An Application of $p$-Adic Convolution
Associated with Daehee Numbers

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Abstract
In this paper, we consider the applications of $p$-adic convolution associated with Daehee numbers and give some relations between Daehee numbers and Bernoulli numbers.

1. Introduction

Let $p$ be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. Let $C(\mathbb{Z}_p, \mathbb{C}_p)$, $UD(\mathbb{Z}_p, \mathbb{C}_p)$ denote the space of all continuous functions on $\mathbb{Z}_p$ with values in $\mathbb{C}_p$ and the space of all uniformly differentiable functions on $\mathbb{Z}_p$ with values in $\mathbb{C}_p$. Let $|·|_p$ be the normalized $p$-adic absolute value with $|p|_p = 1/p$. As is known, the Daehee numbers are defined by the generating function to be

\[\log(1 + t) \sum_{n=0}^{\infty} D_n t^n, \text{ (see [1])} \]

(1)

For $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$, we have an integral $I_0(f)$ with respect to the so called invariant measure $\mu_0$:

\[I_0(f) = \int_{\mathbb{Z}_p} f(x) d\mu_0(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \text{ (see [2–3])} \]

(2)

Let $C_{p^n}$ be the cyclic group consisting of all $p^n$-th roots of unity in $\mathbb{C}_p$ for any $n \geq 0$ and $T_p$ be the direct limit of $C_{p^n}$ with respect to the natural morphism, hence $T_p$ is the union of all $C_{p^n}$ with discrete topology. The Fourier transform $\hat{f}_w$ is given by

\[\hat{f}_w = I_0(f \phi_w), \text{ (see [7, 8])}, \]

(3)

where $\phi_w$ denotes the uniformly differentiable function on $\mathbb{Z}_p$ belonging to $w \in T_p$ defined by $\phi_w(x) = w^x$.

Now, for any $f, g \in UD(\mathbb{Z}_p)$ we define their convolution $f * g$, due to Woodcock, as follows:

\[f * g(x) = \sum_{w} \hat{f}_w \hat{g}_w \phi_{w^{-1}}(x), \text{ (see [7])}. \]

(4)

Then, we have $f * g \in UD(\mathbb{Z}_p, \mathbb{C}_p)$ and \( \left( f * g \right)_w \hat{=} \hat{f}_w \hat{g}_w \). Another convolution $\otimes$ is induced by $*$ above: $f' \otimes g' = -(f * g)'$ for $f, g \in UD(\mathbb{Z}_p, \mathbb{C}_p)$. Thus, we note that

\[f \otimes g(n) = \sum_{k=0}^{n} f(k)g(n-k), \text{ (see [7, 8])}. \]

(5)
An application of $p$-adic convolution

From (2), we have

$$I_0(f_1) - I_0(f) = f'(0), \text{ (see [2]),}$$

(6)

where $f_1(x) = f(x + 1)$. By (6), we easily get

$$\int_{\mathbb{Z}_p} e^{xt}d\mu_0(x) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \text{ (see [1, 2, 3]),}$$

(7)

where $B_n$ are the Bernoulli numbers. Thus, by (7), we have

$$\hat{Z}^{\text{ext}}d\mu_0(x) = \frac{t - 1}{t} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \text{ (see [1, 2, 3]),}$$

(8)

From (1) and (6), we can derive the following equation:

$$\sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x)_n d\mu_0(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} (1 + t)^n d\mu_0(x) = \frac{\log(1 + t)}{t},$$

where $t \in \mathbb{C}_p$ with $|t|_p < p^{-1/(p-1)}$ and $(x)_n = x(x - 1) \cdots (x - n + 1)$. By (8), we get

$$D_n = \int_{\mathbb{Z}_p} (x)_n d\mu_0(x) = \sum_{\ell=0}^{n} S_1(n, \ell) B_\ell, \text{ (n \geq 0),}$$

(9)

where $S_1(n, \ell)$ is the Stirling number of the first kind. C. F. Woodcock proved the following results:

$$(f \otimes g)'(z) = f \otimes g'(z) + f' \otimes g(z) + f \ast g(z)$$

(10)

and

$$f \ast g(z) = I_0(x)(f(x)g(z - x)) - f \otimes g'(z), \text{ (see [7]),}$$

(11)

where $I_0(x)$ means the integration with respect to the variable $x$. Let $A_{m,n} = I_0(x)(z^m \otimes z^{n-1})$. Then, by (10), we get

$$A_{m,n} = \frac{1}{n} \sum_{i=1}^{n} \binom{n}{i} (-1)^i B_{n-i} B_{i+m}, \text{ (m, n \in \mathbb{N}).}$$

(12)

Note that $A_{m,n} = A_{n-1,m+1}$, (see [7, 8]). By (12), we get

$$\sum_{k=2}^{n-2} \binom{n}{k} (-1)^k B_k B_{n-k} = -(n + 1) B_n, \text{ (n \geq 4).}$$

In this paper, we study the applications of $p$-adic convolution associated with Daehee numbers and give some relations between Daehee numbers and Bernoulli numbers.
2. An application of $p$-adic convolution associated with Daehee numbers

By (5), we get
\[ f \otimes g(0) = f(0)g(0). \] (13)

Let us assume that $g(0) = 0$. From (10) and (13), we have
\[ (f \otimes g)'(0) = f \otimes g'(0) + f' \otimes g(0) + f \ast g(0) \]
\[ = f(0)g'(0) + f'(0)g(0) + f \ast g(0) \] (14)
\[ = f(0)g'(0) + f \ast g(0). \]

Let us define the difference operator $\Delta$ by $\Delta f(x) = f(x + 1) - f(x)$. Then, by (14) and (6), we get
\[ I_0((f \otimes \Delta g) = (f \otimes g)'(0) \]
\[ = f(0)g'(0) + f \ast g(0) \]
\[ = f(0)g'(0) + I_0(fg_-) - f \otimes g'(0) \]
\[ = f(0)g'(0) - f(0)g'(0) + I_0(fg_-) \]
\[ = I_0(fg_-), \] (15)
where $g_-(x) = g(-x)$. Let us take $f(z) = (z)_m, g(z) = (z)_n, (m, n \in \mathbb{N})$. Then, by (15), we get
\[ I_0((z)_m \otimes ((z + 1)_n - (z)_n)) = I_0((z)_m(z)_n) \]
\[ = (-1)^n I_0((z)_m z^n), \] (16)
where $z^n = z(z + 1) \cdots (z + n - 1)$. The unsigned Stirling number is defined by
\[ z^n = z(z + 1) \cdots (z + n - 1) = \sum_{\ell_1=0}^{n} \left[ \begin{array}{c} n \\ \ell_1 \end{array} \right] x^{\ell_1}. \] (17)

By (17), we get
\[ I_0((z)_m(z)_n) = (-1)^n I_0((z)_m z^n) \]
\[ = (-1)^n \sum_{\ell_1=0}^{m} S(m, \ell_1) \sum_{\ell_2=0}^{n} \left[ \begin{array}{c} n \\ \ell_2 \end{array} \right] \int_{\mathbb{Z}_p} z^{\ell_1+\ell_2} d\mu_0(z) \]
\[ = \sum_{\ell_1=0}^{m} \sum_{\ell_2=0}^{n} (-1)^n S_1(m, \ell_1) \left[ \begin{array}{c} n \\ \ell_2 \end{array} \right] B_{\ell_1+\ell_2}, \] (18)

Now, we observe that
\[ (z + 1)_n - (x)_n = (z + 1)z(z - 1) \cdots (z - n + 2) - z(z - 1) \cdots (z - n + 1) \]
\[ = nz(z - 1) \cdots (z - n + 2) = n(z)_{n-1}. \] (19)
An application of $p$-adic convolution

By (19), we get
\[ I_0((z)_m \otimes ((z + 1)_n - (z)_n)) = n I_0((z)_m \otimes (z)_{n-1}). \tag{20} \]

Let us define the sequence $T_{m,n}$ as follows:
\[ T_{m,n} = I_0((z)_m \otimes (z)_{n-1}). \tag{21} \]

From (21), we have
\[ T_{m,n} = T_{n-1,m+1}, \quad (m, n \in \mathbb{N}). \tag{22} \]

Therefore, by (16), (18) and (21), we obtain the following theorem.

**Theorem 2.1.** For $m, n \geq 1$, we have
\[ T_{m,n} = \frac{1}{n} \sum_{\ell_1=0}^{m} \sum_{\ell_2=0}^{n} (-1)^n S_1(m, \ell_1) \left[ \frac{n}{\ell_2} \right] B_{\ell_1+\ell_2}, \]
and
\[ T_{m,n} = T_{n-1,m+1}. \]

From Theorem 2.1, we have
\[ T_{0,n} = T_{n-1,1}, \quad (n \in \mathbb{N}). \tag{23} \]

Thus, (23), we have
\[ \frac{1}{n} \sum_{\ell=0}^{n-1} (-1)^{n-1} \left[ \frac{n}{\ell} \right] B_\ell = \sum_{\ell=0}^{n-1} S_1(n-1, \ell) B_{\ell+1}. \]

From (12), we have
\[ I^{(z)}_0(f * g(z)) = I^{(z)}_0(I^{(x)}_0(f(x)g(z-x)) - I^{(z)}_0(f * g'(z)). \tag{24} \]

Thus, by (24), we get
\[ I^{(z)}_0(f)I^{(z)}_0(g) = I^{(z)}_0(I^{(x)}_0(f(x)g(z-x)) - I^{(z)}_0(f \otimes g'(z)). \tag{25} \]

Let us take $f(z) = (z)_m$ and $g(z) = (z)_n$, $(m, n \in \mathbb{N})$. Then, by (8) and (9), (25), we get
\[ D_m D_n = \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} (z)_m(z-x)_n d\mu_0(z) - \int_{\mathbb{Z}_p} (z)_m \otimes (z)'_n d\mu_0(z). \tag{26} \]

Let
\[ K_{m,n} = \int_{\mathbb{Z}_p} (z)_m \otimes (z)'_n d\mu_0(z), \tag{27} \]
where \((z)_n' = d(z)_n/dz = \sum_{\ell=1}^n S_1(n, \ell)\ell z^{\ell-1}\). Now, we observe that
\[
\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} (x)_m(z-x)_n d\mu_0(z)
= \sum_{\ell_1=0}^m S_1(m, \ell_1) \sum_{\ell_2=0}^n S_1(n, \ell_2) \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} x^{\ell_1}(z-x)^{\ell_2} d\mu_0(x) d\mu_0(z)
\]
\[
= \sum_{\ell_1=0}^m \sum_{\ell_2=0}^n S_2(m, \ell_1) S_1(n, \ell_2) \sum_{i=0}^{-\ell_2} \binom{\ell_2}{i} (-1)^i \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} x^{\ell_1+i} z^{\ell_2-i} d\mu_0(x) d\mu_0(z)
\]
\[
= \sum_{\ell_1=0}^m \sum_{\ell_2=0}^n S_1(m, \ell_1) S_1(n, \ell_2) \sum_{i=0}^{-\ell_2} \binom{\ell_2}{i} (-1)^i B_{\ell_1+i} B_{\ell_2-i}
\]
\[
= D_m D_n + \sum_{\ell_1=0}^m \sum_{\ell_2=0}^n \sum_{i=1}^{\ell_2} S_1(m, \ell_1) S_1(n, \ell_2) \binom{\ell_2}{i} (-1)^i B_{\ell_1+i} B_{\ell_2-i}
\]
and
\[
\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} (x)_m(z-x)_n d\mu_0(z) = m!n! \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \binom{x}{m} \binom{z-x}{n} d\mu_0(x) d\mu_0(z).
\]
Therefore, by (28) and (29), we obtain the following theorem.

**Theorem 2.2.** For \(m, n \in \mathbb{N}\), we have
\[
\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \binom{x}{m} \binom{z-x}{n} d\mu_0(x) d\mu_0(z)
= \frac{D_m D_n}{m!n!} + \frac{1}{m!n!} \sum_{\ell_1=0}^m \sum_{\ell_2=0}^n \sum_{i=1}^{\ell_2} S_1(m, \ell_1) S_1(n, \ell_2) \binom{\ell_2}{i} (-1)^i B_{\ell_1+i} B_{\ell_2-i}.
\]
In particular,
\[
K_{m,n} = \sum_{\ell_1=0}^m \sum_{\ell_2=0}^n \sum_{i=1}^{\ell_2} S_1(m, \ell_1) S_1(n, \ell_2) \binom{\ell_2}{i} (-1)^i B_{\ell_1+i} B_{\ell_2-i}
\]
\[
= \sum_{\ell_1=0}^m \sum_{\ell_2=0}^n S_1(m, \ell_1) S_1(n, \ell_2) \ell_2 A_{\ell_1, \ell_2}.
\]

**References**


An application of $p$-adic convolution


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