Advanced Studies in Theoretical Physics
Vol. 8, 2014, no. 17, 745 - 754
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/astp.2014.4686

Identities of Some Special Mixed-Type Polynomials

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Abstract

In this paper, we consider various speical mixed-type polynomials which are related to Bernoulli, Euler, Changhee and Daehee polynomials. From those polynomials, we derive some interesting and new identities.

Mathematics Subject Classification: 05A19; 11B68; 11B83

Keywords: mixed-type polynomial, Bernoulli-Euler, Daehee-Changhee, Cauchy-Daehee, Cauchy-Changhee

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = \frac{1}{p} = p^{-\nu_p(p)}$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$.

For $f \in UD(\mathbb{Z}_p)$, the bosonic p-adic integral is given by

$$I_0(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_0(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad \text{(see [10])}, \tag{1}$$

and the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \quad \text{(see [12])}.$$

In [7, 8], the higher-order Daehee polynomials are defined by

$$\left(\frac{\log(1+t)}{t}\right)^r (1+t)^x = \sum_{n=0}^{\infty} D_n^{(r)}(x) \frac{t^n}{n!}, \quad (r \in \mathbb{N}). \tag{2}$$

When $x = 0$, $D_n^{(r)} = D_n^{(r)}(0)$ are called the Daehee numbers of order r.

When $r = 1$, $D_n^{(1)}(x) = D_n(x)$ are called the Daehee polynomials (see [7]).

As is known, the Changhee polynomials of order $s(\in \mathbb{N})$ are defined by the generating function to be

$$\left(\frac{2}{t+2}\right)^s (1+t)^x = \sum_{n=0}^{\infty} Ch_n^{(s)}(x) \frac{t^n}{n!}, \quad \text{(see [9])}. \tag{3}$$

When $x = 0$, $Ch_n^{(s)} = Ch_n^{(s)}(0)$ are called the Changhee numbers of order s.

For $s = 1$, $Ch_n^{(1)}(x) = Ch_n(x)$ are called the Changhee polynomials.

The Bernoulli polynomials of order $r \in \mathbb{N}$ are given by

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}, \quad \text{(see [10, 11, 13–21])}. \tag{4}$$

When $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$ are called the Bernoulli numbers of order r.

For $r = 1$, $B_n^{(1)}(x) = B_n(x)$ are called the ordinary Bernoulli polynomials.

We recall that the Euler polynomials of order r are defined by the generating function to be

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!}, \quad (\text{see [1–12]}). \tag{5}$$

When $x = 0$, $E_n^{(r)} = E_n^{(r)}(0)$ are called the Euler numbers of order r.

For $r = 1$, $E_n^{(1)}(x) = E_n(x)$ are called the ordinary Euler polynomials.

Finally, the Cauchy polynomials of the first kind of order r are given by

$$\left(\frac{t}{\log(1+t)}\right)^r (1+t)^x = \sum_{n=0}^{\infty} C_n^{(r)}(x)\frac{t^n}{n!}, \quad (\text{see [3, 6]}). \tag{6}$$

When $x = 0$, $C_n^{(r)} = C_n^{(r)}(0)$ are called the Cauchy numbers of the first kind of order r.

For $r = 1$, $C_n^{(1)}(x) = C_n(x)$ are called the ordinary Cauchy polynomials of the first kind (see [3]).

From (1) and (2), we have

$$I_0(f_1) - I_0(f) = f'(0) \tag{7}$$

and

$$I_{-1}(f_1) = -I_{-1}(f) + 2f(0), \tag{8}$$

where $f_1(x) = f(x+1)$ (see [11, 12]).

In this paper, we consider several special polynomials which are derived from the bosonoic or fermionic p-adic integral on $\mathbb{Z}_p$.

Finally, we give some relation or identities of those polynomials.

2. Some special mixed-type polynomials

In this section, we assume that $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$. From (7), we can derive the following equation :

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{(x_1+\cdots+x_r+x)}\, d\mu_0(x_1)\cdots d\mu_0(x_r) \qquad (9)\\
&= \left(\frac{\log(1+t)}{t}\right)^r (1+t)^x = \left(\frac{\log(1+t)}{e^{\log(1+t)}-1}\right)^r e^{x\log(1+t)}\\
&= \sum_{m=0}^{\infty} B_m^{(r)}(x)\frac{(\log(1+t))^m}{m!} = \sum_{m=0}^{\infty} B_m^{(r)}(x)\sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} B_m^{(r)}(x)\, S_1(n,m)\right)\frac{t^m}{n!},
\end{aligned}$$

and

$$\left(\frac{\log(1+t)}{t}\right)^r (1+t)^x = \sum_{n=0}^{\infty} D_n^{(r)}(x)\frac{t^n}{n!}. \tag{10}$$

Therefore, by (9) and (10), we obtain the following equation :

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{x_1+\cdots+x_r+x}{n}d\mu_0\left(x_1\right)\cdots d\mu_0\left(x_r\right) \tag{11}$$
$$=\frac{D_n^{(r)}\left(x\right)}{n!}=\frac{1}{n!}\sum_{m=0}^{n}B_m^{(r)}\left(x\right)S_1\left(n,m\right)$$

where $S_1\left(n,m\right)$ is the Stirling number of the first kind.

From (8), we have

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(1+t\right)^{x_1+\cdots+x_r+x}d\mu_{-1}\left(x_1\right)\cdots d\mu_{-1}\left(x_r\right) \tag{12}$$
$$=\left(\frac{2}{t+2}\right)^r\left(1+t\right)^x=\left(\frac{2}{e^{\log(1+t)}+1}\right)^r e^{x\log(1+t)}$$
$$=\sum_{m=0}^{\infty}E_m^{(r)}\left(x\right)\frac{\left(\log\left(1+t\right)\right)^m}{m!}=\sum_{m=0}^{\infty}E_m^{(r)}\left(x\right)\sum_{n=m}^{\infty}S_1\left(n,m\right)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}E_m^{(r)}\left(x\right)S_1\left(n,m\right)\right\}\frac{t^n}{n!}$$

and

$$\left(\frac{2}{t+2}\right)^r\left(1+t\right)^x=\sum_{n=0}^{\infty}Ch_n^{(r)}\left(x\right)\frac{t^n}{n!}. \tag{13}$$

From (12) and (13)

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{x_1+\cdots+x_r+x}{n}d\mu_{-1}\left(x_1\right)\cdots d\mu_{-1}\left(x_r\right) \tag{14}$$
$$=\frac{Ch_n^{(r)}\left(x\right)}{n!}=\frac{1}{n!}\sum_{m=0}^{n}E_m^{(r)}\left(x\right)S_1\left(n,m\right).$$

Note that

$$\left(1+t\right)^x=\left(\frac{t}{\log\left(1+t\right)}\right)^r\left(1+t\right)^x\left(\frac{\log\left(1+t\right)}{t}\right)^r \tag{15}$$
$$=\left(\sum_{l=0}^{\infty}C_l^{(r)}\left(x\right)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}D_m^{(r)}\frac{t^m}{m!}\right)$$
$$=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}C_l^{(r)}\left(x\right)D_{n-l}^{(r)}\right)\frac{t^n}{n!}$$

and

$$\left(1+t\right)^x=\sum_{n=0}^{\infty}\left(x\right)_n\frac{t^n}{n!}. \tag{16}$$

From (15) and (16), we have

$$(x)_n = \sum_{l=0}^{n} \binom{n}{l} C_l^{(r)}(x) D_{n-l}^{(r)} = \sum_{l=0}^{n} \binom{n}{l} D_{n-l}^{(r)}(x) C_l^{(r)}. \tag{17}$$

That is,

$$\binom{x}{n} = \frac{1}{n!} \sum_{l=0}^{n} \binom{n}{l} C_l^{(r)}(x) D_{n-l}^{(r)}.$$

Let us consider the Bernoulli-Euler mixed-type polynomials of order (r, s) as follows :

$$BE_n^{(r,s)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} E_n^{(s)}(x + y_1 + \cdots + y_r)\, d\mu_0(y_1) \cdots d\mu_0(y_r). \tag{18}$$

Then, we can find the generating function of $BE_n^{(r,s)}(x)$ as follows :

$$\begin{aligned} &\sum_{n=0}^{\infty} BE_n^{(r,s)}(x) \frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} E_n^{(s)}(x + y_1 + \cdots + y_r) \frac{t^n}{n!} d\mu_0(y_1) \cdots d\mu_0(y_r) \\ &= \left(\frac{2}{e^t + 1}\right)^s \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x+y_1+\cdots+y_r)t} d\mu_0(y_1) \cdots d\mu_0(y_r) \\ &= \left(\frac{2}{e^t + 1}\right)^s \left(\frac{t}{e^t - 1}\right)^r e^{xt}. \end{aligned} \tag{19}$$

Note that

$$\begin{aligned} \left(\frac{2}{e^t + 1}\right)^s \left(\frac{t}{e^t - 1}\right)^r e^{xt} &= \left(\sum_{l=0}^{\infty} E_l^{(s)} \frac{t^l}{l!}\right) \left(\sum_{m=0}^{\infty} B_m^{(r)}(x) \frac{t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} E_l^{(s)} B_{n-l}^{(r)}(x)\right) \frac{t^n}{n!}. \end{aligned} \tag{20}$$

From (19) and (20), we have

$$BE_n^{(r,s)}(x) = \sum_{l=0}^{n} \binom{n}{l} E_l^{(s)} B_{n-l}^{(r)}(x). \tag{21}$$

By replacing t by $\log(1 + t)$, we get

$$\sum_{n=0}^{\infty} BE_n^{(r,s)}(x) \frac{(\log(1 + t))^n}{n!} \tag{22}$$

$$= \left(\frac{2}{t+2}\right)^s \left(\frac{\log(1+t)}{t}\right)^r (1+t)^x$$
$$= \left(\sum_{l=0}^{\infty} Ch_l^{(s)} \frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} D_m^{(r)}(x)\frac{t^m}{m!}\right)$$
$$= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m} D_m^{(r)}(x)\, Ch_{n-m}^{(s)}\right\}\frac{t^n}{n!},$$

and

$$\sum_{m=0}^{\infty} BE_m^{(r,s)}(x)\frac{(\log(1+t))^m}{m!} = \sum_{m=0}^{\infty} BE_m^{(r,s)}(x)\sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!} \tag{23}$$
$$= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n} BE_m^{(r,s)}(x)\, S_1(n,m)\right\}\frac{t^n}{n!}.$$

Therefore, by (22) and (23), we obtain the following equation :

$$\sum_{m=0}^{n}\binom{n}{m} D_m^{(r)}(x)\, Ch_{n-m}^{(s)} = \sum_{m=0}^{n} BE_m^{(r,s)}(x)\, S_1(n,m). \tag{24}$$

Let us consider the Daehee-Changhee mixed-type polynomials of order (r, s) as follows :

$$DC_n^{(r,s)}(x) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} D_n^{(r)}(x+y_1+\cdots+y_s)\, d\mu_{-1}(y_1)\cdots d\mu_{-1}(y_r), \tag{25}$$

where $n \geq 0$.

From (25), we can derive the generating function of $DC_n^{(r,s)}(x)$ as follows :

$$\sum_{n=0}^{\infty} DC_n^{(r,s)}(x)\frac{t^n}{n!} \tag{26}$$
$$= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{n=0}^{\infty} D_n^{(r)}(x+y_1+\cdots+y_s)\frac{t^n}{n!} d\mu_{-1}(y_1)\cdots d\mu_{-1}(y_s)$$
$$= \left(\frac{\log(1+t)}{t}\right)^r \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} (1+t)^{x+y_1+\cdots+y_s}\, d\mu_{-1}(y_1)\cdots d\mu_{-1}(y_s)$$
$$= \left(\frac{\log(1+t)}{t}\right)^r \left(\frac{2}{t+2}\right)^s (1+t)^x.$$

We observe that

$$\left(\frac{2}{t+2}\right)^s \left(\frac{\log(1+t)}{t}\right)^r (1+t)^x = \left(\sum_{l=0}^{\infty} Ch_l^{(s)}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} D_m^{(r)}(x)\frac{t^m}{m!}\right) \tag{27}$$
$$= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m} D_m^{(r)}(x)\, Ch_{n-m}^{(s)}\right\}\frac{t^n}{n!}.$$

From (26) and (27), we have

$$DC_n^{(r,s)}(x) = \sum_{m=0}^{n} \binom{n}{m} D_m^{(r)}(x)\, Ch_{n-m}^{(r)}, \tag{28}$$

where $n \geq 0$, $r, s \in \mathbb{N}$.

Now, we define the Cauchy-Daehee mixed-type polynomials of order (r,s) as follows :

$$CD_n^{(r,s)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} C_n^{(r)}(x + y_1 + \cdots + y_s)\, d\mu_0(y_1) \cdots d\mu_0(y_r). \tag{29}$$

From (29), we can derive the generating function of $CD_n^{(r,s)}(x)$ as follows :

$$\begin{aligned}
&\sum_{n=0}^{\infty} CD_n^{(r,s)}(x) \frac{t^n}{n!} \qquad (30)\\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} C_n^{(r)}(x + y_1 + \cdots + y_s) \frac{t^n}{n!} d\mu_0(y_1) \cdots d\mu_0(y_s)\\
&= \left(\frac{t}{\log(1+t)}\right)^r \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{x+y_1+\cdots+y_s}\, d\mu_0(y_1) \cdots d\mu_0(y_s)\\
&= \left(\frac{t}{\log(1+t)}\right)^r \left(\frac{\log(1+t)}{t}\right)^s (1+t)^x .\\
&= \begin{cases} \sum_{n=0}^{\infty} C_n^{(r-s)}(x) \frac{t^n}{n!} & \text{if } r > s \\ \sum_{n=0}^{\infty} D_n^{(s-r)}(x) \frac{t^n}{n!} & \text{if } r < s \\ \sum_{n=0}^{\infty} (x)_n \frac{t^n}{n!} & \text{if } r = s. \end{cases}
\end{aligned}$$

Thus, by (30), we get

$$CD_n^{(r,s)}(x) = \begin{cases} C_n^{(r-s)}(x) & \text{if } r > s \\ D_n^{(s-r)}(x) & \text{if } r < s \\ (x)_n & \text{if } r = s \end{cases} \tag{31}$$

where $n \geq 0$.

By replacing t by $e^t - 1$ in (26), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} DC_n^{(r,s)}(x) \frac{(e^t-1)^n}{n!} &= \left(\frac{t}{e^t-1}\right)^r e^{xt} \left(\frac{2}{e^t+1}\right)^s \qquad (32)\\
&= \left(\sum_{n=0}^{\infty} B_l^{(r)}(x) \frac{t^l}{l!}\right) \left(\sum_{m=0}^{\infty} E_m^{(s)} \frac{t^m}{m!}\right)\\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} B_l^{(r)}(x)\, E_{n-l}\right) \frac{t^n}{n!},
\end{aligned}$$

and

$$\sum_{m=0}^{\infty} DC_n^{(r,s)}(x) \frac{(e^t-1)^m}{m!} = \sum_{m=0}^{\infty} DC_m^{(r,s)}(x) \sum_{n=m}^{\infty} S_2(n,m) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} DC_m^{(r,s)}(x) S_2(n,m) \right) \frac{t^n}{n!}. \tag{33}$$

Therefore, by (32) and (33), we get

$$\sum_{m=0}^{n} DC_m^{(r,s)}(x) S_2(m,n) = \sum_{l=0}^{n} \binom{n}{l} B_l^{(r)}(x) E_{n-l}, \tag{34}$$

where $S_2(n,m)$ is the Stirling number of the second kind.

Finally, we consider the Cauchy-Changhee mixed-type polynomials of order (r,s) as follows :

$$CC_n^{(r,s)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} C_n^{(r)}(x+y_1+\cdots+y_s) \, d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_s), \tag{35}$$

where $n \geq 0$.

By (35), we see that the generating function of $CC_n^{(r,s)}(x)$ are given by

$$\begin{aligned} &\sum_{n=0}^{\infty} CC_n^{(r,s)}(x) \frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} C_n^{(r)}(x+y_1+\cdots+y_s) \frac{t^n}{n!} d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_s) \\ &= \left(\frac{t}{\log(1+t)} \right)^r \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{x+y_1+\cdots+y_s} \, d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_s) \\ &= \left(\frac{t}{\log(1+t)} \right)^r \left(\frac{2}{t+2} \right)^s (1+t)^x \\ &= \left(\sum_{m=0}^{\infty} C_m^{(r)}(x) \frac{t^m}{m!} \right) \left(\sum_{l=0}^{\infty} Ch_l^{(s)} \frac{t^l}{l!} \right) \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} \binom{n}{m} C_m^{(r)}(x) Ch_{n-m}^{(s)} \right\} \frac{t^n}{n!}. \end{aligned} \tag{36}$$

Thus, by (36), we get

$$CC_n^{(r,s)}(x) = \sum_{m=0}^{n} \binom{n}{m} C_m^{(r)}(x) Ch_{n-m}^{(s)}. \tag{37}$$

By replacing t by $e^t - 1$, we get

$$\sum_{n=0}^{\infty} CC_n^{(r,s)}(x) \frac{(e^t-1)^n}{n!} = \left(\frac{e^t-1}{t} \right)^r \left(\frac{2}{e^t+1} \right)^s e^{xt} \tag{38}$$

$$= \left(\sum_{l=0}^{\infty} \frac{S_2(l+r,l)}{\binom{l+r}{r}} \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} E_m^{(s)}(x) \frac{t^m}{m!} \right)$$
$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \frac{S_2(l+r,l) E_{n-l}^{(s)}(x)}{\binom{l+r}{l}} \binom{n}{l} \right) \frac{t^n}{n!},$$

and

$$\sum_{l=0}^{\infty} CC_l^{(r,s)}(x) \frac{(e^t-1)^l}{l!} = \sum_{l=0}^{\infty} CC_l^{(r,s)}(x) \sum_{n=l}^{\infty} S_2(n,l) \frac{t^n}{n!} \tag{39}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} CC_l^{(r,s)}(x) S_2(n,l) \right) \frac{t^n}{n!}.$$

Therefore, by (38) and (39), we obtain the following identities.

$$\sum_{l=0}^{n} CC_l^{(r,s)}(x) S_2(n,l) = \sum_{l=0}^{n} \frac{\binom{n}{l}}{\binom{l+r}{l}} S_2(l+r,l) E_{n-l}^{(s)}(x),$$

where $n \geq 0$.

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Received: June 7, 2014