A Note on the Generalized $q$-Tangent Polynomials

C. S. Ryoo

Department of Mathematics
Hannam University, Daejeon 306-791, Korea

Abstract

In this paper we introduce the generalized $q$-tangent numbers $T_{n,\chi,q}$ and polynomials $T_{n,\chi,q}(x)$.

Mathematics Subject Classification: 11B68, 11S40, 11S80

Keywords: tangent numbers and polynomials, $q$-tangent numbers and $q$-tangent polynomials, generalized $q$-tangent numbers and polynomials

1 Introduction

Recently, many mathematicians have studied in the area of the Euler numbers and tangent numbers(see [1-5]). We defined the $q$-tangent polynomials $T_{n,q}(x)$(see [3]). The $q$-tangent numbers $T_{n,q}$ are defined by the generating function:

$$F_q(t) = \frac{2}{qe^{2t} + 1} = \sum_{n=0}^{\infty} T_{n,q} \frac{t^n}{n!}$$  \hspace{1cm} (1.1)

We consider the $q$-tangent polynomials $T_{n,q}(x)$ as follows:

$$F_q(x, t) = \left( \frac{2}{qe^{2t} + 1} \right) e^{xt} = \sum_{n=0}^{\infty} T_{n,q}(x) \frac{t^n}{n!}$$  \hspace{1cm} (1.2)

The purpose of this paper is to construct the generalized $q$-tangent polynomials $T_{n,\chi,q}(x)$ attached to $\chi$ and derive a new $l$-series which interpolates the generalized $q$-tangent polynomials $T_{n,\chi,q}(x)$. 
Throughout this paper, we always make use of the following notations: \( \mathbb{N} \) denotes the set of natural numbers and \( \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \), \( \mathbb{C} \) denotes the set of complex numbers, \( \mathbb{Z}_p \) denotes the ring of \( p \)-adic rational integers, \( \mathbb{Q}_p \) denotes the field of \( p \)-adic rational numbers, and \( \mathbb{C}_p \) denotes the completion of algebraic closure of \( \mathbb{Q}_p \). Let \( \nu_p \) be the normalized exponential valuation of \( \mathbb{C}_p \) with \( |p|_p = p^{-\nu_p(p)} = p^{-1} \). When one talks of \( q \)-extension, \( q \) is considered in many ways such as an indeterminate, a complex number \( q \in \mathbb{C} \), or \( p \)-adic number \( q \in \mathbb{C}_p \). If \( q \in \mathbb{C} \) one normally assume that \( |q| < 1 \). If \( q \in \mathbb{C}_p \), we normally assume that \( |q - 1|_p < p^{-\frac{1}{p-1}} \) so that \( q^x = \exp(x \log q) \) for \( |x|_p \leq 1 \).

For \( g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \rightarrow \mathbb{C}_p \text{ is uniformly differentiable function} \} \), the fermionic \( p \)-adic invariant integral on \( \mathbb{Z}_p \) is defined by Kim as follows:

\[
I_{-1}(g) = \int_{\mathbb{X}} g(x) d\mu_{-1}(x) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} g(x)(-1)^x, \text{ see [2].}
\]

If we take \( g_n(x) = g(x+n) \) in (1.3), then we see that

\[
I_{-1}(g_n) = (-1)^n I_{-1}(g) + 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} g(l). \quad (1.4)
\]

2 The generalized \( q \)-tangent polynomials

In this section, our goal is to give generating functions of the generalized \( q \)-tangent numbers and polynomials. These numbers will be used to prove the analytic continuation of the \( l \)-series. Let \( q \) be a complex number with \( |q| < 1 \). Let \( \chi \) be Dirichlet’s character with conductor \( d \in \mathbb{N} \) with \( d \equiv 1(\text{mod}2) \). Then the generalized \( q \)-tangent numbers associated with \( \chi \), \( T_{n,\chi,q} \), are defined by the following generating function

\[
F_{\chi,q}(t) = \frac{2 \sum_{a=0}^{d-1} \chi(a)(-1)^a q^a e^{2at}}{q^d e^{2dt} + 1} = \sum_{n=0}^{\infty} T_{n,\chi,q} \frac{t^n}{n!}. \quad (2.1)
\]

We now consider the generalized \( q \)-tangent polynomials associated with \( \chi \), \( T_{n,\chi,q}(x) \), are also defined by

\[
F_{\chi,q}(x, t) = \left( \frac{2 \sum_{a=0}^{d-1} \chi(a)(-1)^a q^a e^{2at}}{q^d e^{2dt} + 1} \right) e^{xt} = \sum_{n=0}^{\infty} T_{n,\chi,q}(x) \frac{t^n}{n!}. \quad (2.2)
\]

When \( \chi = \chi^0 \), above (2.1) and (2.2) will become the corresponding definitions of the tangent \( q \)-numbers \( T_{n,q} \) and polynomials \( T_{n,q}(x) \).
A note on the generalized \(q\)-tangent polynomials

Since
\[
2 \sum_{a=0}^{d-1} \frac{\chi(a)(-1)^a q^a e^{2at}}{q^d e^{2dt} + 1} e^{xt} = \sum_{a=0}^{d-1} \chi(a)(-1)^a q^a \left( 2 e^{(2a+x)dt} \right) \frac{q^d e^{2dt} + 1}{q^d e^{2dt} + 1} \chi(a)(-1)^a q^a \left( \frac{2a + x}{d} \right) \left( \frac{t^m}{m!} \right),
\]
we have the following theorem.

**Theorem 2.1** Let \(\chi\) be Dirichlet’s character with conductor \(d \in \mathbb{N}\) with \(d \equiv 1 (\text{mod} 2)\). Then we have

1. \(T_{n,\chi,q}(x) = d^m \sum_{a=0}^{d-1} \chi(a)(-1)^a q^a T_{m,q} \left( \frac{2a + x}{d} \right) \),
2. \(T_{n,\chi,q} = d^m \sum_{a=0}^{d-1} \chi(a)(-1)^a q^a T_{m,q} \left( \frac{2a}{d} \right) \),
3. \(T_{n,\chi,q}(x) = \sum_{l=0}^{n} \frac{\binom{n}{l}}{l!} T_{l,\chi,q} x^{n-l} \).

For \(n \in \mathbb{N}\) with \(n \equiv 0 (\text{mod} 2)\), we have
\[
\frac{-2 \sum_{a=0}^{d-1} \chi(a)(-1)^a q^a e^{2at}}{q^d e^{2dt} + 1} q^{nd} e^{2ndt} + \frac{2 \sum_{a=0}^{d-1} \chi(a)(-1)^a q^a e^{2at}}{q^d e^{2dt} + 1} = \sum_{m=0}^{\infty} \left( \frac{2 \sum_{a=0}^{d-1} \chi(a)(-1)^a q^a (2a)^m}{m!} \right) \frac{t^m}{m!}.
\]

By comparing coefficients of \(\frac{t^m}{m!}\) in the above equation, we have the following theorem:

**Theorem 2.2** Let \(\chi\) be Dirichlet’s character with conductor \(d \in \mathbb{N}\) with \(d \equiv 1 (\text{mod} 2)\), \(n\) a positive even integer, and \(m \in \mathbb{N}\). Then we have
\[
2 \sum_{a=0}^{nd-1} \chi(a)(-1)^a q^a (2a)^m = -q^{nd} T_{m,\chi,q}(2nd) + T_{m,\chi,q}.
\]

Next, we introduce the \(l\)-series and two variable \(l\)-series.

**Definition 2.3** For \(s \in \mathbb{C}\), define two variable \(l\)-series as
\[
l_q(s, x|\chi) = 2 \sum_{m=0}^{\infty} \frac{(-1)^m \chi(m)q^m}{(2m + x)^s}.
\]
By using (2.2), we easily see that
\[
F_{\chi,q}(x, t) = 2 \sum_{a=0}^{d-1} \chi(a) (-1)^a q^a e^{2at} e^{xt} = 2 \sum_{a=0}^{d-1} \chi(a) (-1)^a q^a e^{(2a+x)t} \sum_{l=0}^{\infty} (-1)^l q^{ld} e^{2dt}
\]
\[
= 2 \sum_{a=0}^{d-1} \sum_{l=0}^{\infty} \chi(a) (-1)^{a+dl} q^{(a+dl)} e^{(2a+x+2dt)t}
\]
\[
= 2 \sum_{m=0}^{\infty} \chi(m)(-1)^m q^m e^{(2m+x)t}.
\]
Then we have
\[
\left( \frac{d}{dt} \right)^k F_{\chi,q}(x, t) \bigg|_{t=0} = 2 \sum_{n=0}^{\infty} \chi(n)(-1)^n q^n (2n + x)^k,
\]
(2.3)
and
\[
\left( \frac{d}{dt} \right)^k \left( \sum_{n=0}^{\infty} T_{n,\chi,q}(x) \frac{t^n}{n!} \right) \bigg|_{t=0} = T_{k,\chi,q}(x), \text{ for } k \in \mathbb{N}.
\]
(2.4)
By (2.3), (2.4), we have the following theorem.

**Theorem 2.4** For any positive integer $k$, we have
\[
T_{k,\chi,q}(x) = l_q(-k, x|\chi).
\]

**Definition 2.5** For $s \in \mathbb{C}$, define $l$-series as
\[
l_q(s \mid \chi) = 2 \sum_{m=1}^{\infty} \frac{(-1)^m \chi(m) q^m}{(2m)^s}.
\]

By simple calculation, we have the following theorem.

**Theorem 2.6** For any positive integer $k$, we have
\[
l_q(-k \mid \chi) = T_{k,\chi,q}.
\]

### 3 Witt-type formulae on $\mathbb{Z}_p$ in $p$-adic number field

Our primary aim in this section is to obtain the Witt-type formulae of the generalized $q$-tangent numbers $T_{n,\chi,q}$ and polynomials $T_{n,\chi,q}(x)$ attached to $\chi$. 
A note on the generalized $q$-tangent polynomials

We assume that $q \in \mathbb{C}_p$ with $|q - 1|^p < 1$. Let $\chi$ be the primitive Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$. Let $g(y) = \chi(y)q^y e^{(2y+x)t}$. By using (1.3), we derive

$$I_1(\chi(y)q^y e^{(2y+x)t}) = \int_X \chi(y)q^y e^{(2y+x)t} d\mu_{-1}(y)$$

$$= \left( \frac{2 \sum_{a=0}^{d-1} \chi(a)(-1)^a q^a e^{2at}}{q^d e^{2dt} + 1} \right) e^{xt}$$

$$= \sum_{n=0}^{\infty} T_{n,\chi,q}(x) \frac{t^n}{n!}. \quad (3.1)$$

By using Taylor series of $e^{(2y+x)t}$ in the above equation (3.1), we obtain

$$\sum_{n=0}^{\infty} \left( \int_X \chi(y)q^y (2y + x)^n d\mu_{-1}(y) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} T_{n,\chi,q}(x) \frac{t^n}{n!}. \quad (3.2)$$

By comparing coefficients of $\frac{t^n}{n!}$ in the above equation, we have the Witt formula for the generalized $q$-tangent polynomials attached to $\chi$ as follows:

**Theorem 3.1** For positive integers $n$, we have

$$T_{n,\chi,q}(x) = \int_X \chi(y)q^y (2y + x)^n d\mu_{-1}(y). \quad (3.2)$$

Observe that for $x = 0$, the equation (3.2) reduces to (3.3).

**Corollary 3.2** For positive integers $n$, we have

$$T_{n,\chi,q} = \int_X \chi(y)q^y (2y)^n d\mu_{-1}(y). \quad (3.3)$$

By (3.1) and (1.4), we have the following theorem:

**Theorem 3.3** For positive integers $n$, we have

$$q^{nd} T_{m,\chi,q}(2nd) - (-1)^n T_{m,\chi,q} = 2^{m+1} \sum_{l=0}^{nd-1} (-1)^{n-1-l} \chi(l) q^l r^m. \quad (3.4)$$

**References**


Received: November 1, 2013