Daehee Polynomials with $q$-Parameter

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Abstract

In this paper, we consider Daehee polynomials with $q$-parameter and investigate some properties of those polynomials.

Mathematics Subject Classification: 05A19; 11B65; 11B83

Keywords: Daehee polynomial with $q$-parameter, $p$-adic invariant integral
1. Introduction

Let \( p \) be a fixed prime. Throughout this paper, \( \mathbb{Z}_p, \mathbb{Q}_p \) and \( \mathbb{C}_p \) will denote the ring of \( p \)-adic integers, the field of \( p \)-adic rational numbers and the completion of the algebraic closure of \( \mathbb{Q}_p \). The \( p \)-adic norm is normalized as \( |p|_p = \frac{1}{p} \).

Let \( UD(\mathbb{Z}_p) \) be the space of uniformly differentiable functions on \( \mathbb{Z}_p \). For \( f \in UD(\mathbb{Z}_p) \), the \( p \)-adic invariant integral on \( \mathbb{Z}_p \) is defined by

\[
I_0(f) = \int_{\mathbb{Z}_p} f(x) \, d\mu_0(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad \text{(see [1–20])}. \tag{1}
\]

Thus, by (1), we get

\[
I_0(f_1) = I_0(f) + f'(0), \quad \text{where } f_1(x) = f(x+1). \tag{2}
\]

The signed Stirling number of the first kind is defined by

\[
(x)_n = x(x-1) \cdots (x-n+1) = \sum_{l=0}^{n} S_1(n, l) x^l, \tag{3}
\]

and the Stirling number of the second kind is defined by the generating function to be

\[
(e^t - 1)^n = n! \sum_{m=n}^{\infty} S_2(m, n) \frac{t^m}{m!}, \quad \text{(see [5, 8])}. \tag{4}
\]

As is well known, the unsigned Stirling number of the first kind is given by

\[
x^n = x(x+1) \cdots (x+n-1) = \sum_{l=0}^{n} |S_1(n, l)| x^l.
\]

The Daehee polynomials are defined by

\[
D_n(x) = \int_{\mathbb{Z}_p} (x+y)_n \, d\mu_0(y), \quad (n \geq 0), \quad \text{(see [5, 8])}. \tag{5}
\]

When \( x = 0 \), \( D_n = D_n(0) \) are called the Daehee numbers.

From (5), we can derive the generating function to be

\[
\left( \frac{\log (1+t)}{t} \right)^{\alpha} (1+t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}, \quad \text{(see [5])}. \tag{6}
\]

For \( \alpha \in \mathbb{N} \), as is well known, the Bernoulli polynomials of order \( \alpha \) are defined by the generating function to be

\[
\left( \frac{t}{e^t - 1} \right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_\alpha^{(n)}(x) \frac{t^n}{n!}, \quad \text{(see [1–20])}. \tag{7}
\]

When \( x = 0 \), \( B_\alpha^{(n)} = B^{(n)}(0) \) are called the Bernoulli numbers of order \( \alpha \).
If \( \alpha = 1 \), \( B_n (x) = B^{(1)}_n (x) \) are called the ordinary Bernoulli polynomials.

We assume that \( q \) is an indeterminate in \( \mathbb{C}_p \) with \( |1 - q|_p < p^{-\frac{1}{p-1}} \).

In this paper, we consider Daehee polynomials with \( q \)-parameter and give some identities of those polynomials which are derived from \( p \)-adic invariant integral on \( \mathbb{Z}_p \).

2. DAEHEE POLYNOMIALS WITH \( q \)-PARAMETER

In this section, we assume that \( t \in \mathbb{C}_p \) with \( |t|_p < |q|_p p^{-\frac{1}{p-1}} \). Now, we define the \( q \)-analogue of the falling factorial sequence as follows:

\[
(x)_{n,q} = x (x-q) (x-2q) \cdots (x-(n-1)q) , \quad (n \geq 0) .
\]

Note that \( \lim_{q \to 1} (x)_{n,q} = (x)_n = \sum_{m=0}^{n} S_1 (n, m) x^m \). Let us consider Daehee polynomials with \( q \)-parameter as follows:

\[
D_{n,q} (x) = \int_{\mathbb{Z}_p} (x+y)_{n,q} \, d\mu_0 (y) , \quad (n \geq 0) .
\]

From (9), we have

\[
D_{n,q} (x) = q^n \int_{\mathbb{Z}_p} \left( \frac{x+y}{q} \right)_n \, d\mu_0 (y)
\]

\[
= q^n \sum_{m=0}^{n} S_1 (m,n) \frac{1}{q^m} \int_{\mathbb{Z}_p} (x+y)^m \, d\mu_0 (x) .
\]

Now, we observe that

\[
\int_{\mathbb{Z}_p} e^{(x+y)t} \, d\mu_0 (y) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n (x) \frac{t^n}{n!} .
\]

By (11), we get

\[
\int_{\mathbb{Z}_p} (x+y)^n \, d\mu_0 (y) = B_n (x) , \quad (n \geq 0) .
\]

From (10) and (12), we have

\[
D_{n,q} (x) = \sum_{m=0}^{n} S_1 (n, m) q^{n-m} B_m (x)
\]

\[
= \sum_{m=0}^{n} |S_1 (n, m)| (-q)^{n-m} B_m (x) .
\]

From (9), we can derive the following generating function of \( D_{n,q} (x) \):

\[
\sum_{n=0}^{\infty} D_{n,q} (x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x+y)_{n,q} \, d\mu_0 (y) \frac{t^n}{n!}
\]
\[ \frac{1}{n^n} \int_{\mathbb{Z}_p} (x+y)^{a_n} d\mu_0 (y) = \frac{1}{n^n} \int_{\mathbb{Z}_p} (x+y)^{a_n} d\mu_0 (y) \]

and, by (2), we get

\[ \frac{1}{n^n} \int_{\mathbb{Z}_p} (1 + qt)^{a_n} d\mu_0 (y) = \frac{1}{n^n} \int_{\mathbb{Z}_p} (1 + qt)^{a_n} d\mu_0 (y) \]

From (15) and (15), we have

\[ \sum_{n=0}^{\infty} D_{n,q} (x) \frac{t^n}{n!} = (1 + qt) \frac{x}{q} \frac{\log (1 + qt)}{q (1 + qt)^{q/2} - 1}. \] \hspace{1cm} (15)

\[ \sum_{n=0}^{\infty} B_{n,q} (x) \frac{1}{n!} = (e^t - 1) \frac{e^{x/t}}{e^{x/t} - 1} = \sum_{n=0}^{\infty} B_n (x) \frac{1}{q^n n!} \] \hspace{1cm} (16)

When \( x = 0 \), \( D_{n,q} = D_{n,q} (0) \) are called the Daehee numbers with \( q \)-parameter.

In (16), by replacing \( t \) by \( \frac{1}{q} \left( e^t - 1 \right) \), we get

\[ \sum_{n=0}^{\infty} D_{n,q} (x) \frac{1}{n!} (e^t - 1)^n = \sum_{n=0}^{\infty} D_{n,q} (x) \frac{1}{q^n n!} = \sum_{m=0}^{\infty} B_n (x) \frac{1}{q^n n!} \] \hspace{1cm} (17)

and

\[ \sum_{n=0}^{\infty} D_{n,q} (x) \frac{1}{n!} (e^t - 1)^n = \sum_{m=0}^{\infty} D_{n,q} (x) \frac{1}{q^n} \sum_{m=0}^{\infty} S_2 (m, n) \frac{t^m}{m!} \] \hspace{1cm} (18)

Thus, by (17) and (18), we see that

\[ B_m (x) = q^m \sum_{n=0}^{m} D_{n,q} (x) \frac{1}{q^n} S_2 (m, n) = \sum_{n=0}^{m} q^{m-n} D_{n,q} (x) S_2 (m, n) \] \hspace{1cm} (19)

\[ = \sum_{n=0}^{m} |S_2 (m, n)| (-q)^{m-n} D_{n,q} (x), \quad (m \geq 0) \]

Now, we consider the \( q \)-Daehee polynomials of order \( k \in \mathbb{N} \) which are derived from the following multivariate \( p \)-adic invariant integral on \( \mathbb{Z}_p \) :

\[ D_{n,q}^{(k)} (x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k + x)_{n,q} d\mu_0 (x_1) \cdots d\mu_0 (x_k) \] \hspace{1cm} (20)

where \( n \in \mathbb{N} \cup \{0\} \) and \( k \in \mathbb{N} \).
From (20), we can derive the generating function of $D_{n,q}^{(k)}(x)$ as follows:

$$\sum_{n=0}^{\infty} D_{n,q}^{(k)}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} q^n \prod_{i=1}^{k} \int_{\mathbb{Z}} \left( \frac{x_1^{q} + \cdots + x_k^{q} + x}{n} \right) d\mu_0(x_1) \cdots d\mu_0(x_k) t^n$$

(21)

$$= \prod_{i=1}^{k} \int_{\mathbb{Z}} (1 + qt) \frac{x_1^{q} + \cdots + x_k^{q}}{q} d\mu_0(x_1) \cdots d\mu_0(x_k)$$

$$= (1 + qt)^{\frac{n}{q}} \prod_{i=1}^{k} \int_{\mathbb{Z}} (1 + qt) \frac{x_1^{q} + \cdots + x_k^{q}}{q} d\mu_0(x_1) \cdots d\mu_0(x_k)$$

$$= (1 + qt)^{\frac{n}{q}} \left( \frac{\log (1 + qt)}{q (1 + qt)^{\frac{1}{q}} - 1} \right)^k.$$

By (20), we easily get

$$D_{n,q}^{(k)}(x) = q^n \sum_{m=0}^{n} \frac{S_1(n,m)}{q^m} \prod_{i=1}^{k} \int_{\mathbb{Z}} (x_1 + \cdots + x_k + x)^m d\mu_0(x_1) \cdots d\mu_0(x_k).$$

(22)

Now, we observe that

$$\int_{\mathbb{Z}} \cdots \int_{\mathbb{Z}} e^{(x_1 + \cdots + x_k + x)t} d\mu_0(x_1) \cdots d\mu_0(x_k)$$

(23)

$$= \left( \frac{t}{e^t - 1} \right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}.$$

Thus, by (23), we get

$$\int_{\mathbb{Z}} \cdots \int_{\mathbb{Z}} (x_1 + \cdots + x_k + x)^n d\mu_0(x_1) \cdots d\mu_0(x_k) = B_n^{(k)}(x), \quad (n \geq 0).$$

(24)

From (22) and (24), we have

$$D_{n,q}^{(k)}(x) = q^n \sum_{m=0}^{n} \frac{S_1(n,m)}{q^m} B_m^{(k)}(x)$$

(25)

$$= \sum_{m=0}^{n} q^{n-m} S_1(n,m) B_m^{(k)}(x)$$

$$= \sum_{m=0}^{n} (-q)^{n-m} |S_1(n,m)| B_m^{(k)}(x).$$

When $x = 0$, $D_{n,q}^{(k)} = D_{n,q}^{(k)}(0)$ are called the $q$-Daehee numbers with order $k$. 

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In (21), by replacing $t$ by $\frac{1}{q} (e^t - 1)$, we get

$$
\sum_{n=0}^{\infty} D_{n,q}^{(k)} (x) \frac{1}{q^n} \frac{(e^t - 1)^n}{n!} = e^{\frac{t}{q}} \left( \frac{1}{e^{\frac{t}{q}} - 1} \right)^k
$$

(26)

and

$$
\sum_{n=0}^{\infty} D_{n,q}^{(k)} (x) \frac{1}{q^n} \frac{(e^t - 1)^n}{n!} = \sum_{n=0}^{\infty} D_{n,q}^{(k)} (x) \frac{1}{q^n} \frac{S_2 (m, n)}{m!} t^m
$$

(27)

By (26) and (27), we get

$$
B_m^{(k)} (x) = \sum_{n=0}^{m} q^{m-n} D_{n,q}^{(k)} (x) S_2 (m, n).
$$

(28)

Now, we consider the Dahee polynomials of the second kind with $q$-parameter as follows :

$$
\hat{D}_{n,q} (x) = \int_{\mathbb{Z}_p} (-y + x)_{n,q} d\mu_0 (x), \quad (n \geq 0).
$$

(29)

Thus, by (29), we get

$$
\hat{D}_{n,q} (x) = q^n \int_{\mathbb{Z}_p} \left( \frac{-y + x}{q} \right)_n d\mu_0 (x).
$$

(30)

From (29) and (30), we can derive

$$
\sum_{n=0}^{\infty} \hat{D}_{n,q} (x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} (1 + qt)^{-\frac{y+x}{q}} d\mu_0 (y)
$$

(31)

$$
= (1 + qt)^{\frac{x}{q}} \frac{\log (1 + qt)}{q \left( (1 + qt)^{\frac{1}{q}} - 1 \right)} (1 + qt)^{\frac{1}{q}}
$$

$$
= (1 + qt)^{\frac{x}{q}} \left( \frac{\log (1 + qt)}{q \left( (1 + qt)^{\frac{1}{q}} - 1 \right)} \right) .
$$

By (30), we get

$$
\hat{D}_{n,q} (x) = q^n \int_{\mathbb{Z}_p} \left( \frac{-y + x}{q} \right)_n d\mu_0 (y)
$$

(32)
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\[
= q^n \sum_{m=0}^{n} \frac{S_1(n, m)}{q^m} \int_{\mathbb{Z}_p} (-y + x)^m d\mu_0(y)
\]

\[
= \sum_{m=0}^{n} S_1(n, m) (-1)^m \int_{\mathbb{Z}_p} (y - x)^m d\mu_0(y) q^{n-m}
\]

\[
= (-1)^n \sum_{m=0}^{n} |S_1(n, m)| B_m(-x) q^{n-m}.
\]

It is easy to show that

\[
B_n(-x) = (-1)^n B_n(x + 1), \quad (n \geq 0).
\]  

From (32) and (33), we have

\[
\hat{D}_{n,q}(x) = (-1)^{m+n} \sum_{m=0}^{n} |S_1(n, m)| B_m(x + 1) q^{n-m}.
\]

When \( x = 0 \), \( \hat{D}_{n,q} = \hat{D}_{n,q}(0) \) are called the Dahee numbers of the second kind with \( q \)-parameter. In (31), by replacing \( t \) by \( \frac{1}{q}(e^t - 1) \), we get

\[
\sum_{n=0}^{\infty} \hat{D}_{n,q}(x) \frac{1}{q^n} \frac{(e^t - 1)^n}{n!} = e^{\left(\frac{x+1}{q}\right)t} \left(\frac{\frac{t}{q}}{e^\frac{t}{q} - 1}\right)
\]

\[
= \sum_{n=0}^{\infty} \frac{B_n(x + 1) t^n}{q^n n!},
\]

and

\[
\sum_{n=0}^{\infty} \hat{D}_{n}(x) \frac{1}{q^n} \frac{(e^t - 1)^n}{n!} = \sum_{n=0}^{\infty} \hat{D}_{n,q}(x) \frac{1}{q^n} \sum_{m=n}^{\infty} S_2(m, n) \frac{t^m}{m!}.
\]

From (35) and (36), we have

\[
B_m(x + 1) = \sum_{n=0}^{m} q^{m-n} \hat{D}_{n,q}(x) S_2(m, n), \quad (m \geq 0).
\]

References


Received: May 9, 2014