Auto-Bäcklund Transformations and New Exact Travelling Wave Solutions of Kadomtsev-Petviashvili Equation for Nonlinear Dust Acoustic Solitary Waves in Dust Plasmas with Variable Dust Charge and Two Temperature Ions

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Abstract

The nonlinear dust acoustic solitary waves (DASWs) in two-dimensional dust plasma with variable dust charge and two temperature ions has been studied analytically. The Kadomtsev-Petviashvili equation (KP) equation for unmagnitized dusty plasmas is derived by using the reductive perturbation theory. It suggests that the nonlinear dust acoustic solitary waves with variable dust charge are stable even there are some higher order transverse perturbations. We used the auto-Bäcklund (BT) Transformation, the sine-cosine expansion method, the sinh-coshine expansion method, the sech-tanh expansion method and the modified $\frac{G}{G_s}$ expansion method for nonlinear partial differential equations (NPDEs) to obtained the new exact travelling wave solutions of the KP equation describing the nonlinear DASWs in dust plasma.
Keywords: dust plasma; KP equation, Reductive perturbation method, solitary waves, Bäcklund transformations, Traveling wave solutions, Nonlinear partial differential equations

1 Introduction

The study of dusty plasmas represents one the most rapidly growing branches of plasma physics. A dusty plasma is an ionized gas containing small particles of solid matter, which acquire a large electric charge by collecting electrons and ions from the plasma. Dust grains or particles are usually highly charged. Charged dust components appear naturally in space environments such as planetary rings, cometary surroundings, interstellar clouds, and lower parts of earths ionospheres [1-4]. The low frequency oscillations in DASWs have been studied in [5-11]. Rao et al [12] first predicted theoretically the existence of extremely low phase velocity dust acoustic waves in unmagnetized dusty plasmas whose constituents were inertial charged dust grains and Boltzmann-distributed ions and electrons. These waves were reported experimentally and their nonlinear features investigated by Barkar et al [13]. Due to their importance, the solitary waves in unmagnetized plasma without geometry distortion and the dissipation effects have been extensively investigated and it was found that the solitary waves could be described by the Korteweg-de Vries (KdV) equation or Kadomtsev-Petviashvili (KP) equation [14-16].

In the presented paper, the DASWs with variable dust charge and two temperature ions has been considered. One can obtain the KP equation using the reductive perturbation method on two dimensional unmagnetized case of this system. We would like to use the homogeneous balance method [17,18] to constructed an auto-Bäcklund Transformation (BT)[19-22] , new exact soliton solutions of KP equation are obtained and we have applied the sine-cosine expansion method, the sinh-coshine expansion method [23-26] , the sech-tanh expansion method [24,27,28] and the modified $\sigma$ expansion method [29-35] to obtained a new travelling wave solutions of KP equation.

The paper is organized as follows :This introduction in Section 1. In Section 2, the model description and the theoretical aspects of the model. The KP equation and its solitary answer are derived and described in this section. In section 3, the auto-Bäcklund Transformation (BT) applying to construct new exact soliton solutions of KP equation. In Section 4, the sine-cosine expansion method, the sinh-coshine expansion method, the sech-tanh expansion method and the modified $\sigma$ expansion method are applied to construct new travelling wave solutions of the KP equation. Finally, physical application of the solutions are given in section 5. Conclusions are given in section 6.
2 Model Description

We now assume that the plasma is unmagnetized and the wave is assumed to propagate in the x direction, however there are higher order transverse perturbations in y direction. In a dusty plasma, the dust grains are much heavier compared to the ordinary ions or electrons. The dusty plasma studied here consists of highly negatively charged dust grains, electrons, and two temperature ions. Charge neutrality at equilibrium gives \( n_{e0} + n_{d0} Z_{d0} = n_{il0} + n_{ih0}, \) where \( n_{e0}, n_{d0}, n_{il0} \) and \( n_{ih0} \) are the unperturbed number density of electrons, low-temperature ions, and high-temperature ions, respectively. \( Z_{d0} \) is the unperturbed number of charges residing on the dust grains measured in the unit of electron charge. For the dust fluid with adiabatic variation of dust charge, we have the following weakly two-dimensional sets of equation of motion

\[
\frac{\partial n_d}{\partial t} + \frac{\partial}{\partial x} (n_d u_d) + \frac{\partial}{\partial y} (n_d v_d) = 0 \tag{1}
\]

\[
\frac{\partial u_d}{\partial t} + u_d \frac{\partial u_d}{\partial x} + v_d \frac{\partial u_d}{\partial y} = Z_d \frac{\partial \phi}{\partial x} \tag{2}
\]

\[
\frac{\partial v_d}{\partial t} + u_d \frac{\partial v_d}{\partial x} + v_d \frac{\partial v_d}{\partial y} = Z_d \frac{\partial \phi}{\partial y} \tag{3}
\]

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = Z_d n_d + n_e - n_{il} - n_{ih} \tag{4}
\]

where \( n_d \) is the number density of dust particles, \( u_d \) and \( v_d \) are the velocities of the dust particles, \( \phi \) is the electrostatic potential and \( Z_d \) is the number charge residing on the dust grain. We define \( T_{\text{eff}} \) as the effective temperature, \( T_{\text{eff}} = \left[ \frac{1}{Z_d n_d} \left( \frac{n_{e0}}{T_e} + \frac{n_{il0}}{T_{il}} + \frac{n_{ih0}}{T_{ih}} \right) \right]^{-1} \), in which \( T_e, T_{il} \) and \( T_{ih} \) are the plasma electron temperature and the temperatures of plasma ions at lower and higher temperatures, respectively. All of these variable are normalized and the number densities of electrons, lower temperature ions and higher temperature ions are respectively as follows:

\[
n_e = \nu \exp(S \beta_1 \phi) \tag{5}
\]

\[
n_{il} = \mu_l \exp(-S \phi) \tag{6}
\]

\[
n_{ih} = \mu_h \exp(-S \beta_2 \phi) \tag{7}
\]

where \( \beta_1 = \frac{T_{il}}{T_e}, \quad \beta_2 = \frac{T_{ih}}{T_{il}}, \quad \beta = \frac{\beta_1}{\beta_2}, \quad S = \frac{T_{\text{eff}}}{T_{il}} = \frac{\delta_1 + \delta_2 - 1}{\delta_1 + \delta_2 \beta + \beta_1}, \quad \nu = \frac{n_{e0}}{n_{d0} Z_d}, \)

\[
\mu_l = \frac{n_{il0}}{n_{d0} Z_d} \quad \mu_h = \frac{n_{ih0}}{n_{d0} Z_d} \quad \delta_1 = \frac{n_{il0}}{n_{e0}} \quad \text{and} \quad \delta_2 = \frac{n_{ih0}}{n_{e0}}.
\]

The space coordinates \( x, t \), velocity \( (u_d, v_d) \), and electrical potential \( \phi \) are normalized by the Debye length \( \lambda_D = \left( \frac{T_{\text{eff}}}{4 \pi n_{d0} Z_d e^2} \right)^{\frac{1}{2}} \), the inverse of dust plasma frequency \( \omega_{pd}^{-1} = \left( \frac{m_d}{4 \pi n_{d0} Z_d e^2} \right)^{\frac{1}{2}} \), the dust acoustic speed \( C_d = \left( \frac{Z_d T_{\text{eff}}}{m_d} \right)^{\frac{1}{2}} \), and \( \frac{T_{\text{eff}}}{e} \) respectively, where \( e \) and \( m_d \) are the charge of an electron and the mass of dust particle respectively.
We now use the reductive perturbation method to obtain the Kadomtsev-Petiauskii (KP) equation that governs the behavior of small amplitude dust acoustic waves. The independent variables are stretched as

\[ X = \epsilon (x - v_0 t), \quad T = \epsilon^3 t, \quad Y = \epsilon^2 y \]  

where \( \epsilon \) is a small dimensionless expansion parameter which characterizes the strength of nonlinearity in the system and \( v_0 \) is the phase velocity of the wave along the \( x \) direction. We can expand physical quantities which have been appeared in (1-4) in term of the expansion parameter \( \epsilon \) as

\[ n_d = 1 + \epsilon^2 n_1 + \epsilon^4 n_2 + \ldots \]  
\[ u_d = \epsilon^2 u_1 + \epsilon^4 u_2 + \ldots \]  
\[ v_d = \epsilon^3 v_1 + \epsilon^5 v_2 + \ldots \]  
\[ \phi = \epsilon^2 \phi_1 + \epsilon^4 \phi_2 + \ldots \]  
\[ Z_d = 1 + \epsilon^2 Z_1 + \epsilon^4 Z_2 + \ldots \]

where \( Z_1 = \gamma_1 \phi_1, \) \( Z_2 = \gamma_1 \phi_2 + \gamma_2 \phi_1^2 \) \[36\]

Substituting from equations (9-13) into equations (1-4). The KP equation is derived in the form

\[ \frac{\partial}{\partial X} \left[ \frac{\partial \phi_1}{\partial T} + A \frac{\partial \phi_1}{\partial X} + B \frac{\partial^3 \phi_1}{\partial X^3} \right] + D \frac{\partial^2 \phi_1}{\partial Y^2} = 0 \]  

where

\[ A = \frac{v_0^3}{2} \left[ (\delta_1 + \delta_2 \beta^2 - \beta_1^2) \frac{\delta_1 + \delta_2 - 1}{(\delta_1 + \delta_2 \beta + \beta_1^2)} - 2 \gamma_2 \right] + \frac{3}{2} \gamma_1 v_0 - \frac{3}{2 v_0}; \quad B = \frac{v_0^3}{2}; \quad D = \frac{v_0}{2} \]  

3 Auto-Bäcklund Transformation (BT) and New Exact Soliton Solutions of KP Equation.

By using the idea of the Homogeneous balance method [37], we seek for Bäcklund transformation (BT) of KP equation (14) in the form

\[ \phi_1(X, Y, T) = \psi(Y, T) \partial^2 X f[\xi(X, Y, T), \eta(X, Y, T)] + \phi_0(X, Y, T), \]  

where \( \psi(Y, T) \) is a differentiable function, \( f(\xi, \eta) \) is a function to be determined later and \( \phi_0 \) be the special (old) solution of KP equation (14).

In the following analysis, we will stay with the following conditions

\[ \psi(Y, T) = 1, \quad \eta_X(X, Y, T) = 0 \Rightarrow \eta(X, Y, T) = \eta(Y, T) \]  

Substituting (16) and (17) into equation (14) yields
\[(Af_{66}^2 + Bf_{66666} + Af_{6666}f_{6\xi})\xi_8^6 + (\xi_{XXX} + 2A\xi_{XX}\phi_0x + B\xi_{XXXXX} + D\xi_{XY}Y + A\xi_{XX}\phi_0x)\]
\[+ A\xi_{XXXXX}\xi_0 + (3\xi_{XX}\xi_{XX} + 3\xi_{XX}\xi_{XX} + 6A\xi_{XX}\xi_0x + 15B\xi_{XXXXX} + 6B\xi_{XX}\xi_{XX})\]
\[+ 2D\xi_2 + 2D\xi_{XX}Y + 10B\xi_{XXXXX} + D\xi_2Y\xi_{XX}X + 2D\xi_2Y\xi_{XY} + 3A\xi_{XX}\phi_0 + 4A\xi_{XX}\xi_0\phi_0)\]
\[+ (3\xi_{XX}\xi_{XX} + 3\xi_{XX}\xi_{XX} + 2A\xi_{XX}\phi_0 + 60B\xi_{XX}\xi_{XX} + 4D\xi_{XX}Y \xi_{XX} + 4D\xi_2Y \xi_{XX})\]
\[+ 15B^2_\xi \xi_{XX} + 6A\xi_{XX} + 20B^2_\xi \xi_{XX} + 3\xi_{XX}) \]
\[+ 15B^2_\xi \xi_{XX} + 15B^3_\xi \xi_{XX} + 12A\xi_{XX} + 12A\xi_{XX} + 12A\xi_{XX} + 12A\xi_{XX} + 12A\xi_{XX} = 0 (18)\]

Setting the coefficients of \(\xi_8^6\) in (18) to zero, we obtain the ordinary differential equation for \(f\); namely
\[Af_{66}^2 + Bf_{66666} + Af_{6666}f_{6\xi} = 0\] (19)

which admits the solution
\[f = c\ln(\xi) + \xi\sigma(\eta)\] (20)

where \(\sigma(\eta)\) is differential function and \(c\) is arbitrary constant. According to (20), we obtain

\[f_\xi = \frac{c}{\xi} + \sigma, f_{6\xi} = \frac{-c}{\xi^2}, f_{66\xi} = \frac{2c}{\xi^2}, f_{666\xi} = \frac{-6c}{\xi^4}, f_{6666\xi} = \frac{24c}{\xi^5}\]

\[f_{6}\xi = \frac{-c}{6}, f_{66\xi} = \frac{-c}{6}, f_{666\xi} = \frac{-c}{6}, f_{6666\xi} = \frac{-c}{6}, f_{66666\xi} = \frac{-c}{6}, f_{66666\xi} = \frac{-c}{6}, f_{66666\xi} = \frac{-c}{6}\]

Substituting (19) and (21) into (18) yields a linear polynomial of \(f_{\xi}, f_{\xi}, f_{\xi}, \ldots\). Equating the coefficients of \(f_{\xi}, f_{\xi}, f_{\xi}, \ldots\) to zero, holds
\[\xi_{XXX} + 2A\xi_{XX}\phi_0x + B\xi_{XXXXX} + D\xi_{XY}Y + A\xi_{XX}\phi_0x + A\xi_{XX}\phi_0x = 0 (22)\]
Case I.

Solving the above system, we have

\[3\xi_{xx}\xi_{xt} + 3\xi_{xxt} + \xi_{txxx} + 6A\xi_{x}\xi_{xxx}\phi_{0x} + 15B\xi_{xx}\xi_{x}xxx + 6B\xi_{x}\xi_{xxx}xxx +
\]

\[+ 2D\xi_{xy}^2 + 2D\xi_{xxy} + 10B\xi_{xx}^2 + D\xi_{yy}^2 + 2D\xi_{y}^2\xi_{xy} + A\epsilon_{X}^2\phi_{0xx}
\]

\[+ 3A\xi_{xx}^2\phi_{0x} + 4A\xi_{x}\xi_{xxx}\phi_{0} - \frac{1}{c}(c + \xi\sigma(\eta))^2(A\epsilon_{X}^2+ A\xi_{xx}\xi_{xxx}xxx) = 0
\]  

(23)

\[3\xi_{X}^2\xi_{XT} + 3\xi_{X}t\xi_{XX} + 2A\epsilon_{X}^3\phi_{0x} + 60B\xi_{X}t\xi_{XX}xxx + D\xi_{YY}^2 + 4D\xi_{Y}^2\xi_{XY}
\]

\[+ D\xi_{Y}^2\xi_{XX} + 15B\xi_{X}^2\xi_{XXX}t + 15B\epsilon_{X}^3 + 6A\xi_{X}^2\xi_{xx}\phi_{0} - \frac{1}{2}(c + \xi\sigma(\eta))
\]

\[= 0
\]  

(24)

\[\epsilon_{X}^3\xi_{T} + 45B\xi_{X}^2\xi_{XX} + 20B\xi_{X}^3\xi_{XXX} + D\xi_{Y}^2\epsilon_{X}^2 + A\epsilon_{X}^4\phi_{0} - \frac{c}{6}(12A\epsilon_{X}^2\xi_{XX} + 4A\epsilon_{X}^3\xi_{XXX})
\]

\[- \frac{1}{3}(c + \xi\sigma(\eta))(2A\epsilon_{X}XX\xi_{X}^3 + 6A\epsilon_{X}^2\xi_{XX}^3) = 0
\]  

(25)

\[15B\epsilon_{X}^4\xi_{XX} - \frac{c}{12}(12A\epsilon_{X}^4\xi_{XX}) - \frac{1}{4}(c + \xi\sigma(\eta))(A\epsilon_{X}^4\xi_{XX}) = 0
\]  

(26)

\[\phi_{0tx} + B\phi_{0xx}xxx + D\phi_{0yy} + A\phi_{0}x + A\phi_{0x} = 0
\]  

(27)

The next crucial step is the assumption that

\[\xi(X, Y, T) = 1 + \exp^{\lambda(T)\pm k_{1}X\pm k_{2}Y} \text{ and } \phi_{0}(X, Y, T) = \phi_{0}
\]  

(28)

where \(k_{1}\) and \(k_{2}\) are arbitrary constants. Substituting into equations (22-27), results in

\[k_{1}\lambda'(T) + BK_{1}^{2} + DK_{2}^{2} + A\phi_{0}k_{1}^{2} = 0
\]  

(29)

\[7k_{1}\lambda'(T) + 31Bk_{1}^{4} + 7DK_{2}^{2} + 7A\phi_{0}k_{1}^{2} - \frac{2A}{c}(c + \xi\sigma(\eta))^2(k_{1}^{4}) = 0
\]  

(30)

\[6k_{1}\lambda'(T) + 90Bk_{1}^{4} + 6DK_{2}^{2} + 6A\phi_{0}k_{1}^{2} - (c + \xi\sigma(\eta))(7Ak_{1}^{4}) = 0
\]  

(31)

\[k_{1}\lambda'(T) + DK_{2}^{2} + A\phi_{0}k_{1}^{2} + (65B - \frac{8}{3}Ac)k_{1}^{4} - \frac{8}{3}(c + \xi\sigma(\eta))(Ak_{1}^{4}) = 0
\]  

(32)

\[60B - 5Ac - A\xi\sigma(\eta) = 0
\]  

(33)

Solving the above system, we have

**Case I.**

\[\lambda = \frac{-Bk_{1}^{4} - DK_{2}^{2} - A\phi_{0}k_{1}^{2}}{k_{1}}T + c_{0}, \quad \sigma = 0, \quad c = \frac{12B}{A},
\]  

(34)
where $c_0$ is an integration constant, and let $c_0 = 0$. Substituting from equation (34) into equation (28), we have

$$
\xi(X, Y, T) = 1 + \exp \frac{-Bk_1^4 - Dk_2^2 - A\phi_0 k_2^2}{k_1} T_{\pm k_1 X \pm k_2 Y}
$$

(35)

Substituting from equation (35) into equation (20), we have

$$
f = \frac{12B}{A} \ln(1 + \exp \frac{-Bk_1^4 - Dk_2^2 - A\phi_0 k_2^2}{k_1} T_{\pm k_1 X \pm k_2 Y})
$$

(36)

Substituting from equation (36) into equation (16), we have the new exact solution solution of KP equation (14) in the form

$$
\phi_1 = \frac{6Bk_1^2}{A + Acosh(Bk_1^3 T - k_2 Y + \frac{Dk_2 T}{k_1} + k_1(-X + A\phi_0 T))} + \phi_0
$$

(37)

**Case II.**

$$
\lambda = \frac{-Dk_2^2}{k_1} T + c_0, \quad \sigma = \frac{-5(-2 + c)}{\xi},
$$

(38)

where $c_0$ is an integration constant, and let $c_0 = 0$. Substituting from equation (38) into equation (28), we have

$$
\xi(X, Y, T) = 1 + \exp \frac{-Dk_2^2}{k_1} T_{\pm k_1 X \pm k_2 Y}
$$

(39)

Substituting from equation (39) into equation (20), we have

$$
f = c \ln(1 + \exp \frac{-Dk_2^2}{k_1} T_{\pm k_1 X \pm k_2 Y}) - 5(-2 + c)
$$

(40)

Substituting from equation (40) into equation (16), we have the new exact solution solution of KP equation (14) in the form

$$
\phi_1 = \frac{ck_1^2}{2 + 2cosh(-\frac{Dk_2^2}{k_1} + k_1 X + k_2 Y)} + \phi_0,
$$

(41)

under the condition

$$
c = \frac{4(-2 + cosh(-Dk_2^2 T/k_1 + k_1 X + k_2 Y))(Bk_2^2 + Ak_1^2 \phi_0)}{(-4Ak_1^4 + 5Ak_1^4 sech(1/2(-Dk_2^2 T/k_1 + k_1 X + k_2 Y)^2))}
$$

(42)
4 Travelling Wave solutions of the KP equation by Applying the sine-cosine expansion method, the sinh-coshine expansion method, the sech-tanh expansion method and the modified \( \frac{G'}{G} \) expansion method

4.1 The sine-cosine expansion method
Suppose that a nonlinear partial differential equation (NPDE) is given by

\[
F(\phi, \phi_X, \phi_Y, \phi_T, \phi_{XX}, \phi_{XY}, \phi_{YY}, \phi_{XY}, \phi_{TT}, \ldots) = 0.
\]

(43)

where \( \phi = \phi(X,Y,T) \) is unknown function and \( F \) is a polynomial in and its partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of the sine-cosine expansion method:

**Step 1.** The traveling wave variable

\[
\phi(X,Y,T) = \phi(\xi), \xi = kX + lY - cT,
\]

(44)

where \( k, l \) and \( c \) are constants, permits us to reduce (43) into the following ordinary differential equation (ODE);

\[
F(\phi, \phi', \phi'', \phi''', \ldots) = 0.
\]

(45)

**Step 2.** Suppose that the solution of (45) can be expressed in the form as follows;

\[
\phi(\xi) = A_0 + \sum_{i=1}^{n} \cos^{i-1}(w(\xi))[A_i \sin(w(\xi))] + B_i \cos(w(\xi)),
\]

(46)

where \( A_i (i = 0, 1, \ldots, n) \) and \( B_i (i = 1, \ldots, n) \) are constants to be determined.

**Step 3.** The parameter "n" in (46) can be found by balancing the highest order derivatives term and the highest nonlinear term in (45);

(i) if \( n \) is a positive integer then go to step 4;

(ii) if \( n \) is not positive integer, we put \( \phi = v^n \) and then return to step 1.

**Step 4.** Substitute (46) with the fixed parameter \( n \) into the obtained ODE using

\[
\frac{d\omega(\xi)}{d\xi} = \mu \sqrt{a + b \sin^2 \omega(\xi)}, \mu = \pm 1,
\]

(47)

where \( a \) and \( b \) are constants. collecting all terms with the same powers of \( w^s \sin^j \cos^i \) together. Set to zero the coefficients of \( w^s \sin^j \cos^i \) \( w(i = 0, 1; s = 0, 1; j = 0,1, 2,\ldots,n) \) to get a set of algebraic equations \( C_{ij}s (A_S, B_S) = 0 \) with respect to the unknowns \( A_0, c, \ldots \)
A_i (i = 1, ..., n) and B_i (i = 1, ..., n).

Step 5. Solve the set of algebraic equations to get $A_0$, $A_i$, and $B_i$, then we have the traveling wave solutions of the given nonlinear equation. In The sine-cosine expansion method there are three cases of (35):

Case I
In these case $a=0$, $b=1$, then the equation reduces to the first-order ODE

$$\frac{dw(\xi)}{d\xi} = \sin(w(\xi)), \quad (48)$$

which has the solutions:

$$\sin(w(\xi)) = \sech(\xi), \quad \text{or} \quad \cos(w(\xi)) = -\tanh(\xi)$$

and

$$\sin(w(\xi)) = \text{sech}(\xi), \quad \text{or} \quad \cos(w(\xi)) = -\coth(\xi),$$

where $i = \sqrt{-1}$

Case II
In these case $a = 1, b = -m^2$, then the equation reduces to the first-order ODE

$$\frac{dw(\xi)}{d\xi} = \pm \sqrt{1 - m^2 \sin^2(w(\xi))}, \quad (49)$$

where $m$ is the modulus of Jacobi elliptic functions, which has the solutions

$$\sin(w(\xi)) = \text{sn}(\xi; m), \quad \text{or} \quad \cos(w(\xi)) = \text{cn}(\xi; m)$$

and

$$\sin(w(\xi)) = \frac{n \sin(\xi; m)}{m}, \quad \text{or} \quad \cos(w(\xi)) = \frac{i \sinh(\xi; m)}{m}$$

Case III
In these case $a = m^2, b = -1$, then the equation reduces to the first-order ODE,

$$\frac{dw(\xi)}{d\xi} = \pm \sqrt{m^2 - \sin^2(w(\xi))}, \quad (50)$$

which has the solutions,

$$\sin(w(\xi)) = m \sin(\xi; m), \quad \text{or} \quad \cos(w(\xi)) = m \cos(\xi; m).$$

and

$$\sin(w(\xi)) = n \sin(\xi; m), \quad \text{or} \quad \cos(w(\xi)) = i \sinh(\xi; m).$$

To obtain traveling wave solutions of KP equation (14), take the traveling wave transformation $\phi(X, Y, T) = \phi(\xi); \xi = kX + lY - cT$, then (14) reduces to

$$Bk^4 \phi_m'' + Ak^2 \phi_m'' + Ak^2 \phi_1'' + (Dl^2 - ck)\phi_1'' = 0, \quad (51)$$

where $A, B, D, k, l$ and $c$ are constants according to step 2, we know that $n=2$ in (46) and suppose that the solution is in the form

$$\phi_1 = A_0 + A_1 \sin(w(\xi)) + B_1 \cos(w(\xi)) + A_2 \sin(w(\xi)) \cos(w(\xi)) + B_2 \cos^2(w(\xi)) \quad (52)$$

with the aid of Mathematica, substituting (52) into ODE (51), then we have the polynomial of $\sin^i w \cos^j w (i = 0, 1; j = 0, 1, ..., n)$. Setting their coefficients to zero yields
a set of algebraic equations. By solving the set of algebraic equations, we can fix those parameters, therefore we have the following doubly periodic solutions, if \( \{ A_0 = \frac{3bck - 3bDl^2 + 12abBk^4 \mu^2 + 24b^2 Bk^4 \mu^2}{3Abk^2}, A_1 = 0, A_2 = 0, B_1 = 0, B_2 = -\frac{12bBk^2 \mu^2}{A} \} \), then

\[
\phi_1 = \frac{3bck - 3bDl^2 + 12abBk^4 \mu^2 + 24b^2 Bk^4 \mu^2}{3Abk^2} - \frac{12bBk^2 \mu^2}{A} \cos^2(w(\xi)) \quad (53)
\]

Case I:

\[
\phi_1 = \frac{3bck - 3bDl^2 + 12abBk^4 \mu^2 + 24b^2 Bk^4 \mu^2}{3Abk^2} - \frac{12bBk^2 \mu^2}{A} \tanh^2(\xi) \quad (54)
\]

or

\[
\phi_1 = \frac{3bck - 3bDl^2 + 12abBk^4 \mu^2 + 24b^2 Bk^4 \mu^2}{3Abk^2} - \frac{12bBk^2 \mu^2}{A} \coth^2(\xi) \quad (55)
\]

Case II:

\[
\phi_1 = \frac{3bck - 3bDl^2 + 12abBk^4 \mu^2 + 24b^2 Bk^4 \mu^2}{3Abk^2} - \frac{12bBk^2 \mu^2}{A} \cn^2(\xi; m) \quad (56)
\]

or

\[
\phi_1 = \frac{3bck - 3bDl^2 + 12abBk^4 \mu^2 + 24b^2 Bk^4 \mu^2}{3Abk^2} - \frac{12bBk^2 \mu^2}{A} \ids^2(\xi; m) \quad (57)
\]

Case III:

\[
\phi_1 = \frac{3bck - 3bDl^2 + 12abBk^4 \mu^2 + 24b^2 Bk^4 \mu^2}{3Abk^2} - \frac{12bBk^2 \mu^2}{A} \dn^2(\xi; m). \quad (58)
\]

or

\[
\phi_1 = \frac{3bck - 3bDl^2 + 12abBk^4 \mu^2 + 24b^2 Bk^4 \mu^2}{3Abk^2} - \frac{12bBk^2 \mu^2}{A} \isc^2(\xi; m). \quad (59)
\]

4.2 The sinh-coshine expansion method

Suppose that NPDE is given by

\[
F(\phi, \phi_X, \phi_Y, \phi_T, \phi_{XX}, \phi_{YY}, \phi_{XY}, \phi_{TT}, \ldots) = 0. \quad (60)
\]

where \( \phi = \phi(X, Y, T) \) is unknown function and F is a polynomial in and its partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In
the following we give the main steps of the sinh-coshine expansion method:

**Step 1.** The travelling wave variable

\[ \phi(X, Y, T) = \phi(\xi), \xi = kX + lY - cT, \quad (61) \]

where \( k, l \) and \( c \) are constants, permits us to reduce (60) into ODE;

\[ F(\phi, \phi', \phi'', \phi''', \ldots) = 0. \quad (62) \]

**Step 2.** Suppose that the solution of (62) can be expressed in the form as follows;

\[ \phi(\xi) = A_0 + \sum_{i=1}^{n} \cosh^{i-1}(\xi)[A_i \sinh(\xi) + B_i \cosh(\xi)], \quad (63) \]

where \( A_i \) (\( i = 0, 1, \ldots, n \)) and \( B_i \) (\( i = 1, \ldots, n \)) are constants to be determined.

**Step 3.** The parameter “n” in (63) can be found by balancing the highest order derivatives term and the highest nonlinear term in (62);

(i) if \( n \) is a positive integer then go to step 4;

(ii) if \( n \) is not positive integer, we put \( \phi = v^n \) and then return to step 1.

**Step 4.** Substitute (63) with the fixed parameter \( n \) into the obtained ODE. Collecting all terms with the same powers of \( \sinh^i \xi \cosh^j \xi \) together. Set to zero the coefficients of \( \sinh^i \xi \cosh^j \xi \) \((i = 0, 1; j = 0, 1, 2, \ldots, n)\) to get a set of algebraic equations \( C_{ij} (A_S, B_S) = 0 \) with respect to the unknowns \( A_0, c, A_i (i = 1, \ldots, n) \) and \( B_i \) \((i = 1, \ldots, n)\).

**Step 5.** Solve the set of algebraic equations to get \( A_0, A_i \) and \( B_i \) then we have the travelling wave solutions of the given NPDE.

The ODE (51) according to steps , we know that \( n = 2 \) in (63) and suppose that the solution take the form

\[ \phi_1 = A_0 + A_1 \sinh(\xi) + B_1 \cosh(\xi) + A_2 \sinh(\xi) \cosh(\xi) + B_2 \cosh^2(\xi) \quad (64) \]

with the aid of mathematica , substituting (64) into (51), we have the polynomial of \( \sinh^i \xi \cosh^j \xi \) \((i = 0, 1; j = 0, 1, \ldots, n)\). Setting their coefficients to zero yields a set of algebraic equations. By solving the set of algebraic equations, we can fix those parameters, therefore we have the doubly periodic solutions, if \( \{A_1 = 0, B_1 = 0, c = \frac{4Bk^2 + Dl^2}{k}\} \), then

\[ \phi_1 = A_0 + A_2 \sinh(\xi) \cosh(\xi) + B_2 \cosh^2(\xi), \quad (65) \]

where \( A_0 = \frac{1}{2}(\frac{2c-k - 8Bk^4 - 2Dl^2}{Ak^2} - 2A_2 \sinh(2\xi) - (1 + 2\cosh(2\xi))B_2 + \frac{-A_2^2 + B_2^2}{A_2 \sinh(2\xi) + B_2 \cosh(2\xi)}) \).
4.3 The sech-tanh expansion method

Suppose that NPDE is given by

\[ F(\phi, \phi_X, \phi_Y, \phi_T, \phi_{XX}, \phi_{YY}, \phi_{XY}, \phi_{TT}, \ldots) = 0. \] (66)

where \( \phi = \phi(X, Y, T) \) is unknown function and \( F \) is a polynomial in and its partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of the sech-tanh expansion method:

**Step 1.** The travelling wave variable

\[ \phi(X, Y, T) = \phi(\xi), \xi = kX + lY - cT, \] (67)

where \( k, l \) and \( c \) are constants, permits us to reduce (66) into ODE;

\[ F(\phi, \phi', \phi'', \phi''', \ldots) = 0. \] (68)

**Step 2.** Suppose that the solution of (68) can be expressed in the form as follows;

\[ \phi(\xi) = A_0 + \sum_{i=1}^{n} \text{sech}^{i-1}(\xi)[A_i\tanh(\xi)] + B_i\text{sech}(\xi), \] (69)

where \( A_i \) (\( i = 0, 1, \ldots, n \)) and \( B_i \) (\( i = 1, \ldots, n \)) are constants to be determined.

**Step 3.** The parameter "n" in (69) can be found by balancing the highest order derivatives term and the highest nonlinear term in (68);

(i) if \( n \) is a positive integer then go to step 4;
(ii) if \( n \) is not positive integer, we put \( \phi = v^n \) and then return to step 1.

**Step 4.** Substitute (69) with the fixed parameter \( n \) into the obtained ODE. Collecting all terms with the same powers of \( \text{sech}^i\xi\tanh^j\xi \) together. Set to zero the coefficients of \( \text{sech}^i\xi\tanh^j\xi \) (\( i = 0, 1; j = 0, 1, 2, \ldots, n \)) to get a set of algebraic equations \( C_{ij} (A_S, B_S) = 0 \) with respect to the unknowns \( A_0, \beta, A_i(i = 1, \ldots, n) \) and \( B_i \) (\( i = 1, \ldots, n \)).

**Step 5.** Solve the set of algebraic equations to get \( A_0, A_i \) and \( B_i \) then we have the traveling wave solutions of the given NPDE.

The ODE(51) according to steps, we know that \( n=2 \) in (69) and suppose that the solution take the form

\[ \phi_1 = A_0 + A_1\tanh(\xi) + B_1\text{sech}(\xi) + A_2\tanh(\xi)\text{sech}(\xi) + B_2\text{sech}^2(\xi) \] (70)

with the aid of mathematica, substituting (70) into (51), we have the polynomial of \( \text{sech}^i\xi\tanh^j\xi \) (\( i = 0, 1; j = 0, 1, \ldots, n \)). Setting their coefficients to zero yields a set of algebraic equations. By solving the set of algebraic equations, we can fix those parameters,
therefore we have the following doubly periodic solutions:
if\( \{ A_0 = \frac{ck - 4Bk^4 - Dl^2}{Ak^2}, A_1 = 0, B_2 = \frac{12Bk^2}{A}, A_2 = 0, B_1 = 0 \} \), then
\[
\phi_1 = \frac{ck - 4Bk^4 - Dl^2}{Ak^2} + \frac{12Bk^2}{A} \text{sech}^2(\xi)
\]  (71)

### 4.4 The modified \( \frac{\mathcal{G}}{G} \) expansion method

Suppose that NPDE is given by
\[
F(\phi, \phi_X, \phi_Y, \phi_T, \phi_{XX}, \phi_{YY}, \phi_{XY}, \phi_{TT}, \ldots) = 0,
\]  (72)
where \( \phi = \phi(X, Y, T) \) is unknown function and \( F \) is a polynomial in and its partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of the modified \( \frac{\mathcal{G}}{G} \) Expansion method:

**Step 1.** The travelling wave variable
\[
\phi(X, Y, T) = \phi(\xi), \xi = kX + lY - cT,
\]  (73)
where \( k, l \) and \( c \) are constants, permits us to reduce (72) into ODE;
\[
F(\phi, \phi', \phi'', \phi''', \ldots) = 0.
\]  (74)

**Step 2.** Suppose that the solution of (74) can be expressed by a polynomial \( \frac{\mathcal{G}}{G} \) in as follows;
\[
\phi(\xi) = \alpha_0 + \sum_{i=1}^{m} [\alpha_i(\frac{\mathcal{G}}{G})^i + \alpha_{-i}(\frac{\mathcal{G}}{G})^{-i}],
\]  (75)
where \( G = G(\xi) \) satisfies the second order linear ODE
\[
G'' + \mu G = 0,
\]  (76)
where \( \alpha_0, \alpha_i, \alpha_{-i} \) and \( \mu \) are constants to be determined.

**Step 3.** The parameter "m" in (75) can be found by balancing the highest order derivatives term and the highest nonlinear term in (74);
(i) if \( m \) is a positive integer then go to step 4;
(ii) if \( m \) is not positive integer, we put \( \phi = v^m \) and then return to step 1.

**Step 4.** Substituting (75) into (74) and using (76), collecting all terms with the same powers of \( \frac{\mathcal{G}}{G} \) together, and then equating each coefficient of the resulted polynomial to zero, yield a system of algebraic equations for \( \alpha_0, \alpha_i, \alpha_{-i} \) and \( \mu \).

**Step 5.** Since the general solutions (74) are well known to us, then substituting \( \alpha_0, \alpha_i, \alpha_{-i} \)
and $\mu$ and the general solutions (75) into (74), we have the traveling wave solutions of the given NPDE. The ODE(51) according to steps ,we know that $n=2$ in (75) and suppose that the solution take the form

$$\phi_1 = \alpha_0 + \alpha_1 \left( \frac{\dot{G}}{G} \right) + \alpha_{-1} \left( \frac{\dot{G}}{G} \right)^{-1} + \alpha_2 \left( \frac{\dot{G}}{G} \right)^2 + \alpha_{-2} \left( \frac{\dot{G}}{G} \right)^{-2}. \quad (77)$$

with the aid of mathematica , substituting (77) into (51) , we have the polynomial of the terms with the same powers of $\frac{\dot{G}}{G}$together.Setting their coefficients to zero yields a set of algebraic equations.By solving the set of algebraic equations, we can fix those parameters, therefore we have the traveling wave solutions:

if \{ $\alpha_0 = \frac{3ck\mu - 3Dl^2 \mu - 24Bk^4 \mu^2}{3Ak^2 \mu}$, $\alpha_1 = 0$, $\alpha_{-1} = 0$, $\alpha_2 = 0$, $\alpha_{-2} = -\frac{12Bk^2 \mu^2}{A}$ \}, then

$$\phi_1 = \frac{3ck\mu - 3Dl^2 \mu - 24Bk^4 \mu^2}{3Ak^2 \mu} - \frac{12Bk^2 \mu^2}{A} \left( -c_1 \sqrt{\mu} \sin \sqrt{\mu} \xi + c_2 \sqrt{\mu} \cos \sqrt{\mu} \xi \right)^2. \quad (78)$$

5 Applications

Now, we shall look at four explicit physical applications of the solutions given above. We investigate Sagdeev potential $\phi_1$ corresponding to the solutions of KP equation (14).The detailed application of these solutions requires a judicious of the free parameters occurring in the solutions.

5.1 First application by using Auto-Bäcklund transformation

Now on inserting the special (old) solution $\phi_0$ [8, 38] of the KP equation (14) given by

$$\phi_0 = \frac{3(c - D)}{A} Sech^2 \left( \frac{X + Y - cT}{W} \right), \quad W = 2\sqrt{\frac{B}{c - D}} \quad (79)$$

The new exact soltion solution (37)of the KP equation (14) can be given as

$$\phi_1 = \frac{6Bk_1^2}{A + A\cosh \left( Bk_1^3 T - k_2 Y + \frac{Dk_2^2 T}{k_1} + k_1 (-X + A\phi_0 T) \right)} + \frac{3(c - D)}{A} Sech^2 \left( \frac{X + Y - cT}{2\sqrt{\frac{B}{c - D}}} \right) \quad (80)$$

Applicable to some relevant values of A, B, D, c, $k_1$ and $k_2$ , the Sagdeev potential $\phi_1$ is displayed in Figure 1.

5.2 Second application by using the sine-cosine expansion method
Now, using the backward substitution of the solution (54) through the backward transformation (44), we obtain the travelling wave solution of the KP equation (14) in the form
\[ \phi_1(X,Y,T) = \frac{3bck - 3bDl^2 + 12abBk^4\mu^2 + 24b^2Bk^4\mu^2}{3Ak^2} - \frac{12bBk^2\mu^2}{A} \tanh^2(kX + lY - cT) \]  
(81)

Applicable to some relevant values of A, B, D, a, b, c, k, l and \( \mu \), the Sagdeev potential \( \phi_1 \) is displayed in Figure 2.

5.3 Third application by using the sinh-coshine expansion method

Now, using the backward substitution of the solution (65) through the backward transformation (61), we obtain the travelling wave solution of the KP equation (14) in the form
\[ \phi_1(X,Y,T) = A_0 + A_2 \sinh(kX + lY - cT) \cosh(kX + lY - cT) + B_2 \cosh^2(kX + lY - cT), \]
(82)

where,
\[ A_0 = \frac{1}{2} \left( \frac{2ck - 8Bk^4 - 2Dl^2}{Ak^2} - 2A_2 \sinh(2(kX + lY - cT)) - (1 + 2\cosh(2(kX + lY - cT)))B_2 + \frac{A_2\sinh(2(kX + lY - cT)) + B_2\cosh(2(kX + lY - cT))}{A_2\sinh(2(kX + lY - cT)) + B_2\cosh(2(kX + lY - cT))} \right) \]

Applicable to some relevant values of A, B, D, c, k, l, \( A_2 \), and \( B_2 \), the Sagdeev potential \( \phi_1 \) is displayed in Figure 3.

5.4 Fourth application by using the sech-tanh expansion method

Now, using the backward substitution of the solution (71) through the backward transformation (67), we obtain the travelling wave solution of the KP equation (14) in the form
\[ \phi_1(X,Y,T) = \frac{ck - 4Bk^4 - Dl^2}{Ak^2} + \frac{12Bk^2}{A} \sech^2(kX + lY - cT) \]  
(83)

Applicable to some relevant values of A, B, D, c, k, and l, the Sagdeev potential \( \phi_1 \) is displayed in Figure 4.

5.5 Fifth application by using the modified \( \zeta \) expansion method

Now, using the backward substitution of the solution (78) through the backward transformation (73), we obtain the travelling wave solution of the KP equation (14) in the form
\[ \phi_1 = \frac{3ck\mu - 3Dl^2\mu - 24Bk^4\mu^2}{3Ak^2\mu} - \frac{12Bk^2\mu^2}{A} \left( \frac{-c_1\sqrt{\mu}\sin(\sqrt{\mu}(kX + lY - cT)) + c_2\sqrt{\mu}\cos(\sqrt{\mu}(kX + lY - cT))}{c_1\cos(\sqrt{\mu}(kX + lY - cT)) + c_2\sin(\sqrt{\mu}(kX + lY - cT))} \right)^{-2}. \]  
(84)

Applicable to some relevant values of A, B, D, c, k, l, \( c_1 \), \( c_2 \), and \( \mu \), the Sagdeev potential \( \phi_1 \) is displayed in Figure 5.

6 Conclusion

Employing the reductive perturbation technique, a KP equation is derived for describing the DASWs in unmagnetized dusty plasma with variable dust charge and two temper-
ature ions. For the KP equation (14), parameters B and D are always positive, but parameter A can be positive or negative; however, it is negative for most of the cases. This means that generally a rarefactive soliton is appeared in the medium. Consequently amplitude of the solitary waves is smaller as compared to the one-dimensional case [39]. Equation (15) with $\gamma_1 = \gamma_2 = 0$, reduces to the results of [20] for warm plasmas with one ion. The effects of dust charge variation and nonthermal ions on dust acoustic solitary wave structure in magnetized dusty plasmas has been studied using the KdV equation in [25]. The solitary solutions and travelling wave solutions of the KP equation are derived. The auto-Bäcklund (BT) transformations, the Sine-Cosine expansion method, the Sinh-Coshine expansion method, the Sech-Tanh expansion method and the modified G expansion method have been successfully applied to find the Sagdeev potential $\phi_1$ for KP equation.
Figure (1) The Sagdeev potential $\phi_1$ (80) in the surface graphic for $T = 1, D = 1, A = 1.5, B = 0.5, k_1 = 0.75, k_2 = 0.74, c = 1.5$

Figure (2) The Sagdeev potential $\phi_1$ (81) in the surface graphic for $T = 1, D = 5, A = 0.4, b = 0.3, B = 0.9, k = 0.3, c = 0.49, l = 1, a = 2, \mu = 0.4$

Figure (3) The Sagdeev potential $\phi_1$ (82) in the surface graphic for $T = 1, D = 1, A = 0.3, B = 0.99, k = 0.7, l = 0.25, A_2 = 2, B_2 = 5$
Figure (4) The Sagdeev potential $\phi_1$ (83) in the surface graphic for $T = 1, D = 5, A = 0.9, B = 3, k = 0.5, c = 0.2, l = 4$

Figure (5) The Sagdeev potential $\phi_1$ (84) in the surface graphic for $T = 1, B = 0.9, k = 0.4, l = 0.2, \mu = 1.6, A = -1.2, c_1 = 0.75, c_2 = 0.74, D = 1, c = 0.32$
References


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