Identities of Symmetry for

Carlitz $q$-Bernoulli Polynomials Arising

from $q$-Volkenborn Integrals on $\mathbb{Z}_p$

under Symmetry Group $S_3$

Dmitry V. Dolgy

Institute of Mathematics and Computer Sciences
Far Eastern Federal University
Vladivostok, 690060, Russia

Dae San Kim

Department of Mathematics, Sogang University
Seoul 121-742, Republic of Korea

Taekyun Kim

Department of Mathematics, Kwangwoon University
Seoul 139-701, Republic of Korea

Jong-Jin Seo

Department of Applied Mathematics
Pukyong National University
Pusan 608-737, Republic of Korea

Copyright © 2014 Dmitry V. Dolgy, Dae San Kim, Taekyun Kim and Jong-Jin Seo. This is an open access article distributed under the Creative Commons Attribution License,
Abstract

In this paper, we give basic identities of symmetry in three variables related to Carlitz $q$-Bernoulli polynomials and $q$-power sums which are derived from $p$-adic $q$-Volkenborn integrals on $\mathbb{Z}_p$.

Mathematics Subject Classification: 11B68; 11S80; 11B83

Keywords: $q$-Volkenborn integrals ; Carlitz $q$-Bernoulli polynomials

1. Introduction

Let $p$ be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let $v_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = \frac{1}{p}$. Let $q$ be an indeterminate in $\mathbb{C}_p$ with $|1 - q|_p < p^{-\frac{1}{p-1}}$. The $q$-number of $x$ is defined by $[x]_q = \frac{1-qx}{1-q}$. Note that $\lim_{q \to 1} [x]_q = x$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the $p$-adic $q$-Volkenborn integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) \, d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \text{ (see [7, 9])}.$$  \hfill (1)

Let $d$ be a fixed positive integer. For $N \in \mathbb{N}$, we set

$$X = X_d = \lim_{N} \frac{\mathbb{Z}}{dp^N \mathbb{Z}}, \quad X^* = \bigcup_{0 < a < dp \atop (a, p) = 1} a + dp \mathbb{Z}_p,$$

$$a + dp^N \mathbb{Z}_p = \{ x \in X \mid x \equiv a \pmod{dp^N} \},$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$. 


From (1), we note that
\[
\int_{\mathbb{Z}_p} f(x) \, d\mu_q(x) = \lim_{N \to \infty} \frac{1}{p^N q} \sum_{x=0}^{p^N-1} f(x) q^x
\]
\[
= \lim_{N \to \infty} \frac{1}{\beta^N q} \sum_{x=0}^{\beta^N-1} f(x) \beta^x
\]
\[
= \int_X f(x) \, d\mu_q(x) \quad \text{(see [8, 9]).}
\]
It is well known that the Bernoulli numbers are given by
\[
B_0 = 1 \text{ and } (B+1)^n - B_n = \begin{cases} 
1 & \text{if } n = 1 \\
0 & \text{if } n > 1,
\end{cases}
\]
with the usual convention about replacing $B^i$ by $B_i$ (see [1-18]).
The Bernoulli polynomials are defined by
\[
B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}.
\]
In [1], L. Carlitz considered the $q$-extension of Bernoulli numbers which are given by
\[
\beta_0 = 1, \quad q (q \beta + 1)^n - \beta_n = \begin{cases} 
1 & \text{if } n = 1 \\
0 & \text{if } n > 1,
\end{cases}
\]
with the usual convention about replacing $\beta^i$ by $\beta_i$.
The $q$-Bernoulli polynomials are given by
\[
\beta_n(x) = \left(q^x \beta + [x]_q\right)^n = \sum_{l=0}^{n} \binom{n}{l} q^{lx} \beta_{l,q}[x]_q^{n-l}.
\]
In [9], Kim gave the Witt’s formula for the Carlitz $q$-Bernoulli polynomials as follows :
\[
\beta_n(x) = \int_{\mathbb{Z}_p} [x + y]_q^n \, d\mu_q(y) \quad \text{(n ≥ 0)}.
\]
In this paper, we give basic identities of symmetry in three variables related to Carlitz $q$-Bernoulli polynomials and $q$-analogue of sums of powers of integers which are derived from $p$-adic $q$-integrals on $\mathbb{Z}_p$ under symmetry group $S_3$. 

Symmetry identities of Carlitz $q$-Bernoulli polynomials under $S_3$
2. Symmetry identities of Carlitz \( q \)-Bernoulli polynomials

Let \( w_1, w_2, w_3 \) be positive integers. Then, by (2), we get

\[
\int_{\mathbb{Z}_p} e^{[w_2 w_3 y + w_1 w_2 w_3 x + w_1 w_3 i + w_1 w_2 j]q^t} d\mu_{q^{w_2 w_3}}(y)
\]

(8)

\[
= \lim_{N \to \infty} \frac{1}{[p^N]_{q^{w_2 w_3}}} \sum_{y=0}^{p^{N-1}-1} e^{[w_2 w_3 y + w_1 w_2 w_3 x + w_1 w_3 i + w_1 w_2 j]q^t} q^{w_2 w_3 y}
\]

\[
= \lim_{N \to \infty} \frac{1}{[w_1 p^N]_{q^{w_2 w_3}}} \sum_{k=0}^{w_1-1} \sum_{y=0}^{p^{N-1}-1} e^{[w_2 w_3 (k+w_1 y) + w_1 w_2 w_3 x + w_1 w_3 i + w_1 w_2 j]q^t}
\]

\[
\times q^{w_2 w_3 (k+w_1 y)}.
\]

From (8), we have

\[
\frac{1}{[w_2 w_3]_q} \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} q^{w_1 w_3 i + w_1 w_2 j} \int_{\mathbb{Z}_p} e^{[w_2 w_3 y + w_1 w_2 w_3 x + w_1 w_3 i + w_1 w_2 j]q^t} d\mu_{q^{w_2 w_3}}(y)
\]

(9)

\[
= \lim_{N \to \infty} \frac{1}{[w_1 w_2 w_3 p^N]_q} \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} \sum_{k=0}^{w_1-1} \sum_{y=0}^{p^{N-1}-1} q^{w_1 w_3 i + w_1 w_2 j + w_2 w_3 k}
\]

\[
\times e^{[w_2 w_3 (k+w_1 y) + w_1 w_2 w_3 x + w_1 w_3 i + w_1 w_2 j]q^t} q^{w_2 w_3 y}.
\]

From (10), we observe that the expression in (9) is invariant under any permutation of \( w_1, w_2, w_3 \). So we would get the full six identities of symmetry. However, here not all permutations of \( w_1, w_2, w_3 \) in (9) yield distinct ones. In fact, only three of them are distinct. So we get Theorem (1).

\textit{Remark.} As was already observed in [5], in general, the expressions obtained from a single one by permuting \( w_1, w_2, w_3 \) can be viewed as a group in a natural manner. Thus the group of such expressions is isomorphic to a quotient of \( S_3 \). In particular, the number of possible distinct expressions are 1, 2, 3, or 6. This explains why we have three distinct ones in the following theorem.
Theorem 1. For $w_1, w_2, w_3 \in \mathbb{Z}$ with $w_1 \geq 1$, $w_2 \geq 1$, $w_3 \geq 1$, we have

\[
\frac{1}{[w_2 w_3]_q} \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} q^{w_1 w_3 i + w_1 w_2 j} \int_{\mathbb{Z}_p} e^{[w_2 w_3 y + w_1 w_2 w_3 x + w_1 w_3 i + w_1 w_2 j]_q} d\mu_{q^{w_2 w_3}}(y)
\]

\[
= \frac{1}{[w_3 w_1]_q} \sum_{i=0}^{w_3-1} \sum_{j=0}^{w_1-1} q^{w_2 w_3 i + w_2 w_3 j} \int_{\mathbb{Z}_p} e^{[w_3 w_1 y + w_1 w_2 w_3 x + w_2 w_3 i + w_2 w_3 j]_q} d\mu_{q^{w_3 w_1}}(y)
\]

\[
= \frac{1}{[w_1 w_2]_q} \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} q^{w_3 w_2 i + w_3 w_1 j} \int_{\mathbb{Z}_p} e^{[w_1 w_2 y + w_1 w_2 w_3 x + w_3 w_2 i + w_3 w_1 j]_q} d\mu_{q^{w_1 w_2}}(y)
\]

Now, we observe that

\[
[w_2 w_3 y + w_1 w_2 w_3 x + w_1 w_3 i + w_1 w_2 j]_q = [w_2 w_3]_q \left[ y + w_1 x + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j \right]_{q^{w_2 w_3}}.
\]

Therefore, by Theorem 1 and (11), we obtain the following corollary.

Corollary 2. For $n \geq 0$, $w_1, w_2, w_3 \geq 1$, we have

\[
[w_2 w_3]_q^{n-1} \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} q^{w_1 w_3 i + w_1 w_2 j} \int_{\mathbb{Z}_p} \left[ y + w_1 x + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j \right]_q^n d\mu_{q^{w_2 w_3}}(y)
\]

\[
= [w_3 w_1]_q^{n-1} \sum_{i=0}^{w_3-1} \sum_{j=0}^{w_1-1} q^{w_2 w_3 i + w_2 w_3 j} \int_{\mathbb{Z}_p} \left[ y + w_2 x + \frac{w_2}{w_3} i + \frac{w_2}{w_1} j \right]_q^n d\mu_{q^{w_3 w_1}}(y)
\]

\[
= [w_1 w_2]_q^{n-1} \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} q^{w_3 w_2 i + w_3 w_1 j} \int_{\mathbb{Z}_p} \left[ y + w_3 x + \frac{w_3}{w_2} i + \frac{w_3}{w_1} j \right]_q^n d\mu_{q^{w_1 w_2}}(y).
\]

Therefore, by (7) and Corollary 2, we obtain the following theorem.

Theorem 3. For $n \geq 0$, we have

\[
[w_2 w_3]_q^{n-1} \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} q^{w_1 w_3 i + w_1 w_2 j} \beta_{n, q^{w_2 w_3}} \left( w_1 x + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j \right)
\]

\[
= [w_3 w_1]_q^{n-1} \sum_{i=0}^{w_3-1} \sum_{j=0}^{w_1-1} q^{w_2 w_3 i + w_2 w_3 j} \beta_{n, q^{w_3 w_1}} \left( w_2 x + \frac{w_2}{w_3} i + \frac{w_2}{w_1} j \right)
\]

\[
= [w_1 w_2]_q^{n-1} \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} q^{w_3 w_2 i + w_3 w_1 j} \beta_{n, q^{w_1 w_2}} \left( w_3 x + \frac{w_3}{w_1} i + \frac{w_3}{w_2} j \right).
\]
It is easy to show that
\[
\left[ y + w_1 x + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j \right]_{q^{w_2 w_3}}^{w_1 x}
\]
\[
= \frac{1 - q^{w_1 w_3 i + w_1 w_2 j}}{1 - q^{w_2 w_3}} + q^{w_1 w_3 i + w_1 w_2 j} [y + w_1 x]_{q^{w_2 w_3}}
\]
\[
= \frac{1 - q^{w_1}}{1 - q^{w_2 w_3}} \times \frac{1 - q^{w_1 (w_3 i + w_2 j)}}{1 - q^{w_1}} + q^{w_1 w_3 i + w_1 w_2 j} [y + w_1 x]_{q^{w_2 w_3}}
\]
\[
= \frac{[w_1]_q}{[w_2 w_3]_q} [w_3 i + w_2 j]_{q} + q^{w_1 w_3 i + w_1 w_2 j} [y + w_1 x]_{q^{w_2 w_3}}.
\]
By (12), we get
\[
\int_{\mathbb{Z}_p} \left[ y + w_1 x + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j \right]_{q^{w_2 w_3}}^{n} d\mu_{q^{w_2 w_3}} (y)
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} \left( \frac{[w_1]_q}{[w_2 w_3]_q} \right)^{n-k} [w_3 i + w_2 j]_{q}^{n-k} q^{k(w_1 w_3 i + w_1 w_2 j)} \beta_{k, q^{w_2 w_3}} (w_1 x).
\]
From (13), we note that
\[
[w_2 w_3]_q^{n-1} \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} q^{w_1 w_3 i + w_1 w_2 j} \int_{\mathbb{Z}_p} \left[ y + w_1 x + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j \right]_{q^{w_2 w_3}}^{n} d\mu_{q^{w_2 w_3}} (y)
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} [w_2 w_3]_q^{k-1} [w_1]_q^{n-k} \beta_{k, q^{w_2 w_3}} (w_1 x) \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} q^{(w_1 w_3 i + w_1 w_2 j)(k+1)} [w_3 i + w_2 j]_{q}^{n-k}
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} [w_2 w_3]_q^{k-1} [w_1]_q^{n-k} \beta_{k, q^{w_2 w_3}} (w_1 x) T_{n,q^{w_1}} (w_2, w_3 | k)
\]
where
\[
T_{n,q} (w_1, w_2 | k) = \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} q^{(w_2 i + w_1 j)(k+1)} [w_3 i + w_1 j]_{q}^{n-k}.
\]
Observe here that $T_{n,q} (w_1, w_2 | k) = T_{n,q} (w_2, w_1 | k)$.
By the same method as (14), we get
\[
[w_3 w_1]_q^{n-1} \sum_{i=0}^{w_3-1} \sum_{j=0}^{w_1-1} q^{w_2 w_1 i + w_2 w_3 j} \int_{\mathbb{Z}_p} \left[ y + w_2 x + \frac{w_2}{w_3} i + \frac{w_2}{w_1} j \right]_{q^{w_2 w_3}}^{n} d\mu_{q^{w_3 w_1}} (y)
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} [w_3 w_1]_q^{k-1} [w_2]_q^{n-k} \beta_{k, q^{w_2 w_1}} (w_2 x) T_{n,q^{w_2}} (w_3, w_1 | k).
\]
Therefore, by (14) and (15), we obtain the following theorem.

**Theorem 4.** For \( n \geq 0, w_1, w_2, w_3 \geq 1 \), we have

\[
\sum_{k=0}^{n} \binom{n}{k} [w_2 w_3]_q^{k-1} [w_1]_q^{n-k} \beta_{k,q}^{w_2 w_3} (w_1 x) T_{n,q,w_1} (w_2, w_3 | k) \\
= \sum_{k=0}^{n} \binom{n}{k} [w_3 w_1]_q^{k-1} [w_2]_q^{n-k} \beta_{k,q}^{w_3 w_1} (w_2 x) T_{n,q,w_2} (w_3, w_1 | k) \\
= \sum_{k=0}^{n} \binom{n}{k} [w_1 w_2]_q^{k-1} [w_3]_q^{n-k} \beta_{k,q}^{w_1 w_2} (w_3 x) T_{n,q,w_3} (w_1, w_2 | k),
\]

where

\[ T_{n,q} (w_1, w_2 | k) = \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} q^{(w_2 i + w_1 j)(k+1)} [w_2 i + w_1 j]_q^{n-k}. \]

**Acknowledgement**

This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MOE) (No.2012R1A1A2003786).

**References**


Dmitry V. Dolgy, Dae San Kim, Taekyun Kim and Jong-Jin Seo


Received: June 7, 2014