi = (1/8) r \sin(\theta) dr d\theta d\varphi d\Phi. \] (3)

We will use the connection (3) from the right to the left hand side. If the variables \((r, \varphi, \theta)\) are fixed, then two different vectors \(u\) belonging to the phase \(\Phi_0 = 0\) and an arbitrary KS phase \(\Phi\), respectively, are connected by the matrix \(T(\Phi)\)
\[ \mathbf{u}(\Phi) = T(\Phi) \mathbf{u}(0) \] (4)

with
\[ T(\Phi) = \begin{pmatrix}
\cos(\Phi), & \sin(\Phi), & 0, & 0 \\
-\sin(\Phi), & \cos(\Phi), & 0, & 0 \\
0, & 0, & \cos(\Phi), & -\sin(\Phi) \\
0, & 0, & \sin(\Phi), & \cos(\Phi)
\end{pmatrix}. \] (5)

In terms of \(\mathbf{u}\), the cartesian components read
\[ x = 2(u_1 u_3 - u_2 u_4), \quad y = 2(u_1 u_4 + u_2 u_3), \quad z = u_1^2 + u_2^2 - u_3^2 - u_4^2, \] (6)
which can be immediately verified by means of (2).

The connection between differential operators is more delicate. In our calculations, we need to transform the velocity operator into \(\mathbf{u}\) space. We start with the cartesian representation
\[ (v_x, v_y, v_z) = \hbar/(im) \left( \partial_x, \partial_y, \partial_z \right), \] (7)

go over to polar coordinates, and eventually to the variables \(u_1, \ldots, u_4\). In terms of polar coordinates, the cartesian gradient components of \(f = f(r, \theta, \varphi)\) read
\[ \partial_x f = \sin(\theta) \cos(\varphi) \partial_r f + (1/r) \cos(\theta) \cos(\varphi) \partial_\theta f - \sin(\varphi) / (r \sin(\theta)) \partial_\varphi f, \]
\[ \partial_y f = \sin(\theta) \sin(\varphi) \partial_r f + (1/r) \cos(\theta) \sin(\varphi) \partial_\theta f + \cos(\varphi) / (r \sin(\theta)) \partial_\varphi f, \]
\[ \partial_z f = \cos(\theta) \partial_r f - (1/r) \sin(\theta) \partial_\theta f. \] (8)
For a check of (8), one may specify consecutively the function \( f \) by the polar coordinates of the position components to produce the unit matrix

\[
(\partial_x, \partial_y, \partial_z) \otimes (x, y, z) = 1.
\]

(9)

In order to transform the right hand sides of (8) into \( u \)-space, we use the 4D vector \( \mathbf{p} := (r, \theta, \varphi, \Phi) \). Denoting the right hand sides of (8) by \( \delta_x, \delta_y \) and \( \delta_z \), we obtain with aid of (2)

\[
D_x \equiv \frac{1}{2u^2} \left[ u_3 \partial_{u_1} - u_4 \partial_{u_2} + u_1 \partial_{u_3} - u_2 \partial_{u_4} \right] = \delta_x + \frac{1}{(2r)} \cot(\theta/2) \sin(\varphi) \partial_{\Phi},
\]

\[
D_y \equiv \frac{1}{2u^2} \left[ u_4 \partial_{u_1} + u_3 \partial_{u_2} + u_2 \partial_{u_3} + u_1 \partial_{u_4} \right] = \delta_y - \frac{1}{(2r)} \cot(\theta/2) \cos(\varphi) \partial_{\Phi},
\]

\[
D_z \equiv \frac{1}{2u^2} \left[ u_1 \partial_{u_1} + u_2 \partial_{u_2} - u_3 \partial_{u_3} - u_4 \partial_{u_4} \right] = \delta_z.
\]

(10)

One proves (10) from the left to the right hand side by deriving from (2) the \( 4 \times 4 \) matrix \( M \): \( d\mathbf{u} = M \, dp \) together with its inverse which gives \( dp = M^{-1} \, d\mathbf{u} \) and the transpose \( N = (M^{-1})^T \), so one finds, for instance,

\[
u_3 \partial_{u_1} = u_3(r, \theta, \varphi, \Phi) \sum_{k=1}^{4} N_{1k} \partial_{p_k}.
\]

(11)

Since our wave function will not depend on the KS phase \( \Phi \), we can rigorously replace the differential operators \( \partial_x, \partial_y, \partial_z \) by the \( u \) forms \( D_i \) with

\[
(v_x, v_y, v_z) = \hbar/(im) (D_x, D_y, D_z).
\]

(12)

We remark that in previous works, e.g. in [2], [3], [7], [5], where the wave function lived on 4D configuration space, a constraint was required to eliminate the differential \( \partial_{\Phi} \). In [7] and [20], the constraint was dealt with approximately only.

3 Construction of wave function in hyperbolic domain

For the initial state \( \psi_0 \), at time \( t = 0 \), we start from the coherent state given in [20] which is defined in the KS space \( \mathbf{u} \in \mathbb{R}^4 \):

\[
\Psi_0 = C \exp \left[ \mathbf{a} \cdot \mathbf{u} - \Gamma_0 u^2 / 2 \right], \quad \Gamma_0 > 0.
\]

(13)

(In the notation of [20], \( \Gamma_0 = \gamma_0 \Gamma_1 \)). The parameter vector \( \mathbf{a} \) was fixed in [20] in such a way that, in the classical limit, the mean initial position and velocity vectors come out as

\[
\mathbf{r}(t = 0) = \{ r_0, 0, 0 \}, \quad \mathbf{v}(t = 0) = \{ 0, v_0, 0 \}, \quad r_0, v_0 > 0.
\]

(14)
The components of $a$ resulted as follows
\[ a_k = \mu_k + i\nu_k, \quad k = 1, \ldots, 4, \] (15)
where, with the abbreviations $C_0 = \cos(\Phi_0)$ and $S_0 = \sin(\Phi_0)$,
\[ \begin{align*}
\mu_1 &= \rho_0 C_0, \quad \mu_2 = \rho_0 S_0, \quad \mu_3 = \rho_0 C_0, \quad \mu_4 = -\rho_0 S_0, \\
\nu_1 &= -\nu \rho_0 S_0, \quad \nu_2 = \nu \rho_0 C_0, \quad \nu_3 = \nu \rho_0 S_0, \quad \nu_4 = \nu \rho_0 C_0
\end{align*} \] (16)
with
\[ \rho_0^2 = \Gamma_0^2 r_0 / 2, \quad \nu > 0. \] (17)
The phase $\Phi_0$ is arbitrary, and $\nu$ is a constant disposable number. In [20], the parameters $\Gamma_0$ and $\nu$ were specifically defined in terms an energy eigenvalue of the Schrödinger equation and the eccentricity of the mean Kepler orbit. In the following, we do not rely on that history and keep $\Gamma_0$ and $\nu$ disposable as long as possible.

After inserting the parameter vector $a$, as specified by (15) and (16), into the expression (13) for $\Psi_0$, and with the aid of (2), we can write
\[ \Psi_0 = C \exp[-\Gamma_0 r/2] \exp[G_0], \quad G_0 = \rho_0 \sqrt{r} \left[ d_1 \cos(\Phi) + d_2 \sin(\Phi) \right], \] (18)
where
\[ \begin{align*}
d_1 &= \sin(\theta/2) + \cos(\theta/2) [\cos(\varphi) + i\nu \sin(\varphi)], \\
d_2 &= i\nu \sin(\theta/2) + \cos(\theta/2) [-i\nu \cos(\varphi) + \sin(\varphi)].
\end{align*} \] (19)
In (18), we have shifted the origin of $\Phi$ from 0 to $-\Phi_0$. It should be noticed that, for a fixed KS phase $\Phi$, the wave function $\Psi_0$ is not invariant under a full rotation in 3D space, one has to go around twice similar to the transformation of a spinor. Formerly, this is due to the occurrence of half of the polar angle, $\theta/2$.

In order to ensure the correct transformation behavior, we form the following linear combination with respect to the KS phase $\Phi$:
\[ \psi_0 = \int_0^{2\pi} d\Phi \Psi_0(\Phi). \] (20)
The wave function $\psi_0$ is now a function of $\sin(\theta)$. As a matter of fact, the $\Phi$ integration is elementary and can be written in terms of the zero order modified Bessel function $I_0$ as follows
\[ \psi_0 = 2\pi C \exp[-\Gamma_0 r/2] I_0(U_0), \quad U_0 = \rho_0 d \sqrt{r} = \kappa d \sqrt{r/(2r_0)} \] (21)
with
\[ d^2 = d_1^2 + d_2^2 = 1 - \nu^2 + \sin(\theta) \left[ (1 + \nu^2) \cos(\varphi) + 2i\nu \sin(\varphi) \right]. \] (22)
The classical limit will be controlled by the dimensionless parameter \( \kappa \gg 1 \) with
\[
\kappa = r_0 \Gamma_0. \tag{23}
\]
By (21), \( U_0 \) is proportional to \( \kappa \), so in the classical limit, with \( |U_0| \gg 1 \), the Bessel function can be approximated by the asymptotic formula [8]
\[
I_0(U_0) \rightarrow \frac{\exp[U_0]}{\sqrt{2\pi U_0}} \left[ 1 + \frac{1}{8U_0} + \mathcal{O}(1/U_0^2) \right], \tag{24}
\]
which gives rise to the asymptotic form of the zero-time wave function
\[
\psi_0 \rightarrow \psi_{as} = C \sqrt{\frac{2\pi}{U_0}} \exp[-\Gamma_0 r/2 + U_0] \left[ 1 + \mathcal{O}(1/U_0) \right], \quad |U_0| \gg 1. \tag{25}
\]

The parameter evolution is borrowed from [20]. Instead of the pseudo time \( \sigma \), we here introduce the curve parameter \( w = \omega \sigma \). According to [20], the parameter dependence enters through the vector \( a \rightarrow a(w) \) and the scalar \( \Gamma_0 \rightarrow \Gamma(w) \) with consequences for \( G_0 \rightarrow G(w) \) and \( U_0 \rightarrow U(w) \) as follows
\[
a(w) = a(0)/f^*(w), \quad G(w) = G_0/f^*(w), \quad U(w) = U_0/f^*(w), \quad \Gamma(w) = \Gamma_R + i \Gamma_I = -i \Gamma_0 \tanh(w + i C_1), \tag{26}
\]
where \( f(w) \) is defined in (60a) below and \( \tanh(C_1) = \gamma_0 \) which implies that
\[
\Gamma_R = \Gamma_0 h(w), \quad \Gamma_I = -\Gamma_0 h(w)(1 + \gamma_0^2)/(2\gamma_0) \sinh(2w) \tag{27}
\]
with \( h(w) \) defined in (60b). The integration constant \( \gamma_0 \), which in [20] was set equal to 1, is kept disposable. The above parameter dependence, with \( w \) being real, refers to hyperbolic orbits. The parameter dependent wave function now reads
\[
\Psi_w(r, \theta, \varphi, \Phi) = C(w) \exp(-\Gamma(w)r/2) \exp[G(w)], \tag{28}
\]
which, since \( f(0) = h(0) = 1 \), reproduces \( \psi_0 \) for \( w = 0 \).

Analogously to (20) and (21), the KS phase is averaged out with the result
\[
\psi_w = \int_0^{2\pi} d\Phi \Psi_w = 2\pi C(w) \exp[-\Gamma(w)r/2] I_0(U). \tag{29}
\]
The magnitude parameter now becomes \( w \) dependent as
\[
\kappa \rightarrow K = r_0 \Gamma_R(w) = \kappa h(w), \tag{30}
\]
which implies that \( K \) decreases with increasing \( w > 0 \), since \( h(0) = 1 \) and \( h(w) < 1 \). In the classical limit, if \( K \gg 1 \), the asymptotic form of the wave function reads
\[
\psi_w \rightarrow \psi_{as} = C(w) \sqrt{\frac{2\pi}{U}} \exp[-\Gamma(w)r/2 + U] \left[ 1 + \mathcal{O}(1/K) \right]. \tag{31}
\]
The state \( \psi_w \) lives in three-dimensional configuration space and is normalizable for all parameters \( w \).
4 Integration Methods

4.1 Mean value theorem

From the maximum of the probability density, \( P = |\psi_{as}|^2 \), we infer that the mean values of the orbit vector \( r = (x, y, z) \), to leading order in \( K \), come out as follows

\[
\bar{x} = a \left[ e - \cosh(2w) \right], \quad \bar{y} = a \sqrt{e^2 - 1} \sinh(2w), \quad \bar{z} = 0, \quad e > 1. \quad (32)
\]

where \( a \) and \( e \) denote the semimajor axis and the eccentricity, respectively. Clearly, (32) is the parametric representation of a hyperbola with parameter \( w \in \mathbb{R} \).

In order to derive the maximum of \( P \), we write the asymptotic probability density in the following form

\[
P = |C(w)|^2 \frac{2\pi}{\sqrt{UU^*}} \exp \left[ G_2 \right], \quad (33)
\]

where

\[
G_2 = -\Gamma_R r + U + U^* = -K \left[ \sqrt{r/r_0} - D \right]^2 + KD^2,
\]

\[
D = \sqrt{2/4 \left[ d(\theta, \varphi) f(w) + d^*(\theta, \varphi) f^*(w) \right]} \quad (34)
\]

with \( d \) defined in (22) and \( U \) in (26) and (21). Since \( D \) is independent of \( r \), \( G_2 \) is maximal at

\[
r_m = r_0 D^2(\theta, \varphi), \quad (35)
\]

with the consequence that

\[
G_2(r_m) = KD^2(\theta, \varphi) \geq 0. \quad (36)
\]

It is easily verified that the only extremum of \( D(\theta, \varphi) \) with respect to \( \theta \in (0, \pi) \) is at \( \theta_m = \pi/2 \), which means that the extremum lies in the \( z = 0 \) plane. Then, from \( \partial_\varphi D(\theta_m, \varphi) = 0 \) we find the further condition

\[
\tan(\varphi_m/2) = \gamma_0 \nu \tanh(w). \quad (37)
\]

As is discussed below and in Appendix A, the extremum of \( G_2 \) is actually a maximum, which for \( K \gg 1 \) is narrowly peaked.

For \( w \geq 0 \), condition (37) is equivalent to

\[
\cos(\varphi_m/2) = \cosh(w) \sqrt{(e - 1)Z_w}, \quad \sin(\varphi_m/2) = \sinh(w) \sqrt{(e + 1)Z_w} \quad (38)
\]

with \( Z_w = \left[ e \cosh(2w) - 1 \right]^{-1} \). Evaluating \( r_m(\theta, \varphi) \) at the maximum (\( \theta = \pi/2, \varphi = \varphi_m \)), we obtain from (35) and (37)

\[
r_m(w) = r_0 \left[ \cosh^2(w) + \gamma_0^2 \nu^2 \sinh^2(w) \right]. \quad (39)
\]
Now, we dispose of the parameter $\gamma_0$ such that $r_m(w)$ is the hyperbolic distance from the force center, i.e.

\[ r_m = a \left[ e \cosh(2w) - 1 \right], \quad r_0 = (e - 1)a, \tag{40} \]

which is true, if

\[ \gamma_0 = (1/\nu) \sqrt{(e + 1)/(e - 1)}, \quad e > 1. \tag{41} \]

With the aid of $\theta_m = \pi/2$, (40), and (38), one easily derives the following vector components of the orbit at the maximum of the probability density for $e > 1$:

\[ x_m \equiv r_m \sin(\theta_m) \cos(\varphi_m) = a \left[ e - \cosh(2w) \right], \]
\[ y_m \equiv r_m \sin(\theta_m) \sin(\varphi_m) = a \sqrt{e^2 - 1} \sinh(2w), \]
\[ z_m \equiv r_m \cos(\theta_m) = 0, \tag{42} \]

which describe a hyperbolic orbit.

In Appendix A, we prove the following mean value lemma for real functions $F$:

\[ \langle \psi_{as} | F(r) \psi_{as} \rangle = F(r_m, \theta_m, \varphi_m) \left[ 1 + \mathcal{O}(1/K) \right]. \tag{43} \]

As a consequence, the vector components at the maximum are actually mean values to leading order in $K$:

\[ (x_m, y_m, z_m) = \mathbf{r} \left[ 1 + \mathcal{O}(1/K) \right]. \tag{44} \]

The proof of the Lemma in Appendix A is based on the Taylor expansion of $G_2$ around the maximum with

\[ r = r_m + \epsilon \delta r, \quad \theta = \theta_m + \epsilon \delta \theta, \quad \varphi = \varphi_m + \epsilon \delta \varphi, \quad \epsilon = 1/\sqrt{K}. \tag{45} \]

We obtain to leading order in $\epsilon$, for details see Appendix A,

\[ G_2 = E_0/\epsilon^2 - \left[ E_r \delta r^2 + E_\varphi \delta \varphi^2 + E_\theta \delta \theta^2 \right] + \mathcal{O}(\epsilon), \]
\[ E_0 = [Z_w(e - 1)]^{-1}, \quad E_r = Z_w(e - 1)/(4r_0^2), \quad E_\varphi = [4Z_w(e - 1)]^{-1}, \]
\[ E_\theta = \frac{(\nu^2 + 1) [2(e - 1)Z_w^2]^{-1}}{(e - 1) (1 + \cosh(2w)) + (e + 1)\nu^2 (\cosh(2w) - 1)}. \tag{46} \]

Clearly, the above $E$ coefficients are positively definite for $e > 1$ and real $w$. They remain positive after the analytic continuation to elliptic orbits, $w \rightarrow i w'$, with the eccentricity confined to the interval $0 \leq e < 1$ and with $w'$ being a real number.

The expansion method implies, that the mean deviations $\delta r, \delta \theta, \delta \varphi$ are of order $\epsilon = 1/\sqrt{K}$. 

4.2 Integration via KS space

The integration in 3D configuration space becomes rather awkward when higher orders in $1/\sqrt{K}$ are needed as it is the case for mean square deviations. In what follows, we adopt a more efficient method by means of a detour in KS space. To this end, one postpones the averaging with respect to the KS phase $\Phi$, and writes the mean value of an observable $O$ as

$$\langle \psi_w | O | \psi_w \rangle = \int_{0}^{2\pi} d\Phi' \int_{0}^{2\pi} d\Phi \langle \Psi_w(\Phi') | O | \Psi_w(\Phi) \rangle,$$  

where

$$\Psi_w(\Phi) = C \exp \left[ a(w) \cdot u(\Phi) - \Gamma(w) u^2 / 2 \right]$$

with $a(w)$ and $\Gamma(w)$ defined in (26) and $C$ denoting the normalization constant; we used the property $r = u^2$ and the fact that $u^2 \equiv u \cdot u$ does not depend on $\Phi$. In the scalar product $a \cdot u(\Phi)$, the $\Phi$ dependence is shifted to the parameter vector $a$ by means of the transposed matrix $T$ with the result

$$\Psi_w(\Phi) = C \exp \left[ a(w, \Phi) \cdot u - \Gamma(w) u^2 / 2 \right],$$

$$\Psi_w^*(\Phi') = C^* \exp \left[ a^*(w, \Phi') \cdot u - \Gamma^*(w) u^2 / 2 \right], \quad u = u(0),$$

where the star denotes the complex conjugation and

$$a(w, \Phi) = T(\Phi) a(w).$$

The explicit form of $a(w, \Phi)$ is, with $\Phi_0 = 0$ and $\rho_0$ defined in (17),

$$a(w, \Phi) = \left[ \rho_0 / f^*(w) \right] \left\{ \cos(\Phi) - i \nu \sin(\Phi), \ i \nu \cos(\Phi) + \sin(\Phi), \ \cos(\Phi) + i \nu \sin(\Phi), \ i \nu \cos(\Phi) - \sin(\Phi) \right\}.$$  

4.3 Normalization

The normalization constant $C$ is calculated by setting in (47) the observable $O \equiv 1$. The corresponding scalar product is written, at first, in the metric of polar coordinates and, then, transformed by means of (3):

$$\langle \psi_w | \psi_w \rangle = C^2 \int_{0}^{2\pi} d\Phi' \int_{0}^{2\pi} dr \, d\theta \, d\phi \, d\Phi \, r^2 \sin(\theta) \exp \left[ A \cdot u - \Gamma_R u^2 \right],$$

$$= C^2 \int_{0}^{2\pi} d\Phi' \int_{-\infty}^{\infty} du_1 du_2 du_3 du_4 (8u^2) \exp \left[ A \cdot u - \Gamma_R u^2 \right],$$

$$= -8C^2 \frac{\partial}{\partial \Gamma_R} \int_{0}^{2\pi} d\Phi' \int_{-\infty}^{\infty} du_1 du_2 du_3 du_4 \exp \left[ A \cdot u - \Gamma_R u^2 \right],$$

$$A = a^*(w, \Phi') + a(w, \Phi), \quad \Gamma_R = \Gamma_0 h(w).$$
The $u$ integrations are Gaussian and immediately give rise to

$$1 = -8|C|^2 \frac{\partial}{\partial \Gamma_R} \frac{\pi^2}{\Gamma_R^2} \int_0^{2\pi} d\Phi' \exp \left[ A \cdot A / (4\Gamma_R) \right]. \tag{53}$$

It is straightforward to show that

$$A \cdot A / (4\Gamma_R) = [k_0 + k_1 \cos (\Phi' - \Phi)] / \Gamma_R, \tag{54}$$

where $k_0$ and $k_1$ are defined in (60d) below. The $\Phi'$ integral in (53) does not depend on $\Phi$ and can be expressed by the zero order modified Bessel function $I_0$ [8],

$$1 = -16\pi^3 C^2 \frac{\partial}{\partial \Gamma_R} \frac{1}{\Gamma_R^2} \exp \left[ \frac{k_0}{\Gamma_R} I_0 \left( \frac{k_1}{\Gamma_R} \right) \right] = 16\pi^3 C^2 \exp \left[ \frac{k_0}{\Gamma_R} \frac{1}{\Gamma_R} \left\{ [2\Gamma_R + k_0] I_0 \left( \frac{k_1}{\Gamma_R} \right) + k_1 I_1 \left( \frac{k_1}{\Gamma_R} \right) \right\} \right] = 16\pi^3 C^2 \exp [K\kappa_0] \frac{r_0^3}{K^3} \left\{ [2 + K\kappa_0] I_0 (K\kappa_1) + K\kappa_1 I_1 (K\kappa_1) \right\}, \tag{55}$$

where in the last equation we set $\Gamma_R = K/r_0$. In macroscopic cases, the argument $Z := K\kappa_1 \gg 1$, so we use the asymptotic approximation of the Bessel functions [8] as

$$I_\nu(Z) \to \frac{\exp[Z]}{\sqrt{2\pi Z}} \left[ 1 + \frac{i_1(\nu)}{Z} + \frac{i_2(\nu)}{Z^2} + \mathcal{O}(Z^{-3}) \right], \quad \nu = 0, 1, 2, \ldots, \tag{56}$$

where, for $\nu = 0, 1, 2$ which is needed further below,

$$i_1(0) = 1/8, \quad i_2(0) = 9/128, \quad i_1(1) = -3/8, \quad i_2(1) = -15/128, \quad i_1(2) = -15/8, \quad i_2(2) = 105/128, \quad i_1(3) = -35/8, \quad i_2(3) = 945/128. \tag{57}$$

The normalization condition now reads, including the first order correction with respect to $1/K$,

$$1 = C^2 8\sqrt{2} r_0^3 \kappa_1^2 (\kappa_0 + \kappa_1) \left[ \pi / (\kappa_1 K) \right]^{5/2} \exp [K (\kappa_0 + \kappa_1)] (1 + \delta n / K), \tag{58}$$

$$\delta n = (\kappa_0 + 13\kappa_1) [8\kappa_1 (\kappa_0 + \kappa_1)]^{-1}, \quad K = \kappa h(w), \tag{59}$$

where $\kappa_0$ and $\kappa_1$ are defined in (60f) below. At the end of Appendix A it is shown, that the zero order result above, with the $\delta n$ term neglected, agrees with result (A13) of the 3D integration.
4.4 List of formulas for the hyperbolic case

For \( e > 1 \), we give a list of functions and symbols used above and further below.

\[
\begin{align*}
f(w) &= \cosh(w) - i\gamma_0 \sinh(w). \quad (60a) \\
h(w) &= (f(w)f^*(w))^{-1}. \quad (60b) \\
\gamma_0 &= \sqrt{(e + 1)/(e - 1)/\nu}. \quad (60c) \\
k_0 &= r_0\Gamma_R^2\kappa_0, \quad k_1 = r_0\Gamma_R^2\kappa_1. \quad (60d) \\
K_1 &= (e - 1)\cosh^2(w) + (1 + e)\nu^2 \sinh^2(w), \quad (e - 1)\nu^2 \cosh^2(w) + (1 + e)\sinh^2(w). \quad (60e) \\
K_2 &= (1/4)(1 - \nu^2) \left[ f^2(w) + (f^*(w))^2 \right], \quad (60f) \\
\kappa_0 &= (1/2)(1 + \nu^2)/h(w), \quad \kappa_0 + \kappa_1 = [e\cosh(2w) - 1]/(e - 1), \quad \kappa_1 = 2\kappa_0(1 + \nu^2)/(1 + \nu^2), \quad \kappa_2 = 2\kappa_0(1 + \nu^2)/(1 + \nu^2), \quad \kappa_3 = 2\kappa_0(1 + \mu^2)/(1 + \mu^2), \quad \kappa_4 = (1/(4\Gamma_0))(1 + \mu^2) \left[ (\Gamma_I + i\Gamma_R)f^2(w) + (\Gamma_I - i\Gamma_R)(f^*(w))^2 \right], \quad (60g) \\
\kappa_5 &= (1/(2\Gamma_0))\Gamma_I(1 - \nu^2)f(w)f^*(w) = \frac{1 + \gamma_0^2}{4\gamma_0}(\nu^2 - 1)\sinh(2w), \quad (60h) \\
\kappa_6 &= i\nu [f^2(w) - (f^*(w))^2] = 2\sqrt{\frac{e + 1}{e - 1}}\sinh(2w). \quad (60i) \\
I_A(\Gamma) &= \exp [(\mathbf{A} \cdot \mathbf{A})/(4\Gamma)] = \exp [(\kappa_0 + \kappa_1 \cos(\Phi' - \Phi))/\Gamma]. \quad (60j) \\
K &= r_0\Gamma_R = \kappa h(w), \quad \kappa = r_0\Gamma_0. \quad (60k) \\
g &= \Gamma_I/\Gamma_R = -(1 + \gamma_0^2)(4\nu\gamma_0^2)^{-1}\kappa_6. \quad (60l) \\
Z_w &= [e\cosh(2w) - 1]^{-1}, \quad Z_e = [e^2 - 1]^{-1/2}. \quad (60m)
\end{align*}
\]

5 Mean values and Kepler orbits

In the following, we list the main results: In subsection A the mean components \( \overline{x_i}, i = 1, 2, 3 \), of the hyperbolic orbit vector \( \mathbf{r} \) together with the mean square deviations \((\Delta x_i)^2 = \overline{x_i^2} - \overline{x_i}^2; \) in B the mean velocity components \( \overline{v_i} \) and their mean square deviations \((\Delta v_i)^2 = \overline{v_i^2} - \overline{v_i}^2; \) in C the uncertainty products at \( w = 0; \) in D the mean Hamiltonian. At the curve parameter \( w = 0 \), where \( \overline{r}(0) = (r_0, 0, 0) \) and \( \overline{v}(0) = (0, v_0, 0) \), the uncertainty products turn to be close to or equal to the quantum mechanical minimum, see subsection D. The mean values are expressed in terms of the functions defined in (60). The results are asymptotic approximations for \( K \gg 1 \). The eccentricity \( e \) is larger 1, and the semi-major axis \( a \) obeys the relation \( a = r_0(e - 1) \). For derivations, we refer to Appendix B and C in the case of position and velocity mean values, respectively.
5.1 Mean values of position variables

We write down the mean values of the cartesian components, including first order corrections with respect to $K$, as

$$\bar{x} = \langle x \rangle_0 [1 + \xi/K + \mathcal{O}(1/K^2)],$$
$$\bar{y} = \langle y \rangle_0 [1 + \eta/K + \mathcal{O}(1/K^2)],$$
$$\bar{z} = 0,$$  \hspace{1cm} (61)

where, in terms of the $\kappa_i$ defined in (60f),

$$\langle x \rangle_0 = \frac{1}{2} r_0 (\kappa_2 + \kappa_3) = a[e - \cosh(2w)],$$
$$\langle y \rangle_0 = \frac{1}{2} r_0 \kappa_6 = a\sqrt{e - 1} \sinh(2w),$$
$$\xi = \frac{4\kappa_1 \kappa_2 - \kappa_0 \kappa_3 + 3\kappa_1 \kappa_3}{2\kappa_1 (\kappa_0 + \kappa_1)(\kappa_2 + \kappa_3)},$$
$$\eta = \frac{2}{(\kappa_0 + \kappa_1)}.$$  \hspace{1cm} (62)

We remark that the result $\bar{z} = 0$ comes out without approximation.

The mean square deviations result as, for the definitions of $K_1$ and $K_2$ see (60e),

$$\langle \Delta x \rangle^2 = \frac{2r_0^2}{\kappa} K_2 \frac{e \cosh(2w) - 1}{(e - 1)^2 \nu^2} [1 + \mathcal{O}(1/K)],$$
$$\langle \Delta y \rangle^2 = \langle \Delta x \rangle^2,$$
$$\langle \Delta z \rangle^2 = \frac{2r_0^2}{\kappa} K_1 K_2 \frac{K_1 K_2}{(e - 1)^2 \nu^2 (1 + \nu^2)} [1 + \mathcal{O}(1/K)].$$  \hspace{1cm} (63)

5.2 Mean values of velocity variables

The mean velocity components come out as

$$\bar{v}_x = \langle v_x \rangle_0 [1 + \xi_v/K + \mathcal{O}(1/K^2)],$$
$$\bar{v}_y = \langle v_y \rangle_0 [1 + \eta_v/K + \mathcal{O}(1/K^2)],$$
$$\bar{v}_z = 0,$$  \hspace{1cm} (64)

with

$$\langle v_x \rangle_0 = -\sqrt{\frac{e - 1}{e + 1}} \frac{\hbar \kappa \nu}{2mr_0} \frac{\sinh(2w)}{e \cosh(2w) - 1},$$
$$\langle v_y \rangle_0 = \frac{\hbar \kappa \nu}{2mr_0} \frac{(e - 1) \cosh(2w)}{e \cosh(2w) - 1},$$
$$\xi_v = -\frac{3\kappa_1 (\kappa_4 + \kappa_5) + \kappa_5 (\kappa_0 + \kappa_1)}{2\kappa_1 (\kappa_0 + \kappa_1)(\kappa_4 + \kappa_5)},$$
$$\eta_v = -3[2(\kappa_0 + \kappa_1)]^{-1}.$$  \hspace{1cm} (65)
From the initial condition \((v_y)_0 = v_0\) at \(w = 0\), one can substitute for \(\hbar \kappa \nu\) the term \(2mr_0v_0\) so that, to leading order in \(\kappa\), the velocity components are the classical expressions without bearing on quantum mechanics through \(\hbar\).

For comparison, the velocity vector may be also derived from the eccentricity (Runge-Lenz) vector by making use of \(r = r_m\) according to (40) and from the mean position components given in (62).

For the mean square deviations of the velocity components, we find

\[
(\Delta v_x)^2 = \kappa F_0 (e - 1)^2 Z_w^3 \sum_{j=0}^{3} c_{2j} \cosh(2jw) \left[ 1 + \mathcal{O}(1/K) \right],
\]

\[
(\Delta v_y)^2 = \kappa F_0 Z_w^3 (e - 1)/(e + 1) \sum_{j=0}^{3} d_{2j} \cosh(2jw) \left[ 1 + \mathcal{O}(1/K) \right],
\]

\[
(\Delta v_z)^2 = \kappa F_0 \frac{(e - 1)Z_w^2}{(1 + \nu^2)(e + 1)} \left[ f_0 + f_4 \cosh(4w) \right] \left[ 1 + \mathcal{O}(1/K) \right],
\]

where

\[
F_0 = \frac{\hbar^2}{16m^2r_0^2},
\]

\[
c_0 = -2(1 + \nu^2), \quad c_2 = e - 1 + (e + 1)\nu^2, \quad c_4 = 0, \quad c_6 = e + 1 + (e - 1)\nu^2,
\]

\[
d_0 = 2 \left( 1 + e + \nu^2(1 - e) \right), \quad d_2 = \nu^2 - 1 - 3e(1 + \nu^2) + 2e^3(1 + \nu^2),
\]

\[
d_4 = -2e(1 + e - \nu^2 + e\nu^2), \quad d_6 = 1 + e + (e - 1)\nu^2,
\]

\[
f_0 = e^2(1 + \nu^2)^2 - 1 - 6\nu^2 - \nu^4, \quad f_4 = e^2(1 + \nu^2)^2 - (\nu^2 - 1)^2.
\]

\[\text{(67)}\]

### 5.3 Uncertainty products at initial time

The uncertainty products

\[
(\Delta x_i)^2(\Delta v_i)^2 = \hbar^2/(4m^2) \, P_i, \quad i = x, y, z,
\]

have to obey the quantum mechanical inequality \(P_i \geq 1\). The initial state, corresponding to \(w = 0\), is connected with a nearly minimum uncertainty which depends on the disposable constant \(\nu > 0\). One easily derives from (63) and (66)

\[
P_x(w = 0) = P_y(w = 0) = 1 + \nu^2, \quad P_z(w = 0) = 1.
\]

As a check of the analytical results, we will prove in Appendix E.4 that \(P_i \geq 1\) in the three-dimensional parameter space \((w, e, \nu)\).

### 5.4 Mean Hamiltonian

The mean energy reads

\[
E = \bar{H} = (m/2)(v_x^2 + v_y^2) - \alpha 1/r,
\]

\[\text{(70)}\]
where $\alpha$ is the interaction constant. To leading order, see Appendix C,
\[
\overline{v_{x,y}^2} = \overline{v_{x,y}^2} \left[1 + \mathcal{O}(1/K)\right],
\]
and by the mean value Lemma (43),
\[
\overline{1/r} = (1/r_m) \left[1 + \mathcal{O}(1/K)\right],
\]
where $r_m$ is stated in (40). With the aid of (65) and (40), we find
\[
E = \frac{e - 1}{8(e + 1)m r_0^2} \frac{C_1 + C_2 \cosh(2w)}{e \cosh(2w) - 1} \left[1 + \mathcal{O}(1/K)\right], \quad e > 1,
\]
\[
C_1 = \hbar^2 \kappa^2 \nu^2 - 8(1 + e)\alpha m r_0, \quad C_2 = e\hbar^2 \kappa^2 \nu^2.
\]

6 Disposing of open parameters and classical limit

In this section, we will fix the half width parameter $\Gamma_0$ from the initial data and from energy conservation. We also relate the magnitude parameter $\kappa$ to the angular momentum and determine the eccentricity $e$ in terms of the initial magnitudes $r_0, v_0$.

The still open parameters are $\Gamma_0$, which has the dimension of a reciprocal length and characterizes the half width of the wave function $\psi_0$ and, furthermore, the number $\nu > 0$. The parameter $\Gamma_0$ appears through the combination
\[
\kappa \nu = r_0 \Gamma_0 \nu \equiv \nu K/h(w).
\]
From the initial condition on $\overline{v_y}$, we can fix $\kappa \nu$ with the aid of (65):
\[
v_0 \equiv \overline{v_y}(w = 0) = \hbar \kappa \nu / (2m r_0) \left[1 + \mathcal{O}(1/K)\right],
\]
which gives rise to
\[
\kappa \nu = 2(m r_0 v_0) / \hbar \left[1 + \mathcal{O}(1/K)\right].
\]
In the result (65) for the mean velocities, the factors $\hbar \kappa \nu$ can be eliminated in terms of $v_0$ by means of (76). As a consequence, the arbitrary constant $\nu$ does not enter the mean values of the Kepler orbit; however, it does affect the mean square deviations, as can be seen from (63) and (66).

As a further condition on $\kappa \nu$, we require that the mean energy $E$ does not depend on the curve parameter $w$ to leading order:
\[
E = \alpha / (2a) \left[1 + \mathcal{O}(1/K)\right], \quad a = r_0 / (e - 1), \quad e > 1.
\]
As is easily verified, condition (77) together with the expression (73) for $E$ is fulfilled, if
\[ \kappa \nu = \frac{2}{\hbar} \sqrt{(e + 1)mr_0 \alpha} [1 + O(1/K)]. \]  
(78)

As a consequence of (78), we can write the coupling parameter $\alpha$ of the potential, $\alpha/r$, as follows
\[ \alpha = \frac{\hbar^2 \kappa^2 \nu^2}{4(e + 1)mr_0} [1 + O(1/K)], \]  
(79)
which tells that the energy $E$ is of leading order $\kappa^2$. With (76) and (78), we now get the following equation for the eccentricity $e$:
\[ e + 1 = \frac{mr_0 v_0^2}{\alpha} [1 + O(1/K)]. \]  
(80)

From the first order corrections in (76) and (77), the eccentricity $e$ can be determined, in principle, to next higher order. For simplicity, we consider the symbol $e$ being correspondingly re-normalized without further specification.

In order to see the connection between the formula (80) for $e$ and a standard formulation of celestial mechanics, one detects the angular momentum in (80),
\[ L = mr_0 \times v_0 = m(r_0, 0, 0) \times (0, v_0, 0) = mr_0 v_0 \hat{z}, \]  
(81)
so we can write
\[ e + 1 = \frac{Lv_0}{\alpha} [1 + O(1/K)]. \]  
(82)

On the other hand, a typical textbook result reads, see e.g. Eq. (15.4) of [14],
\[ e = \sqrt{1 + 2EL^2/(ma^2)}. \]  
(83)

After squaring (83), substituting $\alpha(e - 1)/(2r_0)$ for $E$, and dividing by $(e - 1) > 0$, we recover our result (80).

We remark that the textbook formula (83) is consistent with the sign switch in the connection $r_0 = |1 \mp e| a$ at the initial speed $v_0 = v_c$ of a circle orbit. To see this, one has to insert into (83) $L = mr_0 v_0$ together with the energy $E = -\alpha/(2r_0)(1 - e)$ and $E = -\alpha/(2r_0)(e + 1)$ for $v_0 > v_c$ and $v_0 < v_c$, respectively; see also Sec. IX. We have also checked the sign switch by means of numerical integration of the classical equations of motion at constant $r_0$ and for varying $v_0$.

As to the dynamical magnitude parameter $K = \kappa h(w)$, the function $h(w) \leq 1$ with $h(0) = 1$, see (60b). If $w \gg 1$, then $h(w) \rightarrow 4 \exp[-2w]/(1 + \gamma_0^2)$ gets exponentially small. We have assumed that $K \gg 1$ which, of course, cannot be maintained for arbitrary large curve parameters $w$. Let us assume that $h(w)$ is not too far away from the value 1 and that the still open parameter $\nu$ is of the order 1. Then, large $K$ is equivalent to large $\kappa$ and by (76) amounts to
\[ \kappa \approx mr_0 v_0/\hbar \gg 1, \quad \text{or} \quad L \gg \hbar. \]  
(84)
In words, the classical limit amounts to the orbital angular momentum $L$ being large as compared to $\hbar$ or, equivalently, to the quantum number $l \gg 1$ with $L = \hbar l(l + 1)$.

For large distances $r$ from the force center, the initial probability density $|\psi_0|^2$, see (21), decreases with the exponent $(-\Gamma_0 r) = (-r/\Delta_\kappa)$, where the half width $\Delta_\kappa = r_0/(\Gamma_0 \kappa) \equiv r_0/\kappa$. We now see that the half width essentially is fixed by the angular momentum $L$ and decreases proportional to $1/L$.

7 Time dependence

7.1 From mean values

We introduce the time $t$ from the classical property that the mean velocity is the time derivative of the mean position vector. The mean values are taken to leading order in $K$. In view of the classical behavior, we assume that the $t$ dependence is through the curve parameter $w = w(t)$. Both the $x$ and the $y$ component should lead to the same scalar function $w(t)$, as they actually do. From the relations

$$\frac{d\langle x \rangle_0}{dt} = \langle v_x \rangle_0 [1 + O(1/K)], \quad \frac{d\langle y \rangle_0}{dt} = \langle v_y \rangle_0 [1 + O(1/K)], \quad (85)$$

and from the mean values defined in (62) and (65), one derives the same differential equation for $W \equiv 2w$:

$$\frac{dW}{dt} = \frac{N}{e \cosh(W) - 1}, \quad N = \frac{\hbar \kappa \nu}{2mr_0^2} \frac{(e - 1)^{3/2}}{\sqrt{e + 1}}, \quad e > 1. \quad (86)$$

Integration leads to the standard result for hyperbolic orbits

$$N(t - t_0) = W_0 - W - e \left[ \sinh(W_0) - \sinh(W) \right]. \quad (87)$$

It remains to be shown that $N$ agrees with the standard expression of celestial mechanics,

$$N = \sqrt{\alpha/(ma^3)}, \quad (88)$$

see, for instance, Eqs. (4.101) and (4.97) in [21]. To obtain (88), we use (78) to eliminate the factor $\hbar \kappa \nu$ in the expression of $N$ in (86). Then, substitution of $r_0$ by $r_0 = (e - 1)a$ leads to (88).

To next higher order, if the correction terms of (62) and (65) are included, different functions $W = W(t)$ emerge for the $x$ and $y$ component, respectively. This can be rather easily seen by setting the disposable constant $\nu = 1$, which significantly simplifies the coefficients $\kappa_i$ as a function of $w$. As a consequence, the assumption that time dependence exclusively enters via the curve parameter $w$, can be only maintained in the classical limit $K \to \infty$. 
7.2 Solving the Schrödinger equation

In the following, we show that the time dependence of $w$, as derived above, is consistent with the time dependent Schrödinger equation under certain conditions, which are: (i) keeping only terms to leading order in $K$, (ii) taking the distance variable $r$ close to the maximum of probability density with $r = r_m + \delta r/\sqrt{K}$; the angle variables $\theta$ and $\varphi$ will not be restricted. As more technical assumptions, we take $h(w)$ of order 1 which allows us to equivalently use the magnitude parameter $\kappa$ or $K$; furthermore, we set the constant $\nu = 1$, which will significantly simplify intermediate formulas. Partial derivatives will be indicated by a suffix.

The task is to solve the Schrödinger equation

$$i \hbar \partial_t \psi = H \psi$$

for the initial state at $t = 0$

$$\psi_{as} = 2\pi C_R(w) \sqrt{2\pi/U_0} \exp \left[ i c(w) - \Gamma_0 r/2 + U_0 \right], \quad (90)$$

where, as compared to (25), we wrote the normalization constant $C(w) = C_R(w) \exp[i c(w)]$ in terms of modulus $C_R > 0$ and real phase $c(w)$; the latter will serve as a disposable function. For the time dependent state, we adopt the asymptotic form (31) (neglecting $1/K$ corrections)

$$\psi = C_R(w) \sqrt{2\pi/U} \exp \left[ i c(w) - \Gamma(w) r/2 + U \right] \quad (91)$$

with the assumption that time dependence enters as

$$w = w(t), \quad w(0) = 0. \quad (92)$$

It should be noticed that $f(0) = f^*(0) = 1$, which implies that $U(t = 0) = U_0$, and as a consequence, the time dependent state (91) reduces to the initial state (90).

For the left hand side of (89) we proceed as follows:

$$\psi_t = w_t \psi_w, \quad \psi_w = \left[ ic' + C_R'/C_R + (f^*)'(2f^*) - U(f^*)'/f^* - (1/2) r \Gamma' \right] \psi, \quad (93)$$

where prime denotes the partial derivative with respect to $w$. For $\nu = 1$, we find from the normalization condition (58) to leading order in $K$

$$C_R'/C_R = -(2/3) \left[ f'/f + (f^*)'/f^* \right]. \quad (94)$$

In order to project out of (93) the leading order with respect to $\kappa$, we take into account that both $\Gamma$ and $U$ are proportional to $\Gamma_0 = \kappa/r_0$ with $U = U_0/f^*$, see (21). Since the quotient $f'/f$ is of order 1, we obtain

$$\psi_w/\psi = \left[ ic' - U(f^*)'/f^* - (1/2) r \Gamma' \right] \left[ 1 + \mathcal{O}(1/\kappa) \right]. \quad (95)$$
As to the right hand side of the Schrödinger equation, we find to leading order in $\kappa$
\[ \Delta \psi/\psi = \bigl\{ U_r^2 + (1/r^2) \bigl[ U_\theta^2 + U_\phi^2/\sin^2(\theta) \bigr] - \Gamma(w)U_r + (1/4)\Gamma^2(w) \bigr\} \{1 + O(1/\kappa)\}. \] 
(96)

From (21) and (22), we infer the partial derivatives of $U$ with the result
\[ (\Delta \psi)/\psi = \bigl\{ -U/(2r) \Gamma(w) + (1/4)\Gamma^2(w) \bigr\} \{1 + O(1/\kappa)\}, \] 
(97)
where we used the fact that
\[ U_r^2 + (1/r^2) \bigl[ U_\theta^2 + U_\phi^2/\sin^2(\theta) \bigr] = 0. \] 
(98)

Thus, we obtain to leading order in $\kappa$
\[ H\psi/\psi = -\hbar^2/2m \Delta \psi/\psi - \frac{\alpha}{r} = -\hbar^2/2m \left[ -\frac{U}{2r} \Gamma(w) + \frac{1}{4} \Gamma^2(w) \right] - \frac{\hbar^2\kappa^2}{4(e+1)m r_0 r}. \]
(99)
where, for the potential term $(-\alpha/r)$, we used the relation (79) with $\nu = 1$. The Schrödinger equation now amounts to the condition
\[ S \equiv S_R + i S_I = i\hbar \left[ i c' - U(f^*)'/f^* - (1/2)r \Gamma' \right] - [H\psi]/\psi. \] 
(100)

We have the two real unknown functions $w_t$ and $c'$ to solve the two equations $S_R = 0$ and $R_I = 0$.

To split $S$ into real and imaginary parts, we introduce the following abbreviations
\[ U = U_R + i U_I, \quad \Gamma = \Gamma_R + i \Gamma_I, \quad \Gamma' = \Gamma'_R + i \Gamma'_I, \quad (f^*)'/f^* = F_R + i F_I \] 
(101)
and obtain
\[ S_I = -\hbar/(4mr) \left[ \hbar A + 2mr B w_t \right], \quad A = U_R \Gamma_R + \Gamma_I U_R - r \Gamma_I \Gamma_R, \quad B = 2(F_R U_R - F_I U_I) + r \Gamma'_R; \] 
(102)
\[ S_R = \hbar \left[ 8(1 + e)mr r_0 \right]^{-1} \left[ C w_t + D w_t c' + E \right], \quad C = 4(1 + e)mr r_0 \left[ \Gamma'_I R + 2U_I F_R + 2U_R F_I \right], \quad D = -8(1 + e)mr r_0, \quad E = \hbar \left\{ (1 + e)r_0 \left[ 2U_I \Gamma_I - 2U_R \Gamma_R + r \left( \Gamma'_R - \Gamma'_I \right) \right] + 2\kappa^2 \right\}. \]

From (102), we derive
\[ w_t = -\frac{\hbar A}{2mr B} = \frac{\hbar \kappa}{4mr r_0 \sqrt{e - 1}}. \] 
(104)
Eq. (104) is based on the following auxiliary formulas where we use the abbreviations (60j) for $Z_w$ and $Z_e$:

\[
U_R = c_U \left[ \sqrt{e - 1} \cos(\varphi/2) \cosh(w) + \sqrt{e + 1} \sin(\varphi/2) \sinh(w) \right],
\]
\[
U_I = c_U \left[ \sqrt{e - 1} \sin(\varphi/2) \cosh(w) - \sqrt{e + 1} \cos(\varphi/2) \sinh(w) \right],
\]
\[
c_U = \kappa Z_w \sqrt{r/r_0} \sqrt{(e - 1)} \sin(\theta);
\]
\[
\Gamma_R = (\kappa/r_0) Z_w (e - 1), \quad \Gamma_I = -(\kappa/r_0) Z_w e \sqrt{e - 1} \sinh(2w);
\]
\[
\Gamma'_R = -2(\kappa/r_0) Z_w e (e - 1) \sinh(2w),
\]
\[
\Gamma'_I = -2(\kappa/r_0) Z_w^2 Z_e (e - 1) [e - \cosh(2w)];
\]
\[
F_R = e Z_w \sinh(2w), \quad F_I = Z_w/Z_e.
\]

To evaluate the quotient $A/B$, we write $A = A_0 + A_1 \cos(\varphi/2) + A_2 \sin(\varphi/2)$ and $B = B_0 + B_1 \cos(\varphi/2) + B_2 \sin(\varphi/2)$. Then, it turns out that

\[
A_0/B_0 = A_1/B_1 = A_2/B_2 = -\kappa/(2r_0) Z_e (e - 1),
\]

which implies that $A/B = A_0/B_0$. The dependence on the angles $\theta$ and $\varphi$ has dropped out completely.

With $w_t$ known, we find $c'(w)$ from $S_R = 0$. With the aid of the auxiliary formulas (105) to (108), one can prove that in $S_R$ the coefficients of $U_R$ and $U_I$ vanish. Thus, after inserting the result (104) into (103), we obtain

\[
c' = 1/(2\kappa) Z_e \left[ (1 + e) \Gamma_R^2 + 2\kappa^2 + (1 + e) r \left( r_0 \Gamma_R^2 + \sqrt{(e - 1)/(e + 1)} \Gamma'_R \right) \right].
\]

Further evaluation with the aid of (106) and (107) leads to

\[
c' = (\kappa/2) Z_e [(1 - e)(r/r_0) + 2].
\]

The results (104) for $w_t$ and (111) for $c'$ are now approximated by confining $r$ to the neighborhood of the probability maximum by setting

\[
r = r_m + \epsilon \delta r, \quad \epsilon = 1/\sqrt{K}.
\]

With the aid of (40), we arrive at

\[
w_t = \frac{1}{2} \frac{N_1}{e \cosh(2w) - 1} \left[ 1 + O(1/\sqrt{K}) \right], \quad N_1 = \frac{\hbar \kappa}{2m r_0^2} \frac{(e - 1)^{3/2}}{\sqrt{e + 1}},
\]

which is the result (86) for $\nu = 1$ and $W = 2w$. Furthermore,

\[
c'(w) = -(\kappa/2) Z_e (e \cosh(2w) - 3) \left[ 1 + O(1/\sqrt{K}) \right],
\]

which after integration, with the initial value $c(0) = 0$ and to leading order, gives the result

\[
c(w) = \kappa Z_e [(3/2)w - (e/4) \sinh(2w)].
\]
8 Numerical examples

Let us consider an artificial satellite with mass $m = 10^3$ kg and initial data $r_0 = 4 \times 10^4$ km and $v_0 = 5000$ m/s. One finds from (76), with $\nu = 1$, the parameter $\kappa \approx 10^{48}$ which, for the probability density $|\psi_0|^2$, gives rise to the half width $\Delta_\kappa \equiv r_0/\kappa \approx 10^{-41}$ m and to the angular momentum quantum number $l \approx 10^{24}$.

8.1 Escape velocity

In the gravitational case with $\alpha = GmM$, where $M = 5.979 \times 10^{24}$ kg is the earth mass and $G = 6.673 \times 10^{-11}$ m$^3$kg$^{-1}$s$^{-2}$ the gravitational constant, the eccentricity, as calculated from (80), is given by $e = r_0v_0^2/(GM) - 1 = 1.5064$. By (80) and at the earth surface with $r_0 = R_E = 6.37812 \times 10^3$ km, the parabolic limit with $e = 1$ comes out at the velocity $v_0 = 11.185$ km/s, which is the well known escape velocity from the earth surface.

8.2 Quantum diffusion

For the given example, we use (88) to calculate the time constant $T = 1/N \approx 10$ hour. We introduce the dimensionless time $\zeta$ as

$$\zeta = t/T, \quad T = \sqrt{ma^3/\alpha}. \quad (116)$$

The hyperbolic "Kepler equation" (87), with $t_0 = 0$ and $W_0 = 0$, is then approximately solved by

$$W \approx \log[(2/e)\zeta], \quad \zeta \gg 1. \quad (117)$$

Let us estimate the mean square deviation $(\Delta x)^2 = (\Delta y)^2$ for large time numbers $\zeta$. We use the properties

$$\sinh[W] \to \cosh(W) \to (1/2) \exp[W] \to \zeta/e, \quad K_2 \to \zeta. \quad (118)$$

From (63) we derive for $\nu = 1$:

$$(\Delta x)^2 \to \frac{r_0h}{mv_0(e-1)^2}\zeta^2[1 + O(1/\zeta)], \quad e > 1, \quad \zeta \gg 1. \quad (119)$$

Asking for a mean fluctuation $|\Delta x|$ of one meter, we obtain for the above example $\zeta_1 \approx 10^{16}$ which approximately corresponds to $2 \times 10^{13}$ year. We have to check, whether the dynamic magnitude parameter $K = \kappa h(w)$ is still large for $\zeta_1$. With $\nu = 1$, one finds from (60b) and (60a) that $h(w) \to (e-1)/\zeta_1 > 10^{-18}$ and, thus, $K = \kappa h(w) > 10^{30} \gg 1$, which evidently is within the classical approximation.
8.3 Parabolic limit

For the parabolic limit \( e \to 1 \), we have to scale \( W = \sqrt{e - 1} W' \). To lowest order with respect to \( (e - 1) \to 0 \), (87) amounts to

\[
\zeta_p = W' + (1/6)(W')^3, \quad \zeta_p = t/T_p, \quad T_p = r_0^{3/2} \sqrt{m/\alpha}
\]

(120)

which for large \( \zeta_p \) is approximated by

\[
W' \approx (6 \zeta_p)^{1/3}, \quad \zeta_p \gg 1.
\]

(121)

In the case of the satellite example above (\( v_0 \) has to replaced by the initial speed \( \sqrt{2\alpha/(mr_0)} \) for a parabolic orbit), one finds \( T_p \approx 3.5 \) hour. Furthermore, with the aid of (63) and after taking the limit \( e \to 1 \), a mean square deviation of 1 m would be observable after about \( 10^{21} \) year.

9 Analytic continuation to elliptic orbits

In the following, we continue the curve parameter \( W \equiv 2w \) into the complex plane by the map

\[
W = \sqrt{e - 1} W', \quad W' \in \mathbb{R}, \quad e \geq 0.
\]

(122)

The continuation proceeds from the hyperbolic to the elliptic side. We assume that the initial (apsidal) point \( P_0 \) is fixed together with its distance \( r_0 \) from the force center, which is origin of our cartesian coordinate system. The initial velocity is perpendicular to the apsidal line. As was mentioned in Sec. I, the connection between \( r_0 \) and the semi-major axis \( a \) are \( r_0 = a|e - 1| \) if \( v_0 \geq v_c \), where \( v_c \) is the initial speed for a circular orbit, and \( r_0 = a(e + 1) \) if \( v_0 < v_c \).

By (80), the speed \( v_c \), where \( e = 0 \), is given by

\[
v_c^2 = \alpha/(mr_0).
\]

(123)

9.1 Elliptic orbits, Kepler’s equation, energy conservation

We will confine ourselves, at first, to initial speeds with \( v_0 \geq v_c \), where \( P_0 \) coincides with the peri-center. We start from (32) and use \( a = r_0/(e - 1) \) to write, for \( e > 1 \) and \( W' \in \mathbb{R} \),

\[
x = \frac{r_0}{e - 1} \left[ e - \cosh \left( \sqrt{e - 1} W' \right) \right], \quad y = \frac{r_0}{e - 1} \sqrt{e^2 - 1} \sinh \left( \sqrt{e - 1} W' \right).
\]

(124)

The above equations, differing by a re-scaled curve parameter, describe the same hyperbola as (32), of course. Now, let us assume the eccentricity in the
interval \(0 \leq e < 1\), keeping \(W'\) real. Then, since \(\cosh(iW) = \cos(W)\) and \(\sinh(iW) = i\sin(W)\), where now \(W = \sqrt{1 - eW'}\) and \(W'\) real, we obtain:

\[
x = a [\cos(W) - e], \quad y = a\sqrt{1 - e^2} \sin(W), \quad a = r_0/(1 - e),
\]

(125) which is the parameter representation of an ellipse. In the limit \(e \to 1\), one finds the same parabola both from the hyperbolic and the elliptic side, namely

\[
x = r_0(1 - W'^2/2), \quad y = r_0\sqrt{2} W', \quad \text{or} \quad x = r_0 \left(1 - y^2/(4r_0^2)\right).
\]

(126)

The above continuation also produces Kepler’s equation for elliptic orbits. To see this, we write (88) in the form

\[
N \equiv \frac{1}{a} \sqrt{\frac{\alpha}{ma}} = \frac{e - 1}{r_0} \sqrt{\frac{\alpha(e - 1)}{mr_0}}, \quad e > 1.
\]

(127) Going to the elliptic region with \(0 \leq e < 1\), we obtain

\[
N \to -i N', \quad N' = \frac{1 - e}{r_0} \sqrt{\frac{\alpha(1 - e)}{mr_0}} = \sqrt{\frac{\alpha}{ma^3}}, \quad a = r_0/(1 - e).
\]

(128) In (87), the imaginary unit cancels out and, setting \(W_0 = 0\) at \(t_0 = 0\), we obtain Kepler’s equation

\[
N' t = W - e \sin(W), \quad 0 \leq e < 1.
\]

(129) For the mean distance \(r_m\) and the mean velocity components \(\langle v_{x,y}\rangle_0\) which are given in (40) and (65), respectively, we find the analytically continued expressions, to leading order in \(\dot{K}\),

\[
r_m = a \left[1 - e \cos(W)\right], \quad \langle v_x \rangle_0 = -v_0 \sqrt{\frac{1 - e}{1 + e}} \frac{\sin(W)}{1 - e \cos(W)}, \quad \langle v_y \rangle_0 = v_0 (1 - e) \frac{\cos(W)}{1 - e \cos(W)},
\]

\[
a = r_0/(1 - e), \quad 0 \leq e < 1, \quad v_c \leq v_0 \leq \sqrt{2} v_c.
\]

(130) From (130), we check the energy conservation to leading order in \(\dot{K}\):

\[
E = (m/2) \left[\langle v_x \rangle_0^2 + \langle v_y \rangle_0^2\right] - \alpha/r_m = -\alpha/(2a).
\]

(131) Setting \(a = r_0/(1 - e)\), we obtain

\[
E + \alpha(1 - e)/(2r_0) = F_1 F_2, \quad F_1 = (1 - e)Z_w (1 + e \cos(W)) [8mr_0^2(1 + e)]^{-1}, \quad F_2 = 4(1 + e)mr_0\alpha - (\hbar\nu)^2.
\]

(132)
The above balance vanishes identically in $W$, if $F_2 = 0$ which is equivalent to (80) with (76) and gives the same result for the eccentricity:

$$e = mr_0 v_0^2 / \alpha - 1. \tag{133}$$

Obviously, in order that $0 \leq e < 1$, the initial speed must be confined to the interval $v_c \leq v_0 < \sqrt{2} v_c$.

What happens if $v_0 < v_c$? Then, the eccentricity goes through zero, and we obtain the continued components from (125) and (130) by changing, at fixed $r_0$, $e$ into $-e$ which leads to

$$x = a(\cos(W) + e), \quad y = a\sqrt{1 - e^2} \sin(W), \quad r_m = a [1 + e \cos(W)],$$

$$\langle v_x \rangle_0 = -v_0 \sqrt{\frac{1 + e}{1 - e}} \frac{\sin(W)}{1 + e \cos(W)}, \quad \langle v_y \rangle_0 = v_0 (1 + e) \frac{\cos(W)}{1 + e \cos(W)},$$

$$a = r_0 / (1 + e), \quad 0 \leq e < 1, \quad v_0 < v_c. \tag{134}$$

As a check, the energy conservation is fulfilled with

$$E + \frac{\alpha (1 + e)}{2r_0} = G_1 G_2, \quad G_1 = \frac{(1 + e)(e \cos(W) - 1)}{8mr_0^2(e - 1)(e \cos(W) + 1)},$$

$$G_2 = 4(e - 1)mr_0 \alpha + (\hbar \kappa \nu)^2. \tag{135}$$

The factor $G_2$ vanishes, if

$$e = 1 - (\hbar \kappa \nu)^2 / (4mr_0 \alpha) = 1 - mr_0 v_0^2 / \alpha, \quad v_0 \leq v_c. \tag{136}$$

At zero speed, where $e = 1$, the classical orbit is rectilinear with the angular momentum $L = 0$.

### 9.2 Analytic continuation of the mean square deviations

We substitute $w \to i w$ or, equivalently, $W \to iW$. As compared to the definition of mean square deviations by means of expectation values, which cannot be negative by definition, the derivation by analytic continuation should be checked for positivity in the eccentricity interval $0 \leq e < 1$. In the following, we examine the mean square deviations of the position components; for the velocity components we refer to Appendix E. From (63), one infers for $0 \leq e < 1$:

$$(\Delta x)^2 \to \frac{2r_0^2}{\kappa} (K_2)_{w \to iw} \frac{e \cos(2w) - 1}{(e - 1)^2 \nu^2} [1 + O(1/K)],$$

$$(\Delta y)^2 = (\Delta x)^2,$$

$$(\Delta z)^2 \to \frac{2r_0^2}{\kappa} (K_1 K_2)_{w \to iw} \frac{(K_1 K_2)_{w \to iw}}{(e - 1)^2 \nu^2 (1 + \nu^2)} [1 + O(1/K)]. \tag{137}$$
In the interval $0 \leq e < 1$, the factor $e \cos(2w) - 1$ in the expression of $(\Delta x)^2$ is negative. It remains to be shown that $(K_2)_{w \to iw} < 0$, which immediately follows from the definition in (60e), provided $0 \leq e < 1$. Also $(K_1)_{w \to iw} < 0$ is easily verified, so that the mean square deviations of all three components $x, y, z$ are larger than zero in the elliptic region.

From (66) we derive the following analytic continuations of the mean square deviations of the velocity components, if $0 \leq e < 1$,

$$(\Delta v_x)^2 \rightarrow -\kappa F_0 \frac{(1 - e)^2}{[1 - e \cos(2w)]^3} \sum_{j=0}^{3} c_{2j} \cos(2jw) \left[1 + O(1/K)\right],$$

$$(\Delta v_y)^2 \rightarrow \kappa F_0 \frac{1 - e}{(e + 1)[1 - e \cos(2w)]^3} \sum_{j=0}^{3} d_{2j} \cos(2jw) \left[1 + O(1/K)\right],$$

$$(\Delta v_z)^2 \rightarrow -\kappa F_0 \frac{1 - e}{(1 + \nu^2)(e + 1) [1 - e \cos(2w)]^2} \left[f_0 + f_4 \cos(4w)\right] \left[1 + O(1/K)\right]. \quad (138)$$

In Appendix E, we prove that the above mean square deviations, which were obtained by analytic continuations, are positively definite in the three-dimensional parameter space ($w \in \mathbb{R}$, $0 \leq e < 1$, $\nu > 0$).

## 10 Conclusions

We have demonstrated that the Schrödinger equation of the hydrogen atom, in the classical limit, includes as solutions the Kepler orbits. The elliptic, hyperbolic, and parabolic orbits are obtained from a single wave function in the limit of large quantum numbers of the angular momentum; exceptions are rectilinear and nearby orbits with small angular momentum. The paper unifies and supplements existing results which were based on different methods for elliptic and hyperbolic orbits, respectively. The properties of elliptic orbits were obtained by analytical continuation of the results of the hyperbolic case. The orbits evolve depending on a curve parameter $w$ which, in the elliptic case, corresponds to half of the eccentric anomaly. At zero time, the initial wave function is characterized by a nearly minimum uncertainty product, up to the factor $\sqrt{1 + \nu^2}$, where $\nu > 0$ is a disposable constant. Time is introduced by the assumption that it enters only through the curve parameter as $w = w(t)$. This assumption is justified in the classical limit. The orbits are subject to quantum diffusion which, however, is hardly observable for macroscopic examples like artificial satellites within realistic time spans.

**Appendix A: Proof of mean value lemma (43)**

The following proof is based on the series expansion of $G_2$ according to (34), (35), and (45). Since $D$ does not depend on $r$, by substituting $r = r_m + \epsilon \delta r$
with $\epsilon^2 = 1/K$, we derive from (34):

$$G_2 = KD^2(\theta, \varphi) - E_r \delta r^2 + O(\delta \epsilon), \quad E_r = 1/(4r_0 r_m). \quad (A1)$$

From (22), (34), and (60a), we obtain, by using $\theta_m = \pi/2$,

$$D(\theta_m, \varphi) = \cos(\varphi/2) \cosh(w) + \gamma_0 \nu \sin(\varphi/2) \sinh(w). \quad (A2)$$

The explicit forms of $E_0 = D^2(\theta_m, \varphi_m)$ and $E_r$ are stated in (46).

Next, we expand with respect to $\theta$ and $\varphi$ up to second order in $\epsilon$. To this end, we make use of the following relations:

\[
d(\theta_m, \varphi) = \sqrt{2} \left[ \cos(\varphi/2) + i \nu \sin(\varphi/2) \right],
d_\theta d(\theta_m, \varphi) = d_\varphi d(\theta_m, \varphi) = 0,
\]

\[
d_\varphi d(\theta_m, \varphi) = (1/\sqrt{2}) \left[ i \nu \cos(\varphi/2) - \sin(\varphi/2) \right],
\]

\[
d_\theta d_\varphi d(\theta_m, \varphi) = (1/8) \left[ (1 + \nu^2) \cos(\varphi) + 2i \nu \sin(\varphi) \right] / \left[ d_\varphi d_\varphi d(\theta_m, \varphi) \right],
\]

\[
d_\varphi d_\varphi d(\theta_m, \varphi) = -\sqrt{2/4} \left[ \cos(\varphi/2) + i \nu \sin(\varphi/2) \right]. \quad (A3)
\]

Furthermore, by (A2),

$$\partial_\varphi D(\theta_m, \varphi)_{\varphi=\varphi_m} = 0, \quad (A4)$$

which is a consequence of (37). From (A1) and the first equation of (46), the expansion coefficients of $G_2(r_m, \theta, \varphi)$ are determined at $(\theta_m, \varphi_m)$ as

$$E_0 = D^2, \quad E_\theta = -DD_{\theta\theta}, \quad E_\varphi = -DD_{\varphi\varphi}, \quad (A5)$$

which, with the aid of (A3) and the conditions (37) and (41), are evaluated to give the results stated in (46).

Now, let us explicitly write down the mean value as

$$\overline{F(r)} = C^2 \int_0^\infty dr r^2 \int_0^\pi d\theta \sin(\theta) \int_0^{2\pi} d\varphi F(r) \frac{2\pi}{\sqrt{UU^*}} \exp[G_2] \quad (A6)$$

In view of the definition of $U$ defined by (26) and (21), we introduce the amplitude function $F_1$,

$$F_1(r) = \beta r^2 \sin(\theta) F(r) \left[ r d f(w) d^* f^*(w) \right]^{-1/2}, \quad \beta = 2\sqrt{2\pi} \sqrt{r_0}/K, \quad (A7)$$

and transform the integration variables $(r, \theta, \varphi) \to (\delta r, \delta \theta, \delta \varphi)$ according to (45). Then, for large $K$, the integration limits can be replaced by $(-\infty, \infty)$ which amounts to neglect exponentially small terms of order $\exp[-K]$. The exponent $G_2$ of the integrand has the form given in (46) in terms of $\delta r, \delta \theta, \delta \varphi$.

We expand $F_1$ as

$$F_1(r) = F_1(r_m + \epsilon \delta r, \theta_m + \epsilon \delta \theta, \varphi + \epsilon \delta \varphi) = P_\epsilon F_1(r_m, \theta_m, \varphi_m),$$

$$P_\epsilon = 1 + f_1 \epsilon + f_2 \epsilon^2 + \ldots \quad (A8)$$
and obtain with the aid of (46) and to leading order in $K$

$$\bar{F}(r) = \frac{C^2}{K^{3/2}} F_1(r_m, \theta_m, \varphi_m) \int_{-\infty}^{\infty} (d\delta r) (d\delta \theta) (d\delta \varphi) \exp \{G\}$$

$$= C^2 \pi^{3/2} \left[ K^3 E_r E_\theta E_\varphi \right]^{-1/2} F_1(r_m, \theta_m, \varphi_m) \exp [KE_0], \quad (A9)$$

where we used the fact that the linear $\epsilon$ term of $P_\epsilon$ drops out by parity; the term $\epsilon^2 f_2$ of $P_\epsilon$ is consumed in the error term of relative order $1/K$. We use (A9) and the definition (A7) of $F_1$ to define the normalization constant $C^2$ by the following condition

$$1 = \beta C^2 (r_m \pi)^{3/2} \left[ K^3 d_m f f^* d_m E_r E_\theta E_\varphi \right]^{-1/2} \exp [KE_0], \quad (A10)$$

where $d_m = d(\theta_m, \varphi_m)$. We, thus, arrive at Lemma (43).

In order to show the equivalence of (A10) with the normalization condition (58), we further evaluate (A10) with the aid of (38), (40), (41), (46), and the definition (22) of $d$. We obtain

$$E_r E_\theta E_\varphi = \frac{(\nu^2 + 1) (e \cosh(2w) - 1)^2}{64(e - 1)K_1 r_0^2}, \quad (A11)$$

$$d_m f(w) d^*_m f^*(w) = 2Z_w K_1 K_2 [(e - 1)\nu^2]^{-1} \quad (A12)$$

with $K_1$ and $K_2$ defined in (60e) and $Z_w$ in (60j). After inserting (A11) and (A12) into (A10), we obtain the following normalization condition to zero order with respect to $1/K$

$$1 = C^2 \exp \left[ K \frac{e \cosh(2w) - 1}{e - 1} \right] \frac{16\pi^{3/2} r_0^3 \nu (e \cosh(2w) - 1)}{\sqrt{e - 1} K^{5/2} \sqrt{K_2} \sqrt{1 + \nu^2}}. \quad (A13)$$

On the other hand, we have to insert in (58) the explicit expressions of $\kappa_0$ and $\kappa_1$, as defined in (60f), in order to reproduce the 3D method result (A13).

**Appendix B: Mean values of position variables**

The mean values will be calculated in 4D space to leading and next higher approximation with respect to $K$. The integration method is outlined in Sec. IV. B and exemplified in Sec. IV. C. The $u$ dependence of the position and velocity components $x_i$ and $v_i$ is given in (6) and (12) with (10), respectively.

**B.1 Mean x component**
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\[ \bar{x} = C^2 \int_{0}^{2\pi} d\Phi' \int_{-\infty}^{\infty} (8u^2) du_1 ... du_4 2(u_1 u_3 - u_2 u_4) \exp \left[ \mathbf{A} \cdot \mathbf{u} - \Gamma u^2 \right] \]

\[ = -16C^2 \frac{\partial}{\partial \Gamma} \int_{0}^{2\pi} d\Phi' \int_{-\infty}^{\infty} du_1 ... du_4 \left[ \frac{\partial^2}{\partial A_1 \partial A_3} - \frac{\partial^2}{\partial A_2 \partial A_4} \right] \exp \left[ \mathbf{A} \cdot \mathbf{u} - \Gamma u^2 \right] \]

\[ = -16\pi^2 C^2 \frac{\partial}{\partial \Gamma} \int_{0}^{2\pi} d\Phi' \left[ \frac{\partial^2}{\partial A_1 \partial A_3} - \frac{\partial^2}{\partial A_2 \partial A_4} \right] \frac{1}{\Gamma^2} \exp \left[ \mathbf{A} \cdot \mathbf{A}/(4\Gamma) \right] \]

\[ = -4\pi^2 C^2 \frac{\partial}{\partial \Gamma} \frac{1}{\Gamma^4} \int_{0}^{2\pi} d\Phi' \left[ A_1 A_3 - A_2 A_4 \right] \exp \left[ \mathbf{A} \cdot \mathbf{A}/(4\Gamma) \right], \quad (B1) \]

where the vector \( \mathbf{A} \) is defined in (51) and (52). We obtain

\[ A_1 A_3 - A_2 A_4 = \Gamma^2 r_0 [\kappa_2 + \kappa_3 \cos(\Phi' - \Phi)] \quad (B2) \]

with \( \kappa_2 \) and \( \kappa_3 \) defined in (60f). The \( \Phi' \) integral in (B1) does not depend on \( \Phi \) and can be expressed by modified Bessel functions as follows

\[ \bar{x} = -8C^2 \pi^3 \frac{\partial}{\partial \Gamma} \exp \left[ k_0/\Gamma \right] \frac{r_0}{\Gamma^4} \left[ \kappa_2 I_0(k_1/\Gamma) + \kappa_3 I_1(k_1/\Gamma) \right] \]

\[ = C^2 4\pi^3 r_0^4 K^{-3} \exp[K\kappa_0] \left\{ [2(4 + K\kappa_0)\kappa_2 + K\kappa_1\kappa_3] I_0(K\kappa_1) + 2[K\kappa_1\kappa_2 + 4\kappa_3 + K\kappa_0\kappa_3] I_1(K\kappa_1) + K\kappa_1\kappa_3 I_2(K\kappa_1) \right\}, \quad (B3) \]

where, after differentiation with respect to \( \Gamma \), we replaced \( \Gamma \) by \( \Gamma_R \) and then \( \Gamma_R \) by \( K/r_0 \).

We approximate the Bessel functions by the asymptotic form (56) with (57) to obtain the two leading orders with respect to \( K \)

\[ \bar{x} = 4\sqrt{2} C^2 r_0^4 \left[ \pi/(K\kappa_1) \right]^{5/2} \exp[K(k_0 + \kappa_1)] \kappa_1^2(k_0 + \kappa_1)(\kappa_2 + \kappa_3) [1 + \delta \xi/K] \]

\[ = (1/2)r_0(\kappa_2 + \kappa_3)[1 + \xi/K], \quad (B4) \]

where, for the latter equation, we made use of the normalization condition (58) and (59) which gives rise to

\[ \delta \xi = \frac{-\kappa_0\kappa_2 + 29\kappa_1\kappa_2 - 3\kappa_0\kappa_3 + 25\kappa_1\kappa_3}{8\kappa_1(\kappa_0 + \kappa_1)(\kappa_2 + \kappa_3)}. \quad (B5) \]

With (B4), (B5) and by the definition \( \xi = \delta \xi - \delta n \), we arrive at the result given in (61) and (62).

B.2 Mean y component

In analogy to (B2), the amplitude function comes out as

\[ A_2 A_3 + A_1 A_4 = r_0 \Gamma^2_R \kappa_6, \quad (B6) \]
where $\kappa_6$ is defined in (60f). The amplitude factor (B6) does not depend on the phase $\phi = \Phi - \Phi'$. Analogously to (B4) and (B5), we obtain, after using $\Gamma_0 h(w) = K/r_0$,

$$
\bar{y} = 4\sqrt{2} C^2 r_0 \left[ \pi/(K\kappa_1) \right]^{5/2} \kappa_2^2 (\kappa_0 + \kappa_1) \kappa_6 \exp \left[ K(\kappa_0 + \kappa_1) \right] [1 + \delta \eta/K],
$$

$$
\delta \eta = [\kappa_0 + 19 \kappa_1] / [8 \kappa_1 (\kappa_0 + \kappa_1)].
$$

(B7)

We exploit the normalization condition (58) and (59) to obtain

$$
\bar{y} = (1/2)r_0\kappa_6 \left[ 1 + \eta/K + O(1/K^2) \right],
$$

$$
\eta = (\delta \eta - \delta n) = 2/(\kappa_0 + \kappa_1),
$$

(B8)

as stated in (61) and (62).

**B.3 Mean z component**

In analogy to (B2), the amplitude function reads

$$
A_1^2 + A_2^2 - A_3^2 - A_4^2 = 4i r_0 \Gamma R^2 2\nu/(1 + \nu^2) \kappa_1 \sin(\phi).
$$

(B9)

The function $\kappa_1$ is defined in (60f) and does not depend on the phase $\phi = \Phi - \Phi'$. Thus, the phase integral vanishes identically implying $z = 0$:

$$
\int_0^{2\pi} d\phi \sin(\phi) \exp \left[ k_0 + k_1 \cos(\phi) / \Gamma_R \right] = -\frac{\Gamma R}{k_1} \int_0^{2\pi} d\phi \frac{d}{d\phi} \exp \left[ k_0 + k_1 \cos(\phi) / \Gamma_R \right] = 0.
$$

(B10)

**B.4 Mean square deviation of x**

Analogously to (B1), we have to evaluate

$$
\overline{x^2} = C^2 \int_0^{2\pi} d\Phi' \int_{-\infty}^{\infty} (8u^2) du_1 ... du_4 4(u_1 u_3 - u_2 u_4)^2 \exp \left[ A \cdot u - \Gamma u^2 \right]
$$

$$
= -32C^2 \frac{\partial}{\partial \Gamma} \int_0^{2\pi} d\Phi' \int_{-\infty}^{\infty} du_1 ... du_4 \left[ \frac{\partial^2}{\partial A_1 \partial A_3} - \frac{\partial^2}{\partial A_2 \partial A_4} \right] \exp \left[ A \cdot u - \Gamma u^2 \right]
$$

$$
= -32\pi^2 C^2 \frac{\partial}{\partial \Gamma} \int_0^{2\pi} d\Phi' \left[ \frac{\partial^2}{\partial A_1 \partial A_3} - \frac{\partial^2}{\partial A_2 \partial A_4} \right] \frac{1}{\Gamma^2} \exp \left[ A \cdot A/(4\Gamma) \right]
$$

$$
= -32\pi^2 C^2 \frac{\partial}{\partial \Gamma} \frac{1}{16\Gamma^6} \int_0^{2\pi} d\Phi' \left[ -2A_1 A_2 A_3 A_4 + A_1^2 (A_3^2 + 2\Gamma) + A_2^2 (A_4^2 + 2\Gamma) + 2\Gamma (A_3^2 + A_4^2 + 4\Gamma) \right] \exp \left[ A \cdot A/(4\Gamma) \right].
$$

(B11)

After differentiation with respect to $\Gamma$, we set $\Gamma = \Gamma_R$ and substitute for $A_1, ... A_4$ the explicit expressions which are defined through (51) and (52).
With the abbreviation $\phi = \Phi' - \Phi$ we write

\[
\bar{x}^2 = C^2 \pi^2 r_0^5/(8K^5) \exp[K\kappa_0] \times \int_0^{2\pi} d\phi \left[ c_0 + c_1 \cos(\phi) + c_2 \cos(2\phi) + c_3 \cos(3\phi) \right] \exp[K\kappa_1 \cos(\phi)],
\]

\[
c_0 = 512 + K^3(1 - \nu^2)(1 + \nu^2)^2 \left[ f^6 + (f^*)^6 \right] + 192K(1 - \nu^2) \left[ f^2 + (f^*)^2 \right] + 32K^2(1 + \nu^2 + \nu^4) \left[ f^4 + (f^*)^4 \right] + 128K^2(1 + \nu^4) (ff^*)^2 + K^3(9 + \nu^2 - \nu^4 - 9\nu^6) \left[ f^2(f^*)^4 + f^4(f^*)^2 \right],
\]

\[
c_1 = 2K(1 + \nu^2) ff^* \left[ 192 + 64K(1 - \nu^2) \left[ f^2 + (f^*)^2 \right] + K^2(3 - 2\nu^2 + 3\nu^4) \times \left[ f^4 + (f^*)^4 \right] + K^2(9 - 10\nu^2 + 9\nu^4) (f^*)^2 \right],
\]

\[
c_2 = -2K^2 (ff^*)^2 \left[ -32(1 - \nu^2 + \nu^4) + K(-3 + \nu^2 - \nu^4 + 3\nu^6) \left[ f^2 + (f^*)^2 \right] \right],
\]

\[
c_3 = 2K^3(1 - \nu^2)^2(1 + \nu^2) (f^*)^3.
\]

The $\phi$ integration leads to modified Bessel functions:

\[
\int_0^{2\pi} d\phi \cos(n\phi) \exp[K\kappa_1 \cos(\phi)] = 2\pi I_n(K\kappa_1), \quad n = 0, 1, \ldots \quad (B13)
\]

In view of $K \gg 1$, we make use of the zero and first order asymptotic approximations (56) and (57) to obtain

\[
\bar{x}^2 = F \left[ (9 + 16K\kappa_1 + 128K^2\kappa_1^2) c_0 + (-15 - 48K\kappa_1 + 128K^2\kappa_1^2) c_1 + (105 - 240K\kappa_1 + 128K^2\kappa_1^2) c_2 + (945 - 560K\kappa_1 + 128K^2\kappa_1^2) c_3 \right],
\]

\[
F = C^2(r_0^2 \pi)^{5/2} \left[ 512\sqrt{2}K^5 \right]^{-1} [K\kappa_1]^{-5/2} \exp[K(\kappa_0 + \kappa_1)].
\]

We now insert the coefficients $c_0$ to $c_3$ from (B12), and, in order to take into account normalization, we divide the right hand side of (B14) by the right hand side of (58). The result is given by the two leading orders in $K$ as

\[
\bar{x}^2 = \langle x^2 \rangle_0 + (1/K) \langle x^2 \rangle_1 + O(1/K^2);
\]

\[
\langle x^2 \rangle_0 = \langle x^2 \rangle_0 + (1/K) \langle x^2 \rangle_1 + O(1/K^2);
\]

\[
\langle x^2 \rangle_1 = F_1 \left[ 128\kappa_1(\kappa_0 + \kappa_1)(3 - \nu^2 + 3\nu^4) (ff^*)^2 + 256\kappa_1(\kappa_0 + \kappa_1)(1 - \nu^4) \left[ f(f^*)^3 + f^3 f^* \right] + 64\kappa_1(\kappa_0 + \kappa_1)(1 + \nu^2 + \nu^4) \times \left[ f^4 + (f^*)^4 \right] + (\nu^2 - 1) [8\kappa_0(3 + 2\nu^2 + 3\nu^4) + \kappa_1(69 + 58\nu^2 + 69\nu^4)] \left[ f^4 + (f^*)^4 \right] + 3(\nu^2 - 1)(1 + \nu^2)^2 [f^6 + (f^*)^6] - 2(\kappa_0 + 4\kappa_1)(3 + \nu^2 + \nu^4 + 3\nu^6) \left[ f^5 f^* + f(f^*)^5 \right] - 4(1 + \nu^2) [8\kappa_1(3 - 4\nu^2 + 3\nu^4) + \kappa_0(9 - 14\nu^2 + 9\nu^4)] (ff^*)^3 \right],
\]

\[
F_1 = r_0^2 \left[ 128\kappa_1(\kappa_0 + \kappa_1)^2 \right]^{-1}.
\]
We simplify (B16) by expressing \( f^2 + (f^*)^2 \) and \( ff^* \) in terms of \( \kappa_0 \) and \( \kappa_1 \), respectively, see (60f)and (60b); furthermore, we exploit the connections \( \kappa_2 = \text{const.} \kappa_0 \) and \( \kappa_3 = \text{const.} \kappa_1 \) specified in (60e) to write

\[
\langle x^2 \rangle_0 = (r_0^2/4)(\kappa_2 + \kappa_3)^2, \tag{B18}
\]

which tells that \( \langle x^2 \rangle_0 = (\langle x \rangle_0)^2 \), see (62), and that the mean square deviation \( (\Delta x)^2 \) is of order \( 1/K \):

\[
(\Delta x)^2 = 1/K(\Delta x)_0^2 + O(1/K^2), \quad (\Delta x)_0^2 = \langle x^2 \rangle_1 - 2(\langle x \rangle_0)^2 \xi, \tag{B19}
\]

where \( \xi \) is defined in (62). With the aid of (B17) and (62), we find

\[
(\Delta x)_1^2 = F_1 \{ 32(\kappa_0 + \kappa_1)(\kappa_2 + \kappa_3)(\kappa_0\kappa_3 - 4\kappa_1\kappa_2 - 3\kappa_1\kappa_3) + (5 + 6\nu^2 - 6\nu^4 - 5\nu^8) [f^7 f^* + f (f^*)^7] + 2(15 + 20\nu^2 - 6\nu^4 + 20\nu^6 + 15\nu^8) [f^6(f^*)^2 + f^2(f^*)^6] + (75 + 106\nu^2 - 106\nu^4 - 75\nu^8) [f^5(f^*)^3 + f^3(f^*)^5] + 4(25 + 36\nu^2 + 6\nu^4 + 36\nu^6 + 25\nu^8)f^4(f^*)^4 \}.
\]

To simplify (B20), we first substitute \( \kappa_0, \ldots \kappa_3 \) in terms of \( f \) and \( f^* \) according to (60f) and obtain

\[
(\Delta x)^2 = -2(F_1/K)(1 + \nu^2) f f^* [(\nu^2 - 1) (f^2 + (f^*)^2) - 2(1 + \nu^2)f f^*]^3. \tag{B21}
\]

After substituting the function \( f \) according to (60a) and \( \gamma_0 \) by (41), we find to leading order in \( K \)

\[
(\Delta x)^2 = (F_1/K) K_2 128(1 + \nu^2) [(e - 1)^4\nu^2]^{-1} [e \cosh(2w) - 1]^3. \tag{B22}
\]

We substitute \( \kappa_0, \kappa_1, \) and \( K = \kappa h(w) \) with the result

\[
(F_1/K) = (e - 1)^2 r_0^2 Z_w^2 [64\kappa(1 + \nu^2)]^{-1}, \tag{B23}
\]

With (B22) and (B23), we arrive at the mean square deviation of \( x \) as stated in (63).

**B.5 Mean square deviation of \( y \)**

We have to evaluate

\[
\bar{y}^2 = C^2 \int_0^{2\pi} d\Phi' \int_{-\infty}^{\infty} (8u^2) du_1...du_4 4(u_1 u_4 + u_2 u_3)^2 \exp [A \cdot u - \Gamma u^2]
\]

\[
= -32C^2 \frac{\partial}{\partial \Gamma} \int_0^{2\pi} d\Phi' \int_{-\infty}^{\infty} du_1...du_4 \left[ \frac{\partial^2}{\partial A_1 \partial A_4} + \frac{\partial^2}{\partial A_2 \partial A_3} \right]^2 \exp [A \cdot u - \Gamma u^2]
\]

\[
= -32\pi^2 C^2 \frac{\partial}{\partial \Gamma} \int_0^{2\pi} d\Phi' \left[ \frac{\partial^2}{\partial A_1 \partial A_4} + \frac{\partial^2}{\partial A_2 \partial A_3} \right]^2 \frac{1}{\Gamma^2} \exp [A \cdot A/(4\Gamma)]
\]

\[
= -32\pi^2 C^2 \frac{1}{\Gamma^6} \int_0^{2\pi} d\Phi' \left[ (2A_1 A_2 A_3 A_4 + A_2^2 A_3^2 + 2\Gamma) + A_1^2 (A_4^2 + 2\Gamma) + 2\Gamma(A_3^2 + A_4^2 + 2\Gamma) \right] \exp [A \cdot A/(4\Gamma)]. \tag{B24}
\]
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After differentiation with respect to $\Gamma$, we set $\Gamma = \Gamma_R$ and substitute for $A_1, \ldots A_4$ the explicit expressions (51) and (52). With the abbreviation $\phi = \Phi' - \Phi$, we write

$$
\overline{y^2} = C^2 \pi^2 \frac{5}{(2K^5)} \int_0^{2\pi} d\phi \left[ c_0 + c_1 \cos(\phi) + c_2 \cos(2\phi) \right] \exp[K (\kappa_0 + \kappa_1 \cos(\phi))],
$$

$$
c_0 = 128 + 48K(1 - \nu^2) (f^2 + (f^*)^2) + 8K^2(1 + 6\nu^2 + \nu^4) (ff^*)^2 + 2K^2 (1 - 14\nu^2 + \nu^4) (f^4 + (f^*)^4) + K^3 \nu^2 (1 - \nu^2) (f^4 + (f^*)^4) + K^3 \nu^2 (\nu^2 - 1) (f^6 + (f^*)^6),
$$

$$
c_1 = -2K(1 + \nu^2) ff^* \left[ -48 + 2K^2 \nu^2 (f^4 + (f^*)^4) - 2K^2 \nu^2 (ff^*)^2 + 4K(\nu^2 - 1) (f^2 + (f^*)^2) \right], \quad c_2 = 4K^2 (1 + \nu^2) (ff^*)^2 .
$$

We integrate with respect to $\phi$ and use the asymptotic approximation of the Bessel functions including the first order correction

$$
\overline{y^2} \to F \left[ (9 + 16K\kappa_1 + 128K^2 \kappa_1^2) c_0 + (-15 - 48K\kappa_1 + 128K^2 \kappa_1^2) c_1 + (105 - 240K\kappa_1 + 128K^2 \kappa_1^2) c_2 \right],
$$

$$
F = r_0^5 C^2 \left[ 128\sqrt{2K^5} \right]^{-1} \exp[K(\kappa_0 + \kappa_1)] \left[ \pi/(K\kappa_1) \right]^{5/2} .
$$

We now insert the coefficients $c_0$ to $c_2$ from (B25) and, for normalization, divide the right hand side of (B27) by the right hand side of (58). The result is arranged by orders of $K$ with the result

$$
\overline{y^2} = \langle y^2 \rangle_0 + (1/K) \langle y^2 \rangle_1 + \mathcal{O}(1/K^2),
$$

$$
\langle y^2 \rangle_0 = r_0^2 \nu^2 \left( f^2 - (f^*)^2 \right) \left[ 16(\kappa_0 + \kappa_1) \right]^{-1} \left[ (\nu^2 - 1) (f^2 + (f^*)^2) - 2(1 + \nu^2) ff^* \right],
$$

$$
\langle y^2 \rangle_1 = F_1 \left\{ 3\kappa_1 \nu^2 (1 - \nu^2) (f^6 + (f^*)^6) + 2(\kappa_0 + 4\kappa_1) \nu^2 (1 + \nu^2) (f^5 f^* + f (f^*)^5) + 3\kappa_1 \nu^2 (\nu^2 - 1) (f^4 f^*)^2 + f^2 (f^*)^4 - 4(\kappa_0 + 4\kappa_1) \nu^2 (1 + \nu^2) (ff^*)^3 + 4\kappa_1 (\kappa_0 + \kappa_1) (1 - 14\nu^2 + \nu^4) (f^4 + (f^*)^4) + 8\kappa_1 (\kappa_0 + \kappa_1) (3 + 14\nu^2 + 3\nu^4) (f^6 f^*)^2 + 16\kappa_1 (\kappa_0 + \kappa_1) (1 - \nu^4) \right\} \times \left( f^3 f^* + f (f^*)^3 \right), \quad F_1 = r_0^2 \left[ 32\kappa_1 (\kappa_0 + \kappa_1)^2 \right]^{-1} .
$$

As it turns out, the zero order term amounts to $\langle y^2 \rangle_0 = \langle y \rangle_0^2$. As a consequence, the mean square deviation $(\Delta y)^2$ is of order $1/K$ with

$$
(\Delta y)^2 = 1/K \langle \Delta y \rangle_0^2 + \mathcal{O}(1/K^2), \quad \langle \Delta y \rangle_1^2 = \langle y^2 \rangle_1 - 2(\langle y \rangle_0)^2 \eta ,
$$

where $\eta$ is defined in (62). With the aid of (B28) and (62), we find

$$
(\Delta y)^2 = F_1 \left\{ 3\nu^2 (1 - \nu^2) (f^6 + (f^*)^6) + 2(5 f^5 f^* + f (f^*)^5) (\kappa_0 + 4\kappa_1) \nu^2 (1 + \nu^2) \right\} \times \left( f^3 f^* + f (f^*)^3 \right) (\kappa_0 + \kappa_1) (1 - \nu^4) + 8(ff^*)^2 (\kappa_0 + \kappa_1) (3 + 14\nu^2 + 3\nu^4) - 32(\kappa_0 + \kappa_1) \kappa_0^2 \right\} .
$$

(B30)
After substituting \( \kappa_0, \kappa_1 \), and \( \kappa_6 \) from (60f), we obtain
\[
(\Delta y)_1^2 = (1/2) r_0^2 \left[ (1 - \nu^2) f^2 + 2(1 + \nu^2) f f^* + (1 - \nu^2)(f^*)^2 \right]. \tag{B31}
\]
With the aid of the relation \( \kappa(e - 1) \nu^2 / K_2 = K \), the mean square deviation \( (\Delta y)^2 = (1/K)(\Delta y)_1^2 = (\Delta x)^2 \) as stated in (63).

**B.6 Mean square deviation of \( z \)**

We have to evaluate
\[
\overline{z^2} = C^2 \int_0^{2\pi} d\Phi'\int_{-\infty}^{\infty} (8u^2) du_1...du_4 \left( u_1^2 + u_2^2 - u_3^2 - u_4^2 \right)^2 \exp \left[ A \cdot u - \Gamma u^2 \right]
\]
\[
= -8\pi^2 C^2 \frac{\partial}{\partial T} \int_0^{2\pi} d\Phi' \left[ \frac{\partial^2}{\partial A_1^2} + \frac{\partial^2}{\partial A_2^2} - \frac{\partial^2}{\partial A_3^2} - \frac{\partial^2}{\partial A_4^2} \right]^2 \frac{1}{\Gamma^2} \exp \left[ A \cdot A/(4\Gamma) \right]
\]
\[
= (\pi^2/8)C^2 Q \exp \left[ A \cdot A/(4\Gamma) \right], \tag{B32}
\]
where
\[
Q = \frac{1}{\Gamma_R^4} \left\{ A_1^6 + A_2^6 + A_3^6 + 3A_3^4A_1^2 + 3A_3^2A_1^4 + A_1^6 - A_2^4(A_3^2 + A_4^2 - 32\Gamma_R) + 32A_3^2A_4^2\Gamma_R + 32A_2^4\Gamma_R + 192A_3^2\Gamma_R^2 + 192A_2^4\Gamma_R^2 + 512\Gamma_R^3 + A_1^4 \times (3A_2^2 - A_3^2 - A_4^2 + 32\Gamma_R) + A_1^2 \left[ 3A_3^4 - A_1^4 - A_4^4 - 2A_3^2(A_3^2 + A_4^2 - 32\Gamma_R) - 32A_2^4\Gamma_R + 192\Gamma_R - 2A_3^2(A_2^2 + 16\Gamma_R) \right] - A_2^2 \left[ A_3^4 + A_4^4 + 32A_3^2\Gamma_R - 192\Gamma_R^2 + 2A_3^2(A_2^2 + 16\Gamma_R) \right] \right\}. \tag{B33}
\]
We skip some intermediate steps which are analogous to the calculation of the mean square deviations above. The leading order \( \langle z^2 \rangle_0 = \langle z \rangle_0^2 = 0 \) comes out as expected. Thus, the mean square deviation of \( z \) equals the following first order correction:
\[
(\Delta z)^2 = (1/K)\langle z^2 \rangle_1(1 + \mathcal{O}(1/K)) \tag{B34}
\]
with
\[
\langle z^2 \rangle_1 = F_z f f^* \left[ (\nu^2 - 1) \left( f^2 + (f^*)^2 \right) - 2(1 + \nu^2) f f^* \right] \times \left[ (\nu^4 - 1) \left( f^2 + (f^*)^2 \right) - 2(1 + \nu^4) f f^* \right],
\]
\[
F_z = r_0^2 \left[ 16K\kappa_1(\kappa_0 + \kappa_1) \right]^{-1}. \tag{B35}
\]
Evaluation produces the result as stated in (63).

**Appendix C: Mean values of velocity variables**

**C.1 Mean \( x \) component of velocity**
We calculate the mean value of the velocity operator as defined by (12) and (10).

\[
\overline{D}_x = C^2 \int_0^{2\pi} d\Phi \int dr d\theta d\varphi r^2 \sin(\theta) \exp \left[ \mathbf{a}^*(\Phi) \cdot \mathbf{u} - \Gamma^* u^2 / 2 \right] (1/(2r)) \times \\
[u_3 \partial u_1 - u_4 \partial u_2 + u_1 \partial u_3 - u_2 \partial u_4] \int_0^{2\pi} d\Phi \exp \left[ \mathbf{a}(\Phi) \cdot \mathbf{u} - \Gamma u^2 / 2 \right]. \quad (C1)
\]

By partial integration, we apply the differential operator to the left hand exponent, which produces a minus sign. The factor \((1/r)\) is combined with the metric factor. According to (3), we transform the metric as

\[
d\Phi(r/2) dr d\theta \sin(\theta) d\varphi \rightarrow 4 \, du_1 du_2 du_3 du_4, \quad (C2)
\]

and write

\[
\overline{D}_x = -4C^2 \int_0^{2\pi} d\Phi \int_{-\infty}^{\infty} du_1 du_2 du_3 du_4 \exp \left[ \mathbf{a}(\Phi') \cdot \mathbf{u} - \Gamma u^2 / 2 \right] \times \\
\left\{ [u_3 \partial u_1 - u_4 \partial u_2 + u_1 \partial u_3 - u_2 \partial u_4] \exp \left[ \mathbf{a}^*(\Phi) \cdot \mathbf{u} - \Gamma^* u^2 / 2 \right] \right\}
\]

\[
= -4C^2 \int_0^{2\pi} d\Phi \int_{-\infty}^{\infty} du_1 du_2 du_3 du_4 [u_3 (a_4^* - \Gamma^* u_1) - u_4 (a_2^* - \Gamma^* u_2) + \\
u_1 (a_3^* - \Gamma^* u_3) - u_2 (a_4^* - \Gamma^* u_4)] \exp \left[ \mathbf{A} \cdot \mathbf{u} - \Gamma R u^2 \right], \quad (C3)
\]

where the vector \(\mathbf{A}\) is defined in (52). In the next step, we produce the components \(u_i\) by derivatives with respect to \(A_i\), to be followed by the Gaussian \(u\) integrations. With the notation \(\Gamma = \Gamma_R + i \Gamma_I\), we obtain

\[
\overline{D}_x = -4 \frac{C^2 \pi^2}{\Gamma_R^2} \int_0^{2\pi} d\Phi \, D^x(\mathbf{a}^*, \mathbf{A}) \exp \left[ \mathbf{A} \cdot \mathbf{A} / (4\Gamma_R) \right],
\]

\[
D^x(\mathbf{a}^*, \mathbf{A}) = (2\Gamma_R^2)^{-1} \left\{ i \Gamma_I (A_1 A_3 - A_2 A_4) + \Gamma_R (a_1^* A_3 - A_1 A_3 + a_5^* A_1 + A_2 A_4 - a_4^* A_4 - a_4^* A_2) \right\}. \quad (C4)
\]

We make use of the explicit functions \(\mathbf{a}^*\) and \(\mathbf{A}\), as given in (51) and (52), to obtain

\[
D^x(\mathbf{a}^*, \mathbf{A}) = i \kappa (\kappa_4 + \kappa_5 \cos(\phi)), \quad \phi = \Phi' - \Phi, \quad \kappa = r_0 \Gamma_0 \quad (C5)
\]

with \(\kappa_4\) and \(\kappa_5\) defined in (60f). The phase integral, thus, reads

\[
\overline{D}_x = -i \frac{4\pi^2 C^2 \kappa}{\Gamma_R^2} \int_0^{2\pi} d\phi \left[ \kappa_4 + \kappa_5 \cos(\phi) \right] \exp \left[ (k_0 + k_1 \cos(\phi)) / \Gamma_R \right] \\
= -i \frac{8\pi^3 C^2 \kappa}{\Gamma_R^2} \exp[\kappa_0 K] \left[ \kappa_4 I_0(\kappa_1 K) + \kappa_5 I_1(\kappa_1 K) \right], \quad (C6)
\]

\[
= -i \frac{C^2 \kappa}{16\sqrt{2} \Gamma_R^2} \left( \frac{\pi}{\kappa_1 K} \right)^{5/2} \exp \left[ (k_0 + \kappa_1) K \right] \left[ (9 + 16\kappa_1 K + 128(\kappa_1 K)^2) \kappa_4 + \\
(-15 - 48\kappa_1 K + 128(\kappa_1 K)^2) \kappa_5 \right] [1 + \mathcal{O}(1/K)], \quad (C7)
\]
where in the last equation we applied the asymptotic formulas (56) with (57); furthermore, we use (60d), the normalization condition (58), and the definition (41) of γ₀, in order to get the intermediate expression

\[
\overline{D}_x = -\frac{i \kappa (\kappa_4 + \kappa_5)}{2r_0(\kappa_0 + \kappa_1)} \left[ 1 + (\delta v_x - \delta n)/K + O(1/K^2) \right],
\]

\[
\delta v_x = \left[ 1 + e - \nu^2 + \nu^A - e\nu^A \right] \left[ 8\kappa_1\nu^2 \right]^{-1}.
\] (C8)

Using δn of (59), the auxiliary functions defined in (60f), and \(v_x = h/(im) D_x\), we produce the results as stated in (64) and (65).

C.2 Mean y component of velocity

In (C4) and (C5), we substitute \(D^y\) for \(D^x\) with

\[
D^y(a^*, A) = (2\Gamma_R^2)^{-1} \left\{ i (A_2A_3 + A_1A_4)\Gamma_I + [a^*_2A_3 + A_2(a^*_3 - A_3) - A_1A_4 + a^*_1A_4 + A_1a^*_4]\Gamma_R \right\}
\]

\[= -(1/2)r_0\nu \left[ (\Gamma_I + i\Gamma_R) f^2 - (\Gamma_I - i\Gamma_R)(f^*)^2 \right].\] (C9)

With the aid of the definitions given in (60) and (41), we obtain

\[
D^y = -i \kappa \nu \cosh(2w),
\] (C10)

which does not depend on the phase \(\phi = \Phi - \Phi'\). After integration with respect to \(\phi\), application of the asymptotic approximation of \(I_0(\kappa_1K)\) and taking into account the normalization (58), we obtain the intermediate result

\[
\overline{D}_y = \langle D_y \rangle_0 \left[ 1 + (\delta v_y - \delta n)/K + O(1/K^2) \right],
\]

\[
\langle D_y \rangle_0 = \frac{K\nu}{4r_0(\kappa_0 + \kappa_1)} \left[ \frac{4\kappa_0}{1 - \nu^2} + \frac{1 + \gamma_0^2}{4\nu^2\gamma_0^2 \kappa_6^2} \right],
\]

\[
\delta v_y = 1/(8\kappa_1),
\] (C11)

which, in view of \(v_y = h/(im) D_y\) and with the aid of the formulas of (60), leads to the results stated (64) and (65).

C.3 Mean z component of velocity

In (C4) and (C5), we substitute \(D^z\) for \(D^x\) with

\[
D^z(a^*, A) = (4\Gamma_R^2)^{-1} \left\{ i (A^2_2 + A^2_3 - A^2_4 - A^2_6) \Gamma_I + (-A^2_4 + 2A^*_1A^*_4 - A^2_2 + 2A_2a^*_2 + A^2_3 - 2A_3a^*_3 + A^2_4 - 2A_4a^*_4) \Gamma_R \right\}
\]

\[= -r_0\Gamma_I\nu ff^* \sin(\phi).\] (C12)
Due to the amplitude factor \( \sin(\phi) \), the phase integral with respect to \( \phi \) vanishes identically with the consequence that \( \dot{v}_z = 0 \).

C.4 Mean square deviation of x component of velocity

We use the differential operators (12) and (10) and shift one \( v_x \) to the bra vector by means of partial integration to write (note that \( r = u^2 \))

\[
\overline{v_x^2} = \frac{\hbar^2 C^2}{m^2} \int_0^{2\pi} d\Phi' \int d\Phi \ r^2 dr \sin(\theta) d\theta d\varphi
\]

\[
= \frac{1}{2u^2} \left\{ \left[ u_3 \partial_{u_1} - u_4 \partial_{u_2} + u_1 \partial_{u_3} - u_2 \partial_{u_4} \right] \exp \left[ \mathbf{a}^\ast(\Phi') \cdot \mathbf{u} - \Gamma^* u^2 / 2 \right] \right\} \times
\]

\[
= \frac{1}{2u^2} \left\{ \left[ u_3 \partial_{u_1} - u_4 \partial_{u_2} + u_1 \partial_{u_3} - u_2 \partial_{u_4} \right] \exp \left[ \mathbf{a}(\Phi) \cdot \mathbf{u} - \Gamma u^2 / 2 \right] \right\}.
\]

As it turns out, the integrand depends on the difference \( \Phi' - \Phi \) and, after the integration with respect to \( \Phi' \), does not anymore depend on \( \Phi \). We then use (3) and, after the differentiation with respect to \( u_i \), we get

\[
\overline{v_x^2} = F \int_0^{2\pi} d\Phi' \int_{-\infty}^{\infty} du_1 du_2 du_3 du_4 Q_x \frac{1}{u^2} \exp \left[ \mathbf{A} \cdot \mathbf{u} - \Gamma_R u^2 \right]
\]

\[
= F \int_{-\infty}^{\infty} d\gamma \int_0^{2\pi} d\Phi' \int_{-\infty}^{\infty} du_1 du_2 du_3 du_4 Q_x \exp \left[ \mathbf{A} \cdot \mathbf{u} - \gamma u^2 \right],
\]

where \( \mathbf{A} \) is defined in (52) and

\[
Q_x = q_x^\ast(\Phi')q_x(\Phi),
\]

\[
q_x(\Phi) = u_3 \left[ a_1(\Phi) - (\Gamma_R + i \Gamma_I) u_1 \right] - u_4 \left[ a_2(\Phi) - (\Gamma_R + i \Gamma_I) u_2 \right] +
\]

\[
- u_1 \left[ a_3(\Phi) - (\Gamma_R + i \Gamma_I) u_3 \right] - u_2 \left[ a_4(\Phi) - (\Gamma_R + i \Gamma_I) u_4 \right];
\]

\[
F = 2\hbar^2 C^2 / m^2.
\]

In the second equation of (C14), we produced the factor \( 1/u^2 \) by means of integration with respect to the variable \( \gamma \). The amplitude factor \( Q_x \) is multiplied out, and each monomial in \( \mathbf{u} \) generated by partial differentiation \( u_i \to \partial / \partial A_i \). After carrying out the \( u \) integrations, the derivatives \( \partial / \partial A_i \) act on the function \( I_A \) which is defined in (60g):

\[
\int_{-\infty}^{\infty} du_1 du_2 du_3 du_4 \exp \left[ \mathbf{A} \cdot \mathbf{u} - \gamma u^2 \right] = (\pi^2 / \gamma^2) I_A.
\]

The mean value is now expressed as follows

\[
\overline{v_x^2} = \pi^2 F \int_{-\infty}^{\infty} d\gamma \int_0^{2\pi} d\Phi' Q_x I_A.
\]
We order $\tilde{Q}_x$ by powers of $\Gamma_R/\gamma$ and by $\cos(n\phi)$ terms:

$$
\tilde{Q}_x = r_0^2 \sum_{m=3}^{6} \sum_{n=0}^{2} q^x_{mn} \left( \frac{\Gamma_R}{\gamma} \right)^m \cos(n\phi), \quad (C18)
$$

where,

$$
q^x_{30} = 0, \quad q^x_{31} = \kappa_1/K, \quad q^x_{32} = 0,
$$

$$
q^x_{40} = \frac{1}{2K^2\nu} \left\{ 4g^2\nu - gK(\nu^2 - 1)\kappa_6 + \nu \left[ 4 - 4K\kappa_0 + \frac{3 + 2\nu^2 + 3\nu^4}{(1 + \nu^2)^2}K^2\kappa_1^2 \right] \right\},
$$

$$
q^x_{41} = 2(\kappa_0 - 1/K)\kappa_1, \quad q^x_{42} = \kappa_3^2/8,
$$

$$
q^x_{50} = \left\{ 32\kappa_0(1 + g^2)\nu^2 - K\left[ \kappa_6(\nu^2 - 1)^2 + 8gK\kappa_0\kappa_6\nu(\nu^2 + 1)(\nu^2 - 1)^{-1} + 16\kappa_0^2\nu^2(3 + 2\nu^2 + 3\nu^4)(\nu^2 - 1)^{-2} \right] \right\} \left( 16K\nu^2 \right)^{-1},
$$

$$
q^x_{51} = \frac{\kappa_1}{2K\nu} \left[ 4g^2\nu + 4\nu(1 - 2K\kappa_0) + gK\kappa_6(1 - \nu^2) \right], \quad q^x_{52} = -\kappa_3^2/4,
$$

$$
q^x_{60} = (1 + g^2)(2\kappa_2^2 + \kappa_3^2)/8, \quad q^x_{61} = 2(1 + g^2)\kappa_0\kappa_1, \quad q^x_{62} = (1 + g^2)\kappa_3^2/8. \quad (C19)
$$

For the functions $\kappa_i$ and $g$, see (60f) and (60i), respectively. The $\phi$ integrals are stated in (B13).

There remains the $\gamma$ integration which, after the variable transformation $\gamma \rightarrow s = \Gamma_R/\gamma$, enters as follows

$$
\overline{v^2_x} = 2\pi^3 F r_0 K \sum_{m=3}^{6} \sum_{n=0}^{2} q^x_{mn} W_{mn}, \quad (C20)
$$

$$
W_{mn} = \int_0^1 ds \, s^{m-2} \exp[K\kappa_0 s] I_n((K\kappa_1 s),
$$

where we used $r_0\Gamma_R = K$, $k_0/\Gamma_R = K\kappa_0$ and $k_1/\Gamma_R = K\kappa_1$. The functions $q^x_{mn}$ and $W_{mn}$, as well as $K$, are dimensionless. Since the normalization constant $C^2$ has dimension $1/\text{length}^3$, one verifies immediately that $r_0 F$ has the dimension velocity$^2$. For large $K$, we derive in Appendix D the asymptotic approximation

$$
W_{mn} = \frac{\exp[K(\kappa_0 + \kappa_1)]}{K^{3/2}\sqrt{2\pi\kappa_1(\kappa_0 + \kappa_1)}} \left[ 1 + \frac{1}{K} \frac{i_1(n)(\kappa_0 + \kappa_1) + (5/2 - m)\kappa_1}{\kappa_1(\kappa_0 + \kappa_1)} + \mathcal{O}(1/K^2) \right], \quad (C21)
$$

with $m = 3, \ldots, 6$, $n = 0, 1, 2$, and $i_1(n)$ defined in (57).

In order to eliminate the normalization constant $C^2$, we divide $F$ by (58). After ordering by powers of $K$ and using $K = h(w)\kappa$, we write (C20) as

$$
\overline{v^2_x} = \kappa^2 \left[ V^x_0 + \frac{V^x}{K} + \mathcal{O}(1/K^2) \right], \quad \kappa^2 V^x_0 = \langle v_x \rangle_0^2, \quad (C22)
$$

$$
\kappa^2 V^x_1/K = h^2 \kappa (e - 1) Z^3_w \sum_{j=0}^{3} c^3_{2j} \cosh(2jw) \left[ 32m^2 r_0^2 (e + 1)(1 + \nu^2) \right]^{-1}, \quad (C23)
$$
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where the \( c \) coefficients read:

\[
\begin{align*}
  c_0^x &= -4 \left[ e + 2\nu^2 - e\nu^4 + e^2(1 + \nu^2)^2 \right], \\
  c_2^x &= 5 + 4e\nu^2 - 5\nu^4 + 2e^3(1 + \nu^2)^2 + 3e^2(\nu^4 - 1), \\
  c_4^x &= 4(1 + e + 4\nu^2 + \nu^4 - e\nu^4), \\
  c_6^x &= 2e^3(1 + \nu^2)^2 + 5(\nu^4 - 1) - 3e^2(\nu^4 - 1) - 4e(1 + 3\nu^2 + \nu^4). 
\end{align*}
\] (C24)

To leading order including the first order correction, the mean square deviation is given by

\[
(\Delta v_x)^2 \equiv \bar{v}_x^2 - \bar{v}^2_x = \left[ \kappa^2 V_1^x / K - \langle v_x \rangle_0 \xi_v / K \right] [1 + \mathcal{O}(1/K)] , \] (C25)

which after evaluation with the aid of (C23), (C24), and (65) leads to the result stated in (66).

C.5 Mean square deviation of y component of velocity

In analogy to (C20), we write

\[
\bar{v}_y^2 = 2\pi^3 F r_0 K \sum_{m=3}^6 \sum_{n=0}^2 q_{mn}^y W_{mn} \] (C26)

with

\[
\begin{align*}
  q_{30}^y &= 0, \quad q_{31}^y = \kappa_1 / K, \quad q_{32}^y = 0, \\
  q_{40}^y &= 2(1 + g^2) / K^2 + [-4\kappa_0 + g\kappa_6(1/\nu - \nu)] / (2K) + 4\kappa_1^2\nu^2 / (1 + \nu^2)^2, \\
  q_{41}^y &= -2\kappa_1 / K, \quad q_{42}^y = 0, \\
  q_{50}^y &= 2(1 + g^2)\kappa_0 / K - \kappa_6^2 / 2 + 2g\kappa_0\kappa_6\nu / (\nu^2 - 1), \\
  q_{51}^y &= 2(1 + g^2)\kappa_1 / K, \quad q_{52}^y = 0, \\
  q_{60}^y &= (1/4)(1 + g^2)\kappa_0^2, \quad q_{61}^y = 0, \quad q_{62}^y = 0. \quad \text{(C27)}
\end{align*}
\]

Instead of (C22) and (C23), we obtain

\[
\bar{v}_y^2 = \kappa^2 \left[ V_0^y + V_1^y / K + \mathcal{O}(1/K^2) \right], \quad \kappa^2 V_0^y = \langle v_y \rangle_0^2, \] (C28)

\[
\kappa^2 V_1^y / K = \hbar^2 \kappa (1 - e) Z_w^3 \left[ 32m^2 r_0^2(e + 1) \right]^{-1} \sum_{j=0}^3 c_{2j}^y \cosh(2jw), \] (C29)

with

\[
\begin{align*}
  c_0^y &= 2(3e^2 - 1) \left[ -1 - \nu^2 + e(\nu^2 - 1) \right], \\
  c_2^y &= 7(\nu^2 - 1) - 9e^2(\nu^2 - 1) - 3e(1 + \nu^2) + 5e^3(1 + \nu^2), \\
  c_4^y &= 2 \left[ 5e(1 - \nu^2) + 3e^3(\nu^2 - 1) + 3(1 + \nu^2) - e^2(1 + \nu^2) \right], \\
  c_6^y &= (3e^2 - 5)(1 + e - \nu^2 + e\nu^2). \quad \text{(C30)}
\end{align*}
\]
To leading order including the first order correction, the mean square deviation is given by

\[(\Delta v_y)^2 \equiv \overline{v_y^2} - \overline{v_y^2} = \left(\kappa^2 V_1 y / K - \langle v_y \rangle_0 \eta_e / K \right) \left[1 + \mathcal{O}(1/K) \right], \quad (C31)\]

which after evaluation with the aid of (C29), (C30), and (65) leads to the result stated in (66).

**C.6 Mean square deviation of z component of velocity**

In analogy to (C20), we write

\[v_z^2 = 2\pi^3 F r_0 K \sum_{m=3}^{6} \sum_{n=0}^{2} q_{mn}^z W_{mn}, \quad (C32)\]

with

\[
\begin{align*}
q_{50}^z &= 0, \quad q_{31}^z = \kappa_1 / K, \quad q_{32}^z = 0, \quad (C33) \\
q_{40}^z &= 2(1 + g^2) / K^2 + [-4\kappa_0 + g\kappa_6(1/\nu - \nu)] / (2K) - 2\kappa_1^2 \nu^2 / (1 + \nu^2)^2, \\
q_{41}^z &= -2\kappa_1 / K, \quad q_{42}^z = (2\kappa_1^2 \nu^2) / (1 + \nu^2)^2, \\
q_{50}^z &= 2 \left[2K\kappa_1^2 \nu^2 + (1 + g^2)\kappa_0(1 + \nu^2)\nu \right] / [K(1 + \nu^2)^2], \\
q_{51}^z &= 2(1 + g^2)\kappa_1 / K, \quad q_{52}^z = -(4\kappa_1^2 \nu^2) / (1 + \nu^2)^2, \\
q_{60}^z &= -2(1 + g^2)\kappa_1^2 \nu^2 / (1 + \nu^2)^2, \quad q_{61}^z = 0, \quad q_{62}^z = (2(1 + g^2)\kappa_1^2 \nu^2) / (1 + \nu^2)^2.
\end{align*}
\]

Instead of (C22) and (C23), we obtain

\[
\langle v_z^2 \rangle = \kappa^2 \left[ V_0^z + V_1^z / K + \mathcal{O}(1/K^2) \right], \quad \kappa^2 V_0^z = \langle v_z \rangle^2_0 = 0. \quad (C34)
\]

To leading order including the first order correction, the mean square deviation is given by

\[(\Delta v_z)^2 \equiv \langle v_z^2 \rangle - \langle v_z \rangle^2 = \kappa^2 V_1^z / K \left[1 + \mathcal{O}(1/K) \right], \quad (C35)\]

which leads to the result stated in (66).

**Appendix D: Integrals with Bessel functions**

In the following, we derive an asymptotic approximation of $W_{mn}$ for $K \gg 1$:

\[W_{mn} = \int_0^1 ds \, s^{m-2} \exp[K\kappa_0 s] I_n(K\kappa_1 s), \quad m = 3, 4, 5, 6, \quad n = 0, 1, 2. \quad (D1)\]

For large $K$, the main contribution to the integral comes from $s$ close to 1, where the Bessel function can be approximated by the asymptotic series. However, in the integration interval with small $s$, the asymptotic approximation is unjustified. We, therefore, split the integral into two parts, $A + B$, with

\[A = \int_0^{1/2} ds \ldots, \quad B = \int_{1/2}^1 ds \ldots. \quad (D2)\]
Part $A$ is estimated by an upper bound of the Bessel function, see formula (9.6.18) in [1].

$$0 \leq I_n(x) \leq \sqrt{\pi(x/2)^n \Gamma^{-1}(n+1/2)} \exp(x), \quad n \geq 0, \quad x \geq 0,$$

and can, thus, be estimated as

$$A \leq \sqrt{\pi(1/2)^n \Gamma^{-1}(n+1/2)} (1/2)^{n+m-1} \exp \left[ K (\kappa_0 + \kappa_1) / 2 \right].$$

As compared to $A$, part $B$ has the dominating exponential term with $s = 1/2$ replaced by $s = 1$, so

$$A + B = B \left( 1 + A/B \right), \quad A/B \propto \exp \left[ -K (\kappa_0 + \kappa_1) / 2 \right],$$

which tells that the relative contribution of $A$ is exponentially small for $K \gg 1$; by (60f),

$$\kappa_0 + \kappa_1 \equiv [e \cosh(w) - 1] / (e - 1) \geq 1, \quad e > 1.$$

Figure 1: Numerical check of the accuracy of the asymptotic approximation (D12). $Q = Q(K)$, with $34 < K < 110$, is the quotient of the numerically determined integral (D1) divided by the analytic approximation (D12). Parameters are $\kappa_1 = 0.8$ and $\kappa_0 = 0.4$. The upper three curves refer to $m = 6$ and to the Bessel functions $I_n$ with $n = 0$ (solid curve), $n = 1$ (dashed), and $n = 2$ (dot-dashed), respectively; the lower curves refer to $m = 4$.

In the integrand of $B$, we approximate the Bessel functions by the asymptotic form (56):

$$W_{mn} \to \frac{1}{\sqrt{2\pi K \kappa_1}} \int_{1/2}^{1} ds \exp \left[ K (\kappa_0 + \kappa_1) s \right] \times$$

$$\left[ s^{m-5/2} + i_1(n) s^{m-7/2} / (K \kappa_1) + i_2(n) s^{m-9/2} / ((K \kappa_1)^2) + \ldots \right],$$
where the $i$-coefficients are listed in (57) for $n = 0, 1, 2$. The $s$ integral can be expressed in terms of the incomplete $\Gamma$ functions, see Section 6.5 of [1]; with $m = 3, 4, 5, 6$ and the abbreviation $K_{01} = K(\kappa_0 + \kappa_1)$, we obtain

\[
K_m := \int_{1/2}^{1} ds \exp[K_{01} s] s^{m-5/2} = i (-1)^m K_{01}^{3/2-m} \times \\
[\Gamma(m - 3/2, -K_{01}) - \Gamma(m - 3/2, -K_{01}/2)]. \quad (D8)
\]

The contribution from the lower integration boundary is exponentially small. Using the asymptotic form, see (6.5.32) of [1],

\[
\Gamma(a, s) \sim s^{a-1} \exp(-s) \left[1 + (a - 1)/s + (a - 1)(a - 2)/s^2 + \ldots \right], \quad (D9)
\]

we obtain

\[
K_m \sim \frac{1}{K_{01}} \exp[K_{01}] \left[1 - \frac{m - 5/2}{K_{01}} + \frac{(m - 5/2)(m - 7/2)}{K_{01}^2} + \ldots \right]. \quad (D10)
\]

Thus, we get the following asymptotic approximation:

\[
W_{mn} = \frac{1}{\sqrt{2\pi K \kappa_1}} \left[K_m + \frac{i_1(n)}{K \kappa_1} K_{m-1} + \frac{i_2(n)}{(K \kappa_1)^2} K_{m-2} + \ldots \right]. \quad (D11)
\]

After the amplitude is ordered by powers of $1/K$ and neglecting terms of order $(1/K^2)$, we arrive at

\[
W_{mn} = \frac{\exp[K(\kappa_0 + \kappa_1)]}{K^{3/2} \sqrt{2\pi \kappa_1 \kappa_0 + \kappa_1}} \left[1 + \frac{1}{K} \frac{i_1(n)(\kappa_0 + \kappa_1) + (5/2 - m)\kappa_1}{\kappa_1(\kappa_0 + \kappa_1)} \right]. \quad (D12)
\]

As can be inferred from Fig. 1, for $K > 34$, the difference $Q - 1$, with $Q = (D1)/(D12)$, approaches the value zero proportional to $1/K^b$ with $b \approx 2$, which is consistent with the $O(1/K^2)$ error in (D12).

**Appendix E: Check of mean square deviations and uncertainty products**

As a check of the analytical results, we prove in the following that the analytically continued mean square deviations of the velocity components of elliptic orbits are positive, see subsections 1, 2, and 3 to follow; furthermore, in subsection 4, we show for the hyperbolic case that the uncertainty products obey the quantum mechanical lower bound.

**E.1 Mean square deviation of x component of the velocity**
By (138), it has to be shown that for $0 \leq e < 1$

$$K_x := \sum_{j=0}^{3} c_{2j} \cos(2jw) < 0. \tag{E1}$$

It is convenient to express the functions $\cos(2jw)$ in terms of powers of $X := \cos^2(w)$ by means of the substitutions

$$\cos(2w) = 2X - 1, \ \cos(4w) = 1 - 8X + 8X^2, \ \cos(6w) = 18X - 48X^2 + 32X^3 - 1. \tag{E2}$$

With the aid of the $c$ coefficients defined in (67) and the abbreviation $\nu^2 = n > 0$, we obtain

$$K_x = A_0 + A_1n, \tag{E3}$$

$$A_0 = -2(1 + e) + 4(1 + 5e)X - 48(1 + e)X^2 + 32(1 + e)X^3, \tag{E4}$$

$$A_1 = -2(1 + e) + 4(5e - 4)X + 48(1 - e)X^2 - 32(1 - e)X^3,$$

$$0 \leq e < 1, \quad 0 \leq X \leq 1, \quad n > 0.$$ 

Since $n$ can be arbitrarily small and large, both $A_0$ and $A_1$ have to be negatively definite. At the end points of the $X$ interval, we get the values

$$K_x(X = 0) = -2(1 + e)(1 + n) < 0, \quad K_x(X = 1) = -2(1 - e)(1 + n) < 0. \tag{E5}$$

Let us first consider $A_0$. In the inner interval with $0 < X < 1$, we find a maximum and a minimum at the points

$$X_\pm = (1/12)(1 + e)^{-1}[6(e + 1) \pm d_0], \quad d_0 = \sqrt{6(2 + 3e + e^2)}, \tag{E6}$$

which give rise to the extrema values

$$(A_0)_\pm = -(2/9)(1 + e)^{-1}[9(1 + e) \pm 2(1 + e/2)d_0]. \tag{E7}$$

The minimum, corresponding to the ”+” sign, is negatively definite, obviously. It remains to be shown that the maximum is negatively definite, which is the case, since

$$[9(1 + e)]^2 - [2(1 + e/2)d_0]^2 = 3(1 - e^2)(11 + 14e + 2e^3) > 0, \quad 0 \leq e < 1. \tag{E8}$$

The fact that the internal maximum is negative, together with the negative end point values of $A_0$ (set $n = 0$ in (E4)), proves that $A_0 < 0$.

As to $A_1$, the end point values are negative as can be read from the coefficient of $n$ in (E4). Once more, there exist a maximum and a minimum in the interval $0 < X < 1$; however, the inner extrema exist only if the eccentricity is confined to the interval $0 \leq e \leq 4/5$; in the interval $4/5 < e < 1$, the extrema lie outside the $X$ interval. The maximum lies at

$$X_{\text{max}} = (1/12)(1 - e)^{-1}[6(1 - e) + d_1], \quad d_1 = \sqrt{6(2 - 3e + e^2)}, \tag{E9}$$
and, in the interval $0 \leq e \leq 4/5$, has the value

$$(A_1)_{max} = -(2/9)(1-e)^{-1}[a_1 - a_2], \quad a_1 = 9(1-e), \quad a_2 = 2(1-e/2)d_1.$$  \hfill (E9)

Both $a_1$ and $a_2$ are positive magnitudes. In order that $(A_1)_{max} < 0$, we have to check $a_1 > a_2$, or after squaring

$$a_1^2 - a_2^2 = 3(1-e^2)(11 - 14e + 2e^2) \geq 729/625 > 0, \quad 0 \leq e < 4/5. \hfill (E10)$$

In the eccentricity interval $4/5 < e < 1$, the coefficient $A_1$ monotonically increases with $X$ in the interval $0 \leq X \leq 1$ with the consequence that $A_1 \leq A_1(X = 1) = -2(1-e) < 0$. This completes the proof that $K_x < 0$.

**E.2 Mean square deviation of $y$ component of the velocity**

By (138), it has to be shown that

$$K_y := \sum_{j=0}^{3} d_{2j} \cos(2jw) > 0. \hfill (E11)$$

After expressing the cosine functions according to (E2) and inserting the $d_i$ coefficients from (67), we obtain

$$K_y = 2(1+e)A_0 + 2(1-e)A_1n, \hfill (E12)$$

$$A_0 = 1 - e^2 + 2(4 + 3e + e^2)X - 8(3 + e)X^2 + 16X^3,$$

$$A_1 = (1 + e)^2 - 2(4 + 5e + e^2)X + 8(3 + e)X^2 - 16X^3,$$

$$0 \leq e < 1, \quad 0 \leq X \leq 1, \quad n > 0.$$

In (E12), since $n > 0$ is arbitrary, both $A_0$ and $A_1$ have to be positively definite.

The coefficient $A_0$ can be transformed into the following, evidently positive, expression

$$A_0 = (1 - e^2)(1 - X) + 16X [X - (3 + e)/4]^2 > 0. \hfill (E13)$$

As to $A_1$, we immediately get $A_1(X = 0) = (1+e)^2 > 0$ and $A_1(X = 1) = 1 - e^2 > 0$. In the interval $0 < X < 1$, one finds a maximum and a minimum at

$$X_\pm = (1/12) \left[6 + 2e \pm \sqrt{d_2}\right], \quad d_2 = 2(6 - 3e - e^2). \hfill (E14)$$

The minimum is at $X_-$ with

$$(A_1)_{min} = (1/27) [a_0 - a_1], \quad a_0 = 27 - 9e(1 + e) - 5e^3, \quad a_1 = (6 - 3e - e^2) \sqrt{d_2}. \hfill (E15)$$

It is noticed that both $a_0$ and $a_1$ are positive for $0 \leq e < 1$, so we check

$$a_0^2 - a_1^2 = 27(1-e)(1+e)^2 \left[11 - 5e - 3e^2 - e^3\right] > 0, \hfill (E16)$$
where the square bracket factor is larger than 2 in the eccentricity interval
\(0 \leq e < 1\). Thus, it has been shown that \((A_1)_{\text{min}} > 0\) and since at the end
points \(A_1(X = 0) > 0\) and \(A_1(X = 1) > 0\), it is proved that \(A_1 > 0\) and, thus,
\(K_y > 0\).

**E.3 Mean square deviation of z component of the velocity**

By (138), it has to be shown that
\[
K_z := f_0 + f_4 \cos(4w) < 0.
\]  
(E17)

After expressing the cosine functions according to (E2) and inserting the \(f_i\) coefficients from (67), we obtain
\[
K_z = B_0 + B_1 n + B_2 n^2,
\]
\[
B_0 = -2(1 - e^2)(1 - 2X)^2,
\]
\[
B_1 = 4 \left[-1 + e^2(1 - 2X)^2 - 4X(1 - X)\right],
\]
\[
B_2 = -2(1 - e^2)(1 - 2X)^2.
\]

The coefficients \(B_0\) and \(B_2\) are negatively definite for \(0 \leq e < 1\), and \(B_1\) is
easily estimated as
\[
B_1 \leq 4 \left[-1 + e^2 - 4X(1 - X)\right] < 0.
\]  
(E18)

Thus, it is proved that \(K_z < 0\).

**E.4 Uncertainty products for hyperbolic orbits**

In the following, we prove that the uncertainty products obey the lower
bound condition of quantum mechanics according to (68). In particular, we
show that \(P_i \geq 1\) holds true for all three components \(i = x, y, z\) in the param-
eter domains \((w \in \mathbb{R}, e > 1, n = \nu^2 > 0)\). The proofs to follow are consistent
with (69) which tells that if \(\nu > 0\), then \(P_x(w = 0) > 1\), \(P_y(w = 0) > 1\) and
\(P_z(w = 0) = 1\).

**E.4.1 Uncertainty product of x component**

We obtain from (63), (66) and (67), using the abbreviations \(W = 2w\) and
\(n = \nu^2\),
\[
P_x = N_x / D_x,
\]
\[
N_x = \left[-2(1 + n) + (e - 1 + n(1 + e)) \cosh(W) + (1 + e + n(e - 1)) \cosh(3W)\right] \times
\left[(e - 1)n \cosh^2(W/2) + (e + 1) \sinh^2(W/2)\right],
\]
\[
D_x = 2n \left[e \cosh(W) - 1\right]^2, \quad e > 1, \quad n > 0.
\]  
(E19)
Since \( D_x > 0 \), the condition \( P_x \geq 1 \) is equivalent to the condition \( d_x := N_x - D_x \geq 0 \). We find for this difference

\[
\begin{align*}
d_x &= d_0^x + d_1^x n + d_2^x n^2, \\
d_0^x &= (e + 1) \sinh^2(W/2) \left[ -2 + (e - 1) \cosh(W) + (1 + e) \cosh(3W) \right], \\
d_1^x &= 4(e^2 - 1) \sinh^2(W/2) \left[ \cosh(W/2) + \cosh(3W/2) \right]^2, \\
d_2^x &= (e - 1) \cosh^2(W/2) \left[ -2 + (1 + e) \cosh(W) + (e - 1) \cosh(3W) \right].
\end{align*}
\]

The coefficient \( d_1^x \geq 0 \), obviously. The other coefficients are easily estimated as

\[
\begin{align*}
d_0^x &\geq d_0^x(W = 0) \geq 0, \\
d_2^x &\geq d_2^x(W = 0) > 0.
\end{align*}
\]

As a consequence, \( P_x > 1 \) for \( e > 1 \) and \( n > 0 \).

### E.4.2 Uncertainty product of \( y \) component

From \((63), (66)\) and \((67)\), we obtain

\[
\begin{align*}
P_y &= N_y/D_y, \quad N_y = N_1^y N_2^y \\
N_1^y &= 2(1 + e + n(1 - e)) + \left[ n - 1 - 3e(1 + n) + 2e^3(1 + n) \right] \cosh(W) - 2e(1 + e + (e - 1)n) \cosh(2W) + (1 + e + (e - 1)n) \cosh(3W), \\
N_2^y &= (e - 1)n \cosh^2(W/2) + (e + 1) \sinh^2(W/2), \\
D_y &= 2(e^2 - 1)n \left[ e \cosh(W) - 1 \right]^2, \quad e > 1, \quad n > 0.
\end{align*}
\]

Once more, since \( D_y > 0 \), we can equivalently examine the condition \( d_y := N_y - D_y \geq 0 \), where

\[
\begin{align*}
d_y &= d_0^y + d_1^y n + d_2^y n^2, \\
d_0^y &= (e + 1)^2 \sinh^2(W/2) f_0^y, \\
f_0^y &= 2 + [2e(e - 1) - 1] \cosh(W) - 2e \cosh(2W) + \cosh(3W), \\
d_1^y &= 4(e^2 - 1) \cosh(W) \left[ \cosh(W) - e \right] \sinh^2(W), \\
d_2^y &= (e - 1)^2 \cosh^2(W/2) f_2^y, \\
f_2^y &= -2 + [2e(e + 1) - 1] \cosh(W) - 2e \cosh(2W) + \cosh(3W).
\end{align*}
\]

Since one can choose \( n > 0 \) arbitrarily small or arbitrarily large, both \( f_0^y \) and \( f_2^y \) must not be negative which, however, is not obvious. The coefficient \( d_1^y \), which can be negative, if \( e > 1 \) and \( W \) close to zero, has to be compensated by the \( n^2 \) term. Thus, we examine

\[
d_y = D_0 + d_2^y \left[ n + d_1^y/(2d_2^y) \right]^2, \quad D_0 = d_0^y - (d_1^y)^2/(4d_2^y),
\]

and prove in the following that \( (f_0^y, f_2^y, D_0) > 0 \) which implies that \( d_y > 0 \).
To show this, we make use of the identities
\[ \cosh(W) = Y, \quad \cosh(2W) = 2Y^2 - 1, \quad \cosh(3W) = 4Y^3 - 3Y, \]
\[ \cosh(4W) = 8Y^4 - 8Y^2 + 1, \] (E25)
which allows us to write
\[ f_0^y = 2 \left[ 1 + e + (e^2 - e - 2)Y - 2eY^2 + 2Y^3 \right]. \] (E26)

There are three cases: (a.) \( 1 < e \leq 4 \), (b.) \( 4 < e < e_c \), (c.) \( e > e_c \) with \( e_c = (3 + \sqrt{33})/2 \). In case (a.), there exists one extremum only, a minimum, for \( Y \geq 1 \), in case (b.) there exist a maximum and a minimum, and in case (c.) \( f_0^y \) is monotonically increasing with \( Y \), and \( f_0^y \geq f_0^y(Y = 1) = 2(e - 1)^2 > 0 \).

Since we are interested in a lower bound, in the cases (a.) and (b.), we consider the minimum only for \( 1 \leq e < e_c \), which is at
\[ Y_{min} = (1/6) [2e + s_0], \quad s_0 = \sqrt{12 + 6e - 2e^2}, \quad 1 \leq e \leq e_c, \] (E27)
and has the value
\[ (f_0^y)_{min} = y_0 + y_1s_0 \] (E28)
with
\[ y_0 = (2/27) \left[ 27 + 9e - 9e^2 + 5e^3 \right], \quad y_1 = (2/27) \left[ e^2 - 3e - 6 \right]. \] (E29)

In order to see that this minimum is larger than zero, we observe that \( y_0 > 0 \) and, for \( 1 \leq e \leq e_c \), consider the difference
\[ y_0^2 - y_1^2s_0^2 = (4/27)(e - 1)^2 \left\{ 11 + 16e + e^2 \left[ 1 + (e - 1)^2 \right] \right\} > 0. \] (E30)

Concluding
\[ f_0^y \geq \min \left\{ 2(e - 1)^2, \; (f_0^y)_{min} \right\} > 0, \quad e > 1. \] (E31)

We proceed similarly with
\[ f_2^y = 2 \left[ e - 1 + (e^2 + e - 2)Y - 2eY^2 + 2Y^3 \right], \quad Y \geq 1. \] (E32)

There exists no extremum in the interval \( Y \geq 1 \) for \( e > 1 \), which implies that \( f_2^y \) monotonically increases with \( Y \) with the consequence that
\[ f_2^y \geq f_2^y(Y = 1) = 2(e^2 - 1) > 0, \quad e > 1. \] (E33)

With the positivity of \( f_0^y \) and \( f_2^y \), we have the properties that \( d_0^y \geq 0 \) and \( d_2^y > 0 \), see (E23). It remains to be shown that \( D_0 \geq 0 \). As a matter of fact, we find
\[ D_0 = N_0/f_2^y, \] (E34)
\[ N_0 = 4(e - 1)(e + 1)^3 \sinh^2(W/2) \left[ \cosh(2W) - e \cosh(W) \right]^2 \geq 0. \] (E35)
At \( w = 0 \), we have \( D_0 = 0 \); however, in (E24), the cofactor of \( d_y^2 \) is larger 0, if \( n > 0 \). Thus, it is proved that \( P_y > 1 \) for \( e > 1 \) and \( n > 0 \).

**E.4.3 Uncertainty product of z component**

From (63), (66) and (67), we write after setting \( \nu^2 = n \)

\[
\begin{align*}
P_z &= N_z / D_z = N_1^z N_2^z N_3^z / D_z, \\
N_1^z &= -(1 + 6n + n^2) + e^2(1 + n)^2 + [e^2(1 + n)^2 - (n - 1)^2] \cosh(2W), \\
N_2^z &= (e - 1)n \cosh^2(W/2) + (1 + e) \sinh^2(W/2), \\
N_3^z &= (e - 1) \cosh^2(W/2) + (1 + e)n \sinh^2(W/2), \\
D_z &= 2(e^2 - 1)n(1 + n)^2 [e \cosh(W) - 1]^2, \quad e > 1, \quad n > 0. \tag{E36}
\end{align*}
\]

Since \( D_z > 0 \), the condition \( P_z \geq 1 \) is equivalent with \( d_z = N_z - D_z \geq 0 \) which can be written as

\[
d_z = (1/2) \left[ 4e n + (n - 1)^2 Y - e^2(1 + n)^2 Y \right]^2 (Y^2 - 1). \tag{E37}
\]

Since \( Y = \cosh(W) \geq 1 \), we have \( d_z \geq 0 \), which proves that the uncertainty product amounts to \( P_z \geq 1 \).

To achieve the above compact form of \( d_z \), Mathematica [22] tools were quite helpful: by successively applying the commands Expand, TrigToExp, Expand, ExpToTrig, the difference \( d_z \) comes out in the form \( \sum_j C_j \cosh(jW) \). Then, one uses (E25) followed by the Mathematica command Simplify. As a numerical test with the data \( e = 5/4, n = 4/5, \) and \( W = 1 \), Mathematica delivers the same value to 20 digits, \( N[A, 20] = N[B, 20] = 9.7114786183862078599 \), where \( A = N_z - D_z \) according to (E35) and \( B = d_z \) according to (E36).

**References**


Received: August 25, 2014