Proof of Using Fourier Coefficients for Root Mean Square Calculations on Periodic Signals

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Abstract
The purpose of this work is to academically prove using Fourier coefficients of periodic signals for root mean square (RMS) calculations. In the detail, properties of Fourier series are brought to calculate RMS values of periodic signals obviously. Results from this work can confirm that this method is more convenient, efficient and simple for RMS calculations than Calculus integration technique (if Fourier coefficients are given) and can be cited for further researches.

Keywords: root mean square, RMS calculation, periodic signal, Fourier coefficients, Fourier series

1 Introduction
In order to calculate RMS values of periodic signals, various methods such as direct integration[1], Simpson’s rule[2] and trapezoidal rule[3] are always selected to do. There have been many academic works to show that RMS values of periodic signals are not only able to be calculated by using integration techniques
but also able to be calculated by using Fourier coefficients of their own periodic signals for several years. For the calculation method by using Fourier coefficients, it has been published for several years by many parties[4,5,6], but all works have not been academically proved by using properties of Fourier series.

1.1 RMS Calculations

Although there are many methods for calculating RMS values, the standard method is still Calculus integration technique for continuous data stated as:

\[
\text{f}_{\text{RMS}} = \sqrt{\frac{1}{T} \int_{t_i}^{t_i+T} f^2(t)dt}
\]  

(1.1.1),

where \( f_{\text{RMS}} \) is the RMS value of \( f(t) \) between the domain interval of \( t_i \) to \( t_i+T \), where \( T \) is the period of \( f(t) \)[3].

1.2 Periodic signals

A periodic signal is a signal constructed from many uniformly sinusoidal signals which can be expanded as the following Fourier series:

\[
f(t) = a_0 + \sum_{m=1}^{M} (a_m \cos(m\omega t) + b_m \sin(m\omega t))
\]  

(1.2.1),

where \( f(t) \) is a periodic signal with period of \( T \), \( a_0 \) is the DC signal of \( f(t) \), \( \omega = \frac{2\pi}{T} \) is the basic angular frequency, \( a_m \) and \( b_m \) are Fourier coefficients of harmonic orders run from \( m = 1, 2, 3, \ldots, M-1, M \), where \( M \) is the highest harmonic order of the studied signal[3]. In order to simplify calculation case studies, \( T = 2\pi \) will be the period of all studied signals.

2. RMS Calculations of Periodic Signals Constructed from Fourier Series

From equation (1.2.1), a studied periodic signal with the period of \( T = 2\pi \) can be expanded in terms of Fourier series as following equation (2.1):

\[
f(t) = a_0 + \sum_{m=1}^{M} (a_i \cos(mt) + b_i \sin(mt))
\]  

\[
= (a_0 + a_1 \cos t + a_2 \cos 2t + \ldots + a_M \cos(Mt)) + (b_1 \sin t + b_2 \sin 2t + \ldots + b_M \sin(Mt))
\]  

(2.1).

Therefore, \( f^2(t) \) can be expanded as equation (2.2):
Root mean square calculations on periodic signals

\[ f^2(t) = [(a_0)^2 + (a_1 \cos t)^2 + (a_2 \cos 2t)^2 + ... + (a_M \cos (Mt))^2 + (b_1 \sin t)^2 + (b_2 \sin 2t)^2 + ... + (b_M \sin (Mt))^2 + 2(a_0a_1 \cos t + ... + a_M b_M \cos (Mt) \sin (Mt)) + ... + b_{M-1} b_M \sin((M - 1)t) \sin (Mt))] \]

(2.2).

Then, the definite integral between T period (0 to \(2\pi\)) of \(f^2(t)\) will be done according to the following equation (2.3).

\[
\int_{0}^{2\pi} f^2(t) dt = \int_{0}^{2\pi} [(a_0)^2 + (a_1 \cos t)^2 + (a_2 \cos 2t)^2 + ... + (a_M \cos (Mt))^2 + (b_1 \sin t)^2 + (b_2 \sin 2t)^2 + ... + (b_M \sin (Mt))^2 + 2(a_0a_1 \cos t + ... + a_M b_M \cos (Mt) \sin (Mt)) + ... + b_{M-1} b_M \sin((M - 1)t) \sin (Mt))] dt
\]

(2.3).

By using orthogonal properties of sinusoidal functions, results of integral values between their own periods will be equal to zero or \(\pi\) as following.

\[
\int_{0}^{2\pi} \sin(\alpha t) dt = 0, \text{ where } \alpha \text{ is any integer and not zero.}
\]

\[
\int_{0}^{2\pi} \cos(\beta t) dt = 0, \text{ where } \beta \text{ is any integer and not zero.}
\]

\[
\int_{0}^{2\pi} \sin(\alpha t) \cos(\beta t) dt = 0, \text{ where } \alpha \text{ and } \beta \text{ are integers, not zero.}
\]

\[
\int_{0}^{2\pi} \sin(\alpha t) \sin(\beta t) dt = 0, \text{ where } \alpha \text{ and } \beta \text{ are integers, not zero and } \alpha \neq \beta.
\]

\[
\int_{0}^{2\pi} \cos(\alpha t) \cos(\beta t) dt = 0, \text{ where } \alpha \text{ and } \beta \text{ are integers, not zero and } \alpha \neq \beta.
\]

\[
\int_{0}^{2\pi} (\sin(\alpha t))^2 dt = \pi, \text{ where } \alpha \text{ is any integer and not zero.}
\]
\[ \int_{0}^{2\pi} (\cos(\beta t))^2 \, dt = \pi , \text{ where } \beta \text{ is any integer and not zero.} \]

From reasons above, some terms of equation (2.3) will be eliminated, but all square Fourier coefficients can still remain as equation (2.4).

\[ \int_{0}^{2\pi} f^2(t) \, dt = 2\pi (a_0)^2 + \pi (a_1)^2 + \pi (a_2)^2 + \ldots + \pi (a_M)^2 + \pi (b_1)^2 + \ldots + \pi (b_M)^2 \]

(2.4)

If the RMS value of the periodic function \( f(t) \) is required to know, all Fourier coefficients will be brought to calculate as equation (2.5).

\[ f_{\text{RMS}} = \sqrt{\frac{1}{2\pi} \int_{0}^{2\pi} f^2(t) \, dt} = \sqrt{\frac{(a_0)^2 + (a_1)^2 + (a_2)^2 + \ldots + (a_M)^2 + (b_1)^2 + \ldots + (b_M)^2}{2}} \]

(2.5)

From equation (2.5), it will indicate that RMS does not depend on harmonic frequencies of the periodic signal.

### 3. An Example of RMS Calculation for a Periodic Signal by Using its Fourier Coefficients

A created periodic signal \( v(t) \) has sinusoidal signals as the following equation

\[ v(t) = 7.50 - 5.00 \cos(20\omega t) + 0.500 \sin(100\omega t) + 3.00 \sin(50\omega t) + 2.50 \cos(174\omega t) \]

volt, the RMS value of this mixed signal will be calculated as

\[ v_{\text{RMS}} = \sqrt{\frac{(7.50)^2 + (5.00)^2 + (0.500)^2 + (3.00)^2 + (2.50)^2}{2}} \text{ volt} \]

\[ \approx 8.75 \text{ volt} \]

### 4. Conclusion

In order to calculate an RMS value of a periodic signal by using its Fourier coefficients, it will be more convenient than Calculus integration method, if
Fourier coefficients are known. Furthermore, differences of various frequencies of component harmonic signals will not effect in RMS values. It will mean that RMS values of periodic signals will not depend on their harmonic frequencies, but they will depend on amplitudes of their component harmonics signals which are called Fourier coefficients. The formula of RMS value calculations will be simplified as the equation (2.5).

References


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