The Energy Method on Diffusion Equation with Homogeneous Boundary Conditions

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Abstract

In this paper, we study the diffusion equation under a homogeneous boundary condition. We show that the energy function is a non-increasing function, and it can be used to show that our solution is only one solution to the problem. The separation of variables is used to solve the partial differential equation.

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1 Introduction

The diffusion equation \cite{2} is a partial differential equation which describes density dynamics in a material undergoing diffusion. It is also used to express processes exhibiting diffusive-like behavior, for instance the 'diffusion' of alleles in a population in population genetics. The diffusion equation can be arised in a straightforward way from the continuity equation, which states that a modification in density in any portion of the system is due to inflow and outflow of material into and out of that part of the system. Productively, no material is created or destroyed. Sharma et al. \cite{4} studied Galerkin-finite
element method which is proposed to find the numerical solutions of advection-diffusion equation. The equation is generally used to describe mass, heat, energy, velocity, vorticity. Kalis et al. [3] derived a 2-D stationary boundary value-problem for the diffusion equation with piecewise constant coefficients in a multi-layered domain is considered. Homogeneous boundary conditions of the first kind (BC) or periodic boundary conditions (PBC) are considered in the $x$ direction. In this paper, we study the diffusion equation under a homogeneous boundary condition. We show that the energy function is a non-increasing function, and it can be used to show that our solution is only one solution to the problem. The separation of variables is used to solve the partial differential equation.

2 Preliminaries

We consider the diffusion equation under a homogeneous boundary condition. We begin by showing that the total energy function is a non-increasing function, and the result we obtained will be used in the problem to show that the solution we derive is the only one solution.

**Theorem 2.1. Vanishing Theorem** [1]. Let $f(x)$ be a continuous function in a finite closed interval $[a,b]$. Assume that $f(x) \geq 0$ in the interval and that $\int_a^b f(x)dx = 0$. Then $f(x)$ is identically zero.

**Lemma 2.2. The Total energy function** for $u$ which can be described as

$$
\gamma(t) = \int_0^L \left[ \frac{1}{2} \left[ \tau u_x^2(x,t) + \rho u^2(x,t) \right] \right] dx.
$$

is a non-increasing function of $t$ on $[0, \Delta]$ where $\tau$, $\varphi$ and $\rho$ are positive constants.
Proof. By differentiating equation (1) with respect to \( t \), it leads to

\[
\frac{d\gamma}{dt} = \frac{\partial}{\partial t} \int_{0}^{L} \left[ \frac{1}{2} \left( \tau u_t^2(x,t) + \rho u_x^2(x,t) \right) \right] \, dx
\]

\[
= \int_{0}^{L} \left[ \tau u_t(x,t) u_{tt}(x,t) + \rho u_x(x,t) u_{xt}(x,t) \right] \, dx
\]

\[
= \rho \int_{0}^{L} \left[ u_t(x,t) u_{xx}(x,t) + u_x(x,t) u_{xt}(x,t) \right] \, dx - \varphi \int_{0}^{L} u_t^2(x,t) \, dx
\]

\[
= -\rho \int_{0}^{L} \frac{\partial}{\partial x} \left[ u_t(x,t) u_x(x,t) \right] \, dx - \varphi \int_{0}^{L} u_t^2(x,t) \, dx
\]

\[
= \rho \int_{0}^{L} \left[ u_t(L,t) u_x(L,t) - u_t(0,t) u_x(0,t) \right] \, dx - \varphi \int_{0}^{L} u_t^2(x,t) \, dx
\]

\[
= -\varphi \int_{0}^{L} u_t^2(x,t) \, dx \leq 0. \tag{2}
\]

since \( \frac{d\gamma}{dt} \leq 0 \). Therefore, \( \gamma \) is non-increasing on \( 0 \leq t < \infty \). \( \square \)

Lemma 2.3. \( u(x,t) = \sum_{n=0}^{\infty} \frac{2}{\Omega} \sin \left[ \Omega \pi x \right] e^{-c(\Omega)^2 \pi^2 t} \) is a solution to

\[
\frac{d^2 u}{dt^2} = c u_{xx} \quad \text{in} \quad 0 < x < 1, \quad 0 < t < \infty. \tag{3}
\]

which satisfy the boundary condition

\[
u_x(1,t) = 0 = u(0,t) \quad \text{for} \quad 0 \leq t < \infty, \tag{4}
\]

and subject to the initial condition

\[
u(x,0) = 1 \quad \text{for} \quad 0 \leq x \leq 1, \tag{5}
\]

Proof. We look for a nontrivial solution of equations (3)-(4)-(5) of the form

\[
u(x,t) = X(x)T(t). \tag{6}
\]

Substituting in (3) gives \( X(x)T'(t) - cX''(x)T(t) = 0 \) and therefore \( \frac{X''(x)}{X(x)} = \frac{T'(x)}{T(t)} = -\lambda \). Substituting in (4) leads to \( X(0)T(t) = 0 = X'(1)T(t) \) for \( 0 \leq t < \infty \). Hence,

\[
X''(x) + \lambda X(x) = 0, \quad 0 < x < 1, \quad X(0) = X'(1) = 0, \tag{6}
\]
is the eigenvalue problem. For \( \lambda < 0 \), let \( \lambda = \beta^2 \) where \( \beta > 0 \). We derive
\[
X(x) = c_1 \cos(\beta x) + c_2 \sin(\beta x).
\]
as a general solution of the differential equation. The boundary condition implies \( 0 = X(0) = c_1 \cos(0) + c_2 \sin(0) \) and \( 0 = X'(1) = \beta c_2 \cos(\beta) \). Therefore, \( c_1 = 0 \) and \( \beta = \beta_n = \Omega \pi \) for \( n = 1, 2, ... \). As a consequence, the eigenvalues are \( \lambda_n = \beta_n^2 = (\Omega \pi)^2 \) and the corresponding eigenfunctions are \( X_n(x) = \sin(\Omega \pi x) \) where \( n = 1, 2, 3, ... \). The factors \( T_n = T_n(t) \) satisfy
\[
T_n'(t) + \lambda_n c T_n(t) = 0 \quad \text{for} \quad t > 0.
\]
As a result, \( T_n(t) = e^{-c(\Omega)^2 \pi^2 t} \) where \( n = 1, 2, 3, ... \). Therefore,
\[
\sum_{n=0}^{\infty} c_n X_n(x) T_n(t) = \sum_{n=0}^{\infty} c_n \sin(\Omega \pi x) e^{-c(\Omega)^2 \pi^2 t}.
\]
satisfies equations (3) and (4) for every and every choice of \( c_1, c_2, c_2, ... \) are arbitrary constants. In order to satisfy equation (5), we must have
\[
1 = u(x, 0) = \sum_{n=0}^{\infty} c_n \sin(\Omega \pi x).
\]
for all \( 0 \leq x \leq 1 \). As a result, we take \( c_n = \frac{2}{\Omega} \). As a result, we obtain the solution to the problem as
\[
\sum_{n=0}^{\infty} \frac{2}{\Omega} \sin(\Omega \pi x) e^{-c(\Omega)^2 \pi^2 t}.
\]

3 Main Result

**Lemma 3.1.** \( u(x, t) = \sum_{n=0}^{\infty} \frac{2}{\Omega} \sin(\Omega \pi x) e^{-c(\Omega)^2 \pi^2 t} \) is the only solution to equations (3)-(5) by the energy method.

**Proof.** By Lemma 2.2., we drive that the energy function \( \gamma(t) \) is a non-increasing function on \([0, 1]\) by showing that \( \frac{d\gamma}{dt} \leq 0 \) on \([0, 1]\), and the energy function is non-increasing function of \( t \) on \([0, \infty)\). As a consequence, if there were another solution \( u = \eta(x, t) \) to the problem equations (3)-(5), then the difference between \( u \) and the energy function would solve equations (3) and (4) and \( u(x, 0) = 1 \) for \( 0 \leq x \leq 1 \). The energy of the difference would be
\[
0 \leq \gamma(t) \leq \gamma(0) = 0 \quad \text{for all} \quad t \in [0, \infty).
\]
This follows from the Vanishing Theorem [1] that the difference would be zero for all $0 \leq x \leq 1$ and $t \in [0, \infty)$. As a consequent,

$$
\eta(x, t) = \sum_{n=0}^{\infty} \frac{2}{\Omega} \sin[\Omega \pi x] e^{-c(\Omega)^2 \pi^2 t}.
$$

(13)

for all $0 \leq x \leq 1$ and $t \in [0, \infty)$. As a result, $u = \eta(x, t)$. Therefore, our solution is at most one solution to the problem as desired.

\[\Box\]

\section*{References}


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