A Note on $q$-Analogue of Lambda-Daehee Polynomials

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Abstract

In this paper, we consider the $q$-analogue of lambda-Daehee polynomials and we give some new identities of these polynomials which are derived from $p$-adic invariant integral on $\mathbb{Z}_p$
Keywords: $p$-adic integral on $\mathbb{Z}_p$, lambda-Daehee polynomials, stirling numbers

1. Introduction

As is well known, the lambda-Daehee polynomials are defined by the generating function to be

$$\sum_{n=0}^{\infty} D_{n, \lambda}(x) \frac{t^n}{n!} = \frac{\lambda \log(1 + t)}{(1 + t)^\lambda - 1}(1 + t)^x, \quad (\text{see [7]}).$$

Let $p$ be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively, denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The $p$-adic norm $|\cdot|$ is normalized as $|p|_p = 1/p$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$, the $p$-adic invariant integral on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the $p$-adic invariant integral on $\mathbb{Z}_p$ is defined to be

$$I_0(f) = \int_{\mathbb{Z}_p} f(x) d\mu_0(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) \mu_0(x + p^N \mathbb{Z}_p)$$

$$= \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad (\text{see [1-18]}).$$

By (2), we easily get

$$I_0(f_1) - I_0(f) = f(t(0), \quad (\text{see [8, 10, 11]}))$$

where $f_1(x) = f(x + 1)$.

From (3), we have

$$\int_{\mathbb{Z}_p} e^{xt} d\mu_0(x) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad (\text{see [1-8]}),$$

where $B_n$ are called the Bernoulli numbers.

In particular, the Bernoulli polynomials are given by

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_0(y) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$  \hfill (5)

By (4) and (5), we get

$$B_n(x) = \sum_{\ell=1}^{n} \binom{n}{\ell} B_\ell x^{n-\ell}, \quad (\text{see [10-18]}).$$

$$B_n(x) = \sum_{\ell=1}^{n} \binom{n}{\ell} B_\ell x^{n-\ell}, \quad (\text{see [10-18]}).$$
The Stirling number of the first kind is defined by the falling factorial sequence to be
\[(x)_n = x(x-1) \cdots (x-n+1) = \sum_{\ell=1}^{n} S_1(n, \ell) x^\ell, \quad (n \in \mathbb{Z}_{\geq 0}).\] (7)

As is known, the Stirling number of the second kind is given by
\[(e^t - a)^n = n! \sum_{\ell=n}^{\infty} S_2(\ell, n) t^\ell/\ell!, \quad (\text{see } [8, 16]).\] (8)

In viewpoint of (1), we consider the $q$-analogue of lambda-Daehee polynomials and investigate some properties of those polynomials which are derived from the $p$-adic invariant integral on $\mathbb{Z}_p$.

2. SOME IDENTITIES FOR THE HIGHER-ORDER $q$-BERNOULLI POLYNOMIALS OF THE SECOND KIND

In this section, we assume that $q, t \in \mathbb{C}_p$ with $|t|_p < \frac{1}{2} \frac{1}{p^{\lambda+1}}$ and $\lambda \in \mathbb{Z}_p$ with $\lambda \neq 0$. For $f(x) = (1 + qt)^\lambda x$, by (3), we get
\[\int_{\mathbb{Z}_p} (1 + qt)^{x+\lambda y} d\mu_0(y) = \frac{\lambda \log(1 + qt)}{(1 + qt)^\lambda} (1 + qt)^x.\] (9)

In viewpoint of (1), we define the $q$-analogue lambda-Daehee polynomials as follows:
\[\frac{\lambda \log(1 + qt)}{(1 + qt)^\lambda} (1 + qt)^x = \sum_{n=0}^{\infty} BD_{n,q}(x|\lambda) \frac{t^n}{n!}.\] (10)

When $x = 0, BD_{n,q}(\lambda) = BD_{n,q}(0|\lambda)$ are called the $q$-analogue of lambda-Daehee numbers.

Remark. Note that $\lim_{q \to 1} BD_{n,q}(x|\lambda) = D_{n,\lambda}(x)$.

From (9) and (10), we have
\[\sum_{n=0}^{\infty} BD_{n,q}(x|\lambda) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} (1 + qt)^{\lambda y + x} d\mu_0(y) = \sum_{n=0}^{\infty} q^n \int_{\mathbb{Z}_p} (\lambda y + x)_n d\mu_0(y) \frac{t^n}{n!}.\] (11)

Therefore, by (11), we obtain the following theorem.

**Theorem 2.1.** For $n \geq 0$, we have
\[q^n \int_{\mathbb{Z}_p} (x + \lambda y)_n d\mu_0(dy) = BD_{n,q}(x|\lambda).\]
By replacing $q t$ by $e^t - 1$ in (10), we get
\[
\sum_{n=0}^{\infty} q^{-n} BD_{n,q}(x|\lambda) \frac{(e^t - 1)^n}{n!} = \frac{\lambda t}{e^\lambda - 1} e^{tx} \sum_{n=0}^{\infty} B_n \left( \frac{x}{\lambda} \right) \lambda^n \frac{t^n}{n!}.
\] (12)

and
\[
\sum_{n=0}^{\infty} q^{-n} BD_{n,q}(x|\lambda) \frac{(e^t - 1)^n}{n!} = \sum_{n=0}^{\infty} q^{-n} BD_{n,q}(x|\lambda) \sum_{m=n}^{\infty} S_2(m, n) \frac{t^m}{m!} \sum_{n=0}^{\infty} \left( \sum_{n=0}^{m} q^{-n} BD_{n,q}(x|\lambda) S_2(m, n) \right) \frac{t^m}{m!}.
\] (13)

Therefore, by (12) and (13), we obtain the following theorem.

**Theorem 2.2.** For $m \geq 0$, we have
\[
\sum_{n=0}^{m} q^{-n} BD_{n,q}(x|\lambda) S_2(m, n) = \lambda^m B_m \left( \frac{x}{\lambda} \right).
\]

From Theorem 1, we have
\[
q^{-n} BD_{n,q}(x|\lambda) = \sum_{\ell=0}^{n} S_1(n, \ell) \int_{\mathbb{Z}_p} (x + y \lambda)^\ell d\mu_0(y)
= \sum_{\ell=0}^{n} S_1(n, \ell) \lambda^\ell \int_{\mathbb{Z}_p} \left( \frac{x}{\lambda} + y \right)^\ell d\mu_0(y)
= \sum_{\ell=0}^{n} S_1(n, \ell) \lambda^\ell B_\ell \left( \frac{x}{\lambda} \right).
\] (14)

**Theorem 2.3.** For $n \geq 0$, we have
\[
q^{-n} BD_{n,q}(x|\lambda) = \sum_{\ell=0}^{n} S_1(n, \ell) \lambda^\ell B_\ell \left( \frac{x}{\lambda} \right).
\]

Let us consider the $q$-analogue of lambda-Daehee polynomials of order $k \in \mathbb{N}$ as follows:
\[
q^{-n} BD_{n,q}^{(k)}(x|\lambda) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( \lambda \sum_{i=1}^{k} x_i + x \right) d\mu_0(x_1) \cdots d\mu_0(x_k).
\] (15)

Thus, by (15), we get
\[
q^{-n} BD_{n,q}^{(k)}(x|\lambda) = \sum_{\ell=1}^{n} S_1(n, \ell) \lambda^\ell \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( \sum_{i=1}^{k} x_i + \frac{x}{\lambda} \right)^\ell d\mu_0(x_1) \cdots d\mu_0(x_k).
\] (16)
Now, we observe that
\[
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{\left(\sum_{i=1}^{k} x_i + x\right)} d\mu_0(x_1) \cdots d\mu_0(x_k) = \left(\frac{t}{e^t + 1}\right)^k e^{xt}
\]
\[
= \sum_{n=0}^{\infty} B^{(k)}_n(x) \frac{t^n}{n!}
\]
where \(B^{(k)}_n(x)\) are called Bernoulli polynomials of order \(k\).

By (17), we get
\[
B^{(k)}_n(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(\sum_{i=1}^{k} x_i + x\right)^n d\mu_0(x_1) \cdots d\mu_0(x_k).
\]

From (16) and (18), we have
\[
q^{-n} BC^{(k)}_{n,q}(x|\lambda) = \sum_{\ell=1}^{n} S_1(n, \ell) \lambda^\ell B^{(k)}_{\ell}\left(\frac{x}{\lambda}\right).
\]

From (15), we can derive the generating function of \(BD^{(k)}_{n,q}(x|\lambda)\) as follows:
\[
\sum_{n=0}^{\infty} BD^{(k)}_{n,q}(x|\lambda) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + qt)^{\lambda \sum_{i=1}^{k} x_i + x} d\mu_0(x_1) \cdots d\mu_0(x_k)
\]
\[
= \left(\frac{\lambda \log(1 + qt)}{(1 + qt)^\lambda - 1}\right)^k (1 + qt)^x.
\]

by replacing \(qt\) by \(e^t - 1\), we get
\[
\sum_{n=0}^{\infty} q^{-n} BD^{(k)}_{n,q}(x|\lambda) \frac{1}{n!} (e^t - 1)^n = \left(\frac{\lambda t}{e^t - 1}\right)^k e^{xt}
\]
\[
= \sum_{n=0}^{\infty} B^{(k)}_n\left(\frac{x}{\lambda}\right) \lambda^n \frac{t^n}{n!}
\]
and
\[
\sum_{n=0}^{\infty} q^{-n} BD^{(k)}_{n,q}(x|\lambda) \frac{1}{n!} (e^t - 1)^n = \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} q^{-n} BD^{(k)}_{n,q}(x|\lambda) S_2(m, n)\right) \frac{t^m}{m!}.
\]

Therefore, by (21) and (22), we obtain the following theorem.

**Theorem 2.4.** For \(m \geq 0\), we have
\[
\sum_{n=0}^{\infty} BD^{(k)}_{n,q}(x|\lambda) S_2(m, n) q^{-n} = \lambda^m B^{(k)}_m\left(\frac{x}{\lambda}\right).
\]
For \( n \geq 0 \), the rising factorial sequence is defined by
\[
x^{n!} = x(x - 1) \cdots (x - n + 1) = (-1)^n (-x)_n
\]
\[
= \sum_{\ell=0}^{n} |S_1(n, \ell)| x^\ell,
\]
where \( |S_1(n, \ell)| = (-1)^{n-\ell} S_1(n, \ell) \).

We consider the \( q \)-analogue of lambda-Daehee polynomials of the second kind as follows:
\[
\hat{BD}_n,q(x|\lambda) = q^n \int_{Z_p} (-\lambda y + x)_n d\mu_0(y), \quad (n \geq 0).
\] (24)

From (24), we have
\[
q^n \hat{BD}_n,q(x|\lambda) = \sum_{\ell=0}^{n} S_1(n, \ell) (-1)^\ell \lambda^\ell \int_{Z_p} \left( -\frac{x}{\lambda} + y \right)^\ell d\mu_0(y)
\]
\[
= \sum_{\ell=0}^{n} S_1(n, \ell) (-1)^\ell \lambda^\ell B_\ell \left( -\frac{x}{\lambda} \right).
\] (25)

When \( x = 0 \), \( \hat{BD}_{n,q}(\lambda) = \hat{BD}_{n,q}(0|\lambda) \) are called the \( q \)-analogue of lambda-Daehee numbers of the second kind. The generating function of \( \hat{BD}_{n,q}(x|\lambda) \) is given by
\[
\sum_{n=0}^{\infty} \hat{BD}_{n,q}(x|\lambda) t^n n! = \int_{Z_p} (1 + qt)^{-\lambda y + x} d\mu_0(y)
\]
\[
= \frac{\lambda \log(1 + qt)}{(1 + qt)^\lambda - 1} (1 + qt)^{\lambda + x}.
\] (26)

By replacing \( qt \) by \( e^t - 1 \), we get
\[
\sum_{n=0}^{\infty} q^n \hat{BD}_{n,q}(x|\lambda) \frac{1}{n!} (e^t - 1)^n = \sum_{m=0}^{\infty} \lambda^m B_m \left( \frac{\lambda + x}{\lambda} \right) \frac{t^m}{m!}
\] (27)

and
\[
\sum_{n=0}^{\infty} \hat{BD}_{n,q}(x|\lambda) q^n \frac{1}{n!} (e^t - 1)^n = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} \hat{BD}_{n,q}(x|\lambda) S_2(m, n) q^n \right) \frac{t^m}{m!}.
\] (28)

Therefore, by (27) and (28), we obtain the following theorem.

**Theorem 2.5.** For \( m \geq 0 \), we have
\[
q^{-m} \hat{BD}_{m,q}(x|\lambda) = \sum_{\ell=0}^{m} S_1(m, \ell) (-1)^\ell \lambda^\ell B_\ell \left( -\frac{x}{\lambda} \right)
\]
and
\[
\lambda^m B_m \left( \frac{\lambda + x}{\lambda} \right) = \sum_{n=0}^{m} \overline{BD}_{n,q}(x|\lambda) S_2(m, n) q^{-n}.
\]

For \( k \in \mathbb{N} \), let us consider the \( q \)-analogue of lambda-Daehee polynomials of the second kind with order \( k \) as follows:
\[
\overline{BD}_{n,q}^{(k)}(x|\lambda) = q^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left( -\lambda \sum_{i=1}^{k} x_i + x \right)_n \, d\mu_0(x_1) \cdots d\mu_0(x_k),
\]
where \( n \geq 0 \).

From (29), we have
\[
q^{-n} \overline{BD}_{n,q}^{(k)}(x|\lambda) = \sum_{\ell=0}^{n} S_1(n, \ell)(-1)^\ell B_\ell^{(k)} \left( -\frac{x}{\lambda} \right) \lambda^\ell.
\]

The generating function of \( \overline{BD}_{n,q}^{(k)}(x|\lambda) \) is given by
\[
\sum_{n=0}^{\infty} \overline{BD}_{n,q}^{(k)}(x|\lambda) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + qt)^{-\lambda \sum_{i=1}^{k} x_i + x} \, d\mu_0(x_1) \cdots d\mu_0(x_k)
\]
\[
= \left( \frac{\lambda \log(1 + qt)}{1 + qt} \right)^k (1 + qt)^{\lambda k + x}.
\]

By replacing \( qt \) by \( e^t - 1 \) in (31), we get
\[
\sum_{n=0}^{\infty} q^{-n} \overline{BD}_{n,q}^{(k)}(x|\lambda) \frac{1}{n!} (e^t - 1)^n = \left( \frac{\lambda t}{e^M - 1} \right)^k e^{(\lambda k + x)t}
\]
\[
= \sum_{m=0}^{\infty} \lambda^m B_m^{(k)} \left( k + \frac{x}{\lambda} \right) \frac{t^m}{m!}
\]
and
\[
\sum_{n=0}^{\infty} q^{-n} \overline{BD}_{n,q}^{(k)}(x|\lambda) \frac{1}{n!} (e^t - 1)^n = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \overline{BD}_{n,q}^{(k)}(x|\lambda) S_2(m, n) q^{-n} \frac{t^m}{m!}.
\]

Therefore, by (32) and (33), we obtain the following theorem.

**Theorem 2.6.** For \( m \geq 0 \), we have
\[
(-q)^m \overline{BD}_{m,q}^{(k)}(x|\lambda) = \sum_{\ell=0}^{m} |S_1(m, \ell)| \lambda^\ell B_\ell^{(k)} \left( -\frac{x}{\lambda} \right)
\]
and
\[
\lambda^m B_m^{(k)} \left( k + \frac{x}{\lambda} \right) = \sum_{n=0}^{m} \overline{BD}_{n,q}^{(k)}(x|\lambda) S_2(m, n) q^{-n}.
\]
Now, we observe that
\[ q^{-n}(-1)^n \frac{BD_{n,q}(x|\lambda)}{n!} = (-1)^n \int_{\mathbb{Z}_p} \binom{x + \lambda y}{n} d\mu_q(y) \]
\[ = \int_{\mathbb{Z}_p} \binom{-\lambda y - x + n - 1}{n} d\mu_q(y) \]
\[ = \sum_{m=0}^{n} \binom{n-1}{m-1} \int_{\mathbb{Z}_p} \binom{-y\lambda - x}{m} d\mu_q(y) \]
\[ = \sum_{m=1}^{n} \binom{n-1}{m-1} q^{-m} \frac{BD_{m,q}(-x|\lambda)}{m!} \]  
(36)

and
\[ (-1)^n q^{-n} \frac{BD_{n,q}(x|\lambda)}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{BD_{m,q}(-x|\lambda)}{m!} q^{-m}. \]  
(37)

Therefore, by (36) and (37), we obtain the following theorem.

**Theorem 2.7.** For \( n \geq 1 \), we have
\[ q^{-n}(-1)^n \frac{BD_{n,q}(x|\lambda)}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{BD_{m,q}(-x|\lambda)}{m!} q^{-m}, \]  
(38)

and
\[ q^{-n}(-1)^n \frac{BD_{n,q}(x|\lambda)}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{BD_{m,q}(-x|\lambda)}{m!} q^{-m}. \]  
(39)

**References**


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