Some Symmetric Identities of Generalized Carlitz’s $q$-Bernoulli Polynomials of the First Kind

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Abstract

In this paper, we consider the generalized Carlitz $q$-Bernoulli polynomials of the first kind which are derived from $q$-Volkenborn integral on $\mathbb{Z}_p$. Finally, we investigate some symmetric identities of those polynomials.

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1. Introduction

Let $p$ be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. The $p$-adic norm $|\cdot|_p$ is normalized as $|p|_p = 1/p$. We assume that $q$ is an indeterminate in $\mathbb{C}_p$ with $|1 - q|_p < p^{-1/(p-1)}$. For $d \in \mathbb{N}$ with $(d, p) = 1$, we set

$$X = \lim_{N \to \infty} \mathbb{Z}/dp^NZ, \quad X^* = \bigcup_{0 < a < dp \atop (a, dp) = 1} (a + dp\mathbb{Z}_p)$$

and

$$a + dp^NZ_p = \{ x \in X | x \equiv a \, (\text{mod} \, dp^N) \},$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$.

The $q$ number of $x$ is defined by $[x]_q = (1 - q^x)/(1 - q)$. Note that $\lim_{q \to 1} [x]_q = x$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$ the $q$-Volkenborn integral on $\mathbb{Z}_p$ is defined by T. Kim to be

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x)$$

$$= \lim_{N \to \infty} \sum_{x=0}^{p^{N-1}} f(x)\mu_q(x + p^NZ_p)$$

$$= \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^{N-1}} f(x)q^x, \quad (\text{see } [13, 14]). \tag{1}$$

From (1), we can easily derive the following integral equation:
\begin{align*}
q I_q(f_1) &= I_q(f) + (q - 1)f(0) + \frac{q - 1}{\log q} f'(0), \\
\text{(2)}
\end{align*}

where \( f_1(x) = f(x + 1) \).

Let \( \chi \) be a Dirichlet’s character with conductor \( d \in \mathbb{N} \). As is well known, the generalized Bernoulli polynomials attached to \( \chi \) are defined by the generating function to be

\begin{align*}
\frac{1}{e^{dt} - 1} \left( \sum_{a=0}^{d-1} \chi(a)e^{at} \right) e^{xt} &= \sum_{n=0}^{\infty} B_{n,\chi}(x) \frac{t^n}{n!}, \quad \text{(see [3-19]).} \\
\text{(3)}
\end{align*}

When \( x = 0 \), \( B_{n,\chi} = B_{n,\chi}(0) \) are called the generalized Bernoulli numbers attached to \( \chi \). From (1) and (2), we have

\begin{align*}
\int_X \chi(y)e^{(x+y)t}d\mu_0(y) &= \frac{t}{e^{dt} - 1} \sum_{a=0}^{d-1} \chi(a)e^{(a+x)t} \\
&= \sum_{n=0}^{\infty} B_{n,\chi}(x) \frac{t^n}{n!}, \\
\text{(4)}
\end{align*}

where \( \lim_{q \to 1} \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \int_{\mathbb{Z}_p} f(x)d\mu_0(x) \).

Thus, by (4), we get

\begin{align*}
\int_X \chi(y)(x+y)^n d\mu_0(y) = B_{n,\chi}(x), \quad (n \geq 0). \\
\text{(5)}
\end{align*}

The generalized Carlitz \( q \)-Bernoulli polynomials of the first kind attached to \( \chi \) with viewpoint of (5) are defined by

\begin{align*}
\int_X \chi(y)[x+y]^n d\mu_0(y) = \beta_{n,\chi,q}(x), \quad (n \geq 0). \\
\text{(6)}
\end{align*}

When \( x = 0 \), \( \beta_{n,\chi,q} = \beta_{n,\chi,q}(0) \) are called the first kind attached to \( \chi \) (see [3, 13]). From (6), we note that the generating function of \( \beta_{n,\chi,q}(x) \) is given by

\begin{align*}
\int_X \chi(y)e^{[x+y]q^t}d\mu_q(y) &= \sum_{n=0}^{\infty} \beta_{n,\chi,q}(x) \frac{t^n}{n!}. \\
\text{(7)}
\end{align*}

From (6) and (7), we can derive

\begin{align*}
\beta_{n,\chi,q}(x) &= \sum_{\ell=0}^{n} \binom{n}{\ell} \beta_{\ell,\chi,q} q^{\ell x} [x]^n = [d]^{n-1} \sum_{a=0}^{d-1} \chi(a) q^a \beta_{n,q^d} \left( \frac{x + a}{d} \right), \\
\text{(8)}
\end{align*}
where $\beta_{n,q}(x)$ are the Carlitz $q$-Bernoulli polynomials of the first kind (see [3, 13, 14, 15]).

In this paper, we investigate some properties of the generalized Carlitz $q$-Bernoulli polynomials of the first kind attached to $\chi$ and give some interesting symmetric identities of those polynomials.

2. SOME IDENTITIES OF THE GENERALIZED CARLITZ $q$-BERNOULLI POLYNOMIALS OF THE FIRST KIND.

Let $w_1, w_2, w_3$ be positive integers. From (7), we consider the following integral equation;

\[
\int_X \chi(y) \exp \{ [w_2 w_3 y + w_1 w_2 w_3 x + w_1 w_3 i + w_1 w_2 j]_q t \} d\mu_{q w_2 w_3} (y) = \lim_{N \to \infty} \frac{1}{[dw_1 p^N]_{q w_2 w_3}} \sum_{k=0}^{d w_1 - 1} \sum_{y=0}^{p^N - 1} \chi(k) \\
\times \exp \{ [w_2 w_3 (k + dw_1 y) + w_1 w_2 w_3 x + w_1 w_3 i + w_1 w_2 j]_q t \} q^{w_2 w_3 (k + dw_1 y) t}.
\]

(9)

From (9), we can derive the following equation;

\[
I = \frac{1}{[w_2 w_3]_q} \sum_{i=0}^{d w_2 - 1} \sum_{j=0}^{d w_3 - 1} \chi(i) \chi(j) q^{w_1 w_3 i + w_1 w_2 j} \\
\times \int_X \chi(y) \exp \{ [w_2 w_3 y + w_1 w_2 w_3 x + w_1 w_3 i + w_1 w_2 j]_q t \} d\mu_{q w_2 w_3} (y) = \lim_{N \to \infty} \frac{1}{[d(w_1 w_2 w_3)]_q} \sum_{i=0}^{d w_2 - 1} \sum_{j=0}^{d w_3 - 1} \sum_{k=0}^{d w_1 - 1} \sum_{y=0}^{p^N - 1} q^{w_1 w_3 i + w_1 w_2 j + w_2 w_3 k} \chi(i) \chi(j) \chi(k) \\
\times \exp \{ [w_2 w_3 (k + dw_1 y) + w_1 w_2 w_3 x + w_1 w_3 i + w_1 w_2 j]_q t \} q^{w_1 w_3 w_2 w_3 y}.
\]

(10)

By the same method as (10), we get
Theorem 2.1. Let $\sigma$ be the same for any permutations $\sigma \in S_3$.

Therefore, by (10) and (11), we obtain the following theorem.

Theorem 2.1. Let $d, w_1, w_2, w_3$ be positive integers. Then the following expressions

\[
[\sum_{i=0}^{d-1} \sum_{j=0}^{d-1} \chi(i)\chi(j)q^{w_1i+w_2j}]_q = \lim_{N \to \infty} \frac{1}{[d! w_1 w_2 w_3]_q} \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} \sum_{k=0}^{N-1} \sum_{y=0}^{w_2 w_1 + w_2 w_3 + w_1 w_3 k} \chi(i)\chi(j) \exp \{ [w_3 w_1(k + d w_3 y) + w_1 w_2 w_3 x + w_2 w_1 i + w_2 w_3 j]_q t \}.
\]

(11)

are the same for any permutations $\sigma \in S_3$.

It is easy to show that

\[
[w_1 w_2 w_3 y + w_1 w_2 w_3 x + w_1 w_3 i + w_1 w_2 j]_q = [w_2 w_3]_q \left[ y + w_1 x + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j \right]_q^{w_2 w_3}.
\]

(12)

By (12), we get

\[
\int_X \left[ w_2 w_3 y + w_1 w_2 w_3 x + w_1 w_3 i + w_1 w_2 j \right]_q^n \chi(y) d\mu_{q^{w_2 w_3}}(y) = [w_2 w_3]_q^n \int_X \chi(y) \left[ y + w_1 x + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j \right]_q^n d\mu_{q^{w_2 w_3}}(y).
\]

(13)

Therefore, by Theorem 1 and (13), we obtain the following theorem.

Theorem 2.2. Let $d, w_1, w_2, w_3 \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, the following expressions
are the same for any permutation $\sigma \in S_3$.

We observe that

$$
\left[ y + w_1 x + \frac{w_1 i}{w_2} + \frac{w_1 j}{w_3} \right]^n
$$

is the same as

$$
\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \left( \frac{w_1 i}{w_2 w_3} \right)^{n-k} [w_3 i + w_2 j]^{n-k} q^{k(w_1 w_3 i + w_1 w_2 j)} [y + w_1 x]^k
$$

Thus, by (14), we get

$$
\int_X \chi(y) \left[ y + w_1 x + \frac{w_1 i}{w_2} + \frac{w_1 j}{w_3} \right]^n d\mu_{q^{w_2 w_3}}(y)
$$

$$
= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \left( \frac{[w_1 q]}{[w_2 w_3]_q} \right)^{n-k} [w_3 i + w_2 j]^{n-k} q^{k(w_1 w_3 i + w_1 w_2 j)} \int_X \chi(y) [y + w_1 x]^k d\mu_{q^{w_2 w_3}}(y)
$$

$$
= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \left( \frac{[w_1 q]}{[w_2 w_3]_q} \right)^{n-k} [w_3 i + w_2 j]^{n-k} q^{k(w_1 w_3 i + w_1 w_2 j)} \beta_{k, x, q^{w_2 w_3}}(w_1 x)
$$

From (15), we have

$$
I = [w_2 w_3]_q^{n-1} \sum_{i=0}^{d w_2 - 1} \sum_{j=0}^{d w_3 - 1} \chi(i) \chi(j) q^{w_1 w_3 i + w_1 w_2 j}
$$

$$
\times \int_X \chi(y) \left[ y + w_1 x + \frac{w_1 i}{w_2} + \frac{w_1 j}{w_3} \right]^n d\mu_{q^{w_2 w_3}}(y)
$$

$$
= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} [w_2 w_3]_q^{k-1} [w_1 q]^{n-k} \beta_{k, x, q^{w_2 w_3}}(w_1 x)
$$

$$
\times \sum_{i=0}^{d w_2 - 1} \sum_{j=0}^{d w_3 - 1} \chi(i) \chi(j) q^{(w_1 w_3 i + w_1 w_2 j)(k+1)} [w_3 i + w_2 j]^{n-k}
$$

$$
= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} [w_2 w_3]_q^{k-1} [w_1 q]^{n-k} \beta_{k, x, q^{w_2 w_3}}(w_1 x) S_{n, k, q^{w_1}}(w_1, w_2, \ldots, d|\chi),
$$
where
\[
S_{n,k,q}(w_1, w_2, : d|\chi) = \sum_{i=0}^{dw_1-1} \sum_{j=0}^{dw_2-1} \chi(i)\chi(j)q^{(w_2i+w_1j)(k+1)}[w_2i + w_1j]_q^{n-k}.
\]

By the same method as (16), we get
\[
I = [w_3w_1]_q^{n-1} \sum_{i=0}^{dw_3-1} \sum_{j=0}^{dw_1-1} \chi(i)\chi(j)q^{w_2w_1i+w_2w_3j}
\times \int_X \chi(y) \left[ y + w_2x + \frac{w_2i}{w_3} + \frac{w_2j}{w_1} \right]^n d\mu_{q^{w_1w_3}}(y)
= \sum_{k=0}^{n} \binom{n}{k} [w_3w_1]_q^{k-1} [w_2]_q^{n-k} \beta_{k,\chi,q^{w_1w_3}} (w_2x) S_{n,k,q^{w_1w_3}}(w_1, w_2, : d|\chi).
\]

Therefore, by (16) and (17), we obtain the following theorem.

**Theorem 2.3.** For \( d, w_1, w_2, w_3 \in \mathbb{N} \) and \( n \geq 0 \), the following expressions
\[
\sum_{k=0}^{n} \binom{n}{k} [w_{\sigma(2)}w_{\sigma(3)}]_q^{k-1} [w_{\sigma(1)}]_q^{n-k} \beta_{k,\chi,q^{w_{\sigma(2)}w_{\sigma(3)}}} (w_{\sigma(1)}x) S_{n,k,q^{w_{\sigma(1)}}}(w_{\sigma(2)}, w_{\sigma(3)}, : d|\chi)
\]
are all the same for any \( \sigma \in S_3 \).

**Remark 2.4.** Recently, several authors have studied the \( q \)-extension of Bernoulli polynomials and identities of symmetry for Bernoulli numbers and polynomials (see [1-22]).

**References**


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