Derivation of Friedmann’s Acceleration Equation

from Canonical Quantum Cosmology

Ch’ng Han Siong and Shahidan Radiman

School of Applied Physics, Faculty of Science and Technology
Universiti Kebangsaan Malaysia 43600 UKM Bangi
Selangor D.E. Malaysia
chng_hs@yahoo.com

Abstract

It is well known that in Standard Cosmology, the Friedmann equations are derived from Einstein’s field equations for a spatially homogeneous and isotropic universe. They are important in describing the evolution of the universe. The Friedmann equations consist of two independent equations which are usually called the first and second Friedmann equations. The first and second Friedmann equations contain, respectively, the first and second derivatives of scale factor with respect to time. Hence, the second Friedmann equation is also named as Friedmann’s acceleration equation. In a previous paper [16], we have derived the first Friedmann equation ( $k = 0$ ) from de Broglie-Bohm interpretation in canonical quantum cosmology. In this paper, we derive the second Friedmann equation (Friedmann’s acceleration equation) as well as presenting the derivation of the first Friedmann equation ( $k = 0$ ).

Keywords: Canonical quantum cosmology, de Broglie-Bohm interpretation, Friedmann equations, Wheeler-DeWitt equation
1 Introduction

The Einstein’s field equation in general theory of relativity was developed by Albert Einstein in 1915. Einstein’s field equation provides the relationship between the geometry of space-time and the source of energy-momentum. Einstein’s field equation is thus very important in describing the geometry of space-time of the universe and given as follows [7]:

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}, \]  

where \( R_{\mu\nu} \), \( R \), \( g_{\mu\nu} \) and \( T_{\mu\nu} \) are Ricci tensor, Ricci scalar, metric tensor and energy-momentum tensor respectively. In addition, \( G \) and \( c \) are Newton’s gravitational constant and the speed of light. In general relativity, the energy-momentum is taken to originate from a perfect fluid. A perfect fluid is one where all anti slipping forces are zero and the only interaction between fluid elements is pressure.

People tend to take the universe to be spatially homogeneous and isotropic (Cosmological principle), whose metric is given by

\[ ds^2 = -c^2 dt^2 + a^2 \left( \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right), \]

where \( a = a(t) \) and \( k \) are the scale factor and curvature parameter respectively. The curvature parameter, \( k \) can take one of the three values, \(-1\), \(0\) or \(+1\) to give the spatially open, flat or closed universe respectively. On the other hand, the scale factor, \( a \) here takes the units of length and the coordinate \( r \) is unitless. The metric (2) is called the Robertson-Walker metric. By plugging the Robertson-Walker metric into Einstein’s field equation, we obtain the following two equations [7]:

\[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{c^2 k}{a^2} = \frac{8\pi G \epsilon}{3c^2}, \]  

and

\[ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2} (\epsilon + 3p). \]

where the symbol dot denotes the differentiation with respect to time while \( \epsilon = \epsilon(a) \) and \( p = p(\epsilon) \) are the energy density and pressure of the perfect fluid in the universe. \( p = p(\epsilon) \) is also called the equation of state. In addition, equation (3) is usually called first Friedmann equation or just simply Friedmann equation, whereas equation (4) is called second Friedmann equation or acceleration equation. Besides this, the acceleration equation (4) does not depend on the curvature parameter. We would like to know whether the universe is undergoing accelerated expansion from equation (4) for a type of energy density which is dominating the universe.
In the study, we derive and obtain the Friedmann equations by applying the de Broglie-Bohm interpretation to the wave function of the universe. The first Friedmann equation \((k = 0)\) was derived and shown in our previous paper \([16]\). In this paper, we derive the second Friedmann equation (Friedmann’s acceleration equation) by using the same method used in the previous paper. In canonical quantum cosmology, the central equation is the Wheeler-DeWitt equation which is treated as the Schrödinger equation for the universe. Since quantum mechanics is believed to be the universal theory, in principle we should be able to recover the theory of Standard Cosmology, such as Friedmann equations from the solution of Wheeler-DeWitt equation. The solution of Wheeler-DeWitt equation is the wave function of the universe and contains the information of evolution of the universe. However in this paper, we apply the de Broglie-Bohm interpretation to the wave function of the universe. The de Broglie-Bohm interpretation will be able to provide definite and continuous values of a variable. Hence, this interpretation is suitable to be applied in canonical quantum cosmology where the universe evolves continuously with time.

In the next section, we present a spatially flat cosmological model and illustrate how the energy density varies with the scale factor. After that, we obtain the Wheeler-DeWitt equation and wave function of the universe. In section 3, we illustrate the de Broglie-Bohm interpretation of the wave function. In section 4, the derivation of Friedmann equations is presented. Finally, we end with our discussion and conclusions in section 5.

2 A spatially flat cosmological model

We begin by splitting a spatially flat, homogeneous and isotropic space-time into space and time variables and are thus led to the following spatially flat Robertson-Walker metric in Cartesian coordinate \([5]\):

\[
ds^2 = -N^2 c^2 dt^2 + a^2 \left( dx^2 + dy^2 + dz^2 \right),
\]

where \(N = N(t)\) is the lapse function and unit less. The scale factor, \(a = a(t)\) here takes the units of length and the coordinates \(x, y, z\) are unit less. Then, the Ricci scalar and determinant of this metric are computed and given as follows:

\[
R = \frac{6\dot{a}}{N^2 c^2 a} - \frac{6N\ddot{a}}{N^3 c^2 a} + \frac{6\dot{a}^2}{N^2 c^2 a^2},
\]

\[
g = -N^2 a^6.
\]

Next, we plug the Ricci scalar (6) and determinant of the metric into the following Einstein-Hilbert action \([8,11,12,17]\):
\[ S = \int \sqrt{-g} \left[ \frac{Re^4}{16\pi G} - \varepsilon \right] dx \, dy \, dz \, cdt, \] (8)

Consequently, the Einstein-Hilbert action becomes

\[ S = \int \left[ \frac{c^2}{16\pi G} \left( \frac{6\dot{a}^2}{N} - \frac{6\ddot{\dot{a}} a^2}{N^2} + \frac{6\dot{a}^2 a}{N} \right) - Na^3 \varepsilon \right] dx \, dy \, dz \, cdt. \] (9)

After integrating the first term by parts with respect to \( t \), equation (9) becomes

\[ S = V_o \int \left[ -\frac{3\dot{a}^2 c^4}{8\pi N G} - Na^3 \varepsilon \right] dt, \] (10)

where \( V_o = \int dx \, dy \, dz \). From now, we set \( V_o \) equal to one. The reason is that for any given value of \( t \), the geometry of our universe is the same everywhere. Hence, the \( V_o = \int dx \, dy \, dz \) is constant and can be set equal to one by integrating over an appropriate compact region of space [3]. The energy density \( \varepsilon = \varepsilon(a) \) in equation (10) can be written in the form as follows (for details, see [4,7]):

\[ \varepsilon = \frac{k_{(w)}}{a^{3(1+w)}}, \] (11)

where \( w \) and \( k_{(w)} \) are some constants. The values of \( w \) are obtained from the equation of state for a type of perfect fluid that dominates the universe at certain era. Different type of perfect fluid has their own value of \( w \). In general, the equation of state is given as follows:

\[ p = w \varepsilon, \] (12)

where \( p \) and \( \varepsilon \) are the pressure and energy density for a type of perfect fluid having a value \( w \) in the equation of state. We now rewrite the equation (11) as follows:

\[ \varepsilon = \frac{k_{(w)}}{a^n}. \] (13)

Hence, we have

\[ n = 3(1 + w). \] (14)
Here, we list out some types of perfect fluid and their corresponding values of $w$ and $n$ in the Table 1.

<table>
<thead>
<tr>
<th>Type of perfect fluid</th>
<th>$w$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radiation</td>
<td>$1/3$</td>
<td>4</td>
</tr>
<tr>
<td>Pressureless matter</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>Vacuum energy</td>
<td>$-1$</td>
<td>0</td>
</tr>
</tbody>
</table>

If we take the energy density $\varepsilon$ in Einstein-Hilbert action (10) to be the energy density of the type of perfect fluid that dominates the universe, we can substitute (13) into (10). Hence, Einstein-Hilbert action turns out to be

$$S = \int \left[ -\frac{3\dot{a}^2 c^3}{8\pi G N} - \frac{Na^3 c k_{(w)}}{a^n} \right] dt.$$ \hspace{1cm} (15)

The Lagrangian $L$ from Einstein-Hilbert action (15) will be as follows:

$$L = -\frac{3\dot{a}^2 c^3}{8\pi G N} - \frac{Na^3 c k_{(w)}}{a^n}.$$ \hspace{1cm} (16)

The momentum conjugate to $a$ is obtained as

$$P_a = \frac{\partial L}{\partial \dot{a}} = -\frac{3a\dot{c} c^3}{4\pi G N}.$$ \hspace{1cm} (17)

Therefore, the Hamiltonian is given by

$$H = P_a \dot{a} - L = N \left( -\frac{2\pi G P_a^2}{3ac^2} + \frac{a^3 c k_{(w)}}{a^n} \right).$$ \hspace{1cm} (18)

As from the Hamiltonian constraint (for details, see[9]), we obtain $H = 0$. Hence, we have the following equation:

$$-\frac{2\pi G P_a^2}{3ac^2} + \frac{a^3 c k_{(w)}}{a^n} = 0.$$ \hspace{1cm} (19)

So far, what we have worked out is purely classical. We next apply the canonical quantization procedure to equation (19). This is done by replacing $P_a$ by $-i\hbar \frac{d}{da}$, where $i$ is an complex number and $\hbar$ is the reduced Planck constant and imposing $\hat{H}\psi = 0$. Hence we have the following Wheeler-DeWitt equation:
\[
\left[ \frac{d^2}{da^2} + \frac{3k(\phi)a^{4-n}c^4}{2\pi G h^2} \right] \psi = 0,
\]

(20)

where \( \psi \) is the wave function of the universe. The wave function \( \psi \) of equation (20) is given as follows [15]:

\[
\psi = a^2 \left[ C_1 J_{\frac{1}{6-n}} \left( \frac{2}{6-n} \sqrt{\frac{3k(\phi)c^4}{2\pi G h^2} a^{3-n}} \right) + C_2 Y_{\frac{1}{6-n}} \left( \frac{2}{6-n} \sqrt{\frac{3k(\phi)c^4}{2\pi G h^2} a^{3-n}} \right) \right],
\]

(21)

where \( J_{\frac{1}{6-n}} \) and \( Y_{\frac{1}{6-n}} \) are Bessel functions of first and second kind respectively of order \( \frac{1}{6-n} \). Now, we would like to replace the Bessel functions with its asymptotic forms when the arguments of Bessel function are taken to have a large value. This can be the case when the scale factor is to have a sufficiently large value as for our current large scale universe. To be more specific, we require the following condition [10]:

\[
\frac{2}{6-n} \sqrt{\frac{3k(\phi)c^4}{2\pi G h^2} a^{3-n}} \gg 1.
\]

The asymptotic forms of Bessel functions of first and second kind for large argument are given as

\[
J_m(u) \approx \sqrt{\frac{2}{\pi u}} \cos \left( u - \frac{m\pi}{2} - \frac{\pi}{4} \right) \quad \text{and} \quad Y_m(u) \approx \sqrt{\frac{2}{\pi u}} \sin \left( u - \frac{m\pi}{2} - \frac{\pi}{4} \right)
\]

respectively. Both are convincing when the value of \( u \gg 1 \). Let us now determine the value of \( C_1 \) and \( C_2 \). People tend to choose the value of constants \( C_1 \) and \( C_2 \) which are to satisfy the assumed condition such as initial condition of the universe. However, we do not know the real initial condition for the universe. Therefore, this allows us to choose the values of \( C_1 \) and \( C_2 \) as long as it brings physical meaning to us. We are led to choose \( C_1 = A \) and \( C_2 = -iA \) where \( A \) is a positive constant. Consequently, choosing these values of constants and replacing Bessel functions in asymptotic forms, the wave function of the universe (21) becomes

\[
\psi = A \sqrt{\frac{a^n}{\pi a^2 \sqrt{3k(\phi)c^4 / 2\pi G h^2}}} \cdot \exp \left\{ i \left[ \frac{-2}{6-n} \sqrt{\frac{3k(\phi)c^4}{2\pi G h^2}} a^{3-n} + h\pi \left( \frac{1}{4} + \frac{1}{12-2n} \right) \right] \right\}.
\]

(22)
3 De Broglie-Bohm interpretation

Let us now make the following replacements:

\[ R = A \sqrt[2^n]{\frac{a^2}{\pi a^2} \frac{(6-n)}{\sqrt{3k_{(w)}^2c^4/2\pi\hbar^2}}} \]  

(23)

and

\[ S = -\frac{2}{6-n} \sqrt{\frac{3k_{(w)}^2c^4}{2\pi G}} a^{3-n} + \hbar \pi \left( \frac{1}{4} + \frac{1}{12 - 2n} \right) \]  

(24)

where \( R \) and \( S \) are functions of the scale factor \( a \). Then, the wave function (22) can be written as follows:

\[ \psi = R \cdot \exp \left( \frac{iS}{\hbar} \right) \]  

(25)

To illustrate the de Broglie-Bohm interpretation [1,2,6,13,14], we substitute the wave function (25) into (20). Taking the derivatives and separating into the real and imaginary parts, equation (20) reduces to the following equations:

\[ \frac{2\pi G}{3c^3 a} (\nabla S)^2 + a^3 c^2 k_{(w)}^2 + \frac{2\pi G\hbar^2}{3c^3 a} \nabla^2 R = 0, \]  

(26)

\[ \nabla \left( R^2 \nabla S \right) = 0. \]  

(27)

The notation \( \nabla \) denotes the differentiation with respect to the scale factor \( a \). Equation (26) is viewed as modified Hamilton-Jacobi equation, while equation (27) is known as the continuity equation for probability. Equation (26) differs from the usual classical Hamilton-Jacobi equation only by the addition of an extra term which is called the quantum potential

\[ Q = \frac{2\pi G\hbar^2}{3c^3 a} \frac{\nabla^2 R}{R}. \]  

(28)

Quantum potential is responsible for the quantum effects. However, it is expected to have a very small value and effect here because the scale factor \( a \) we considered is taken to be sufficiently large. The modified Hamilton-Jacobi equation (26) is an equation describing dynamic of a cosmological field. To make equation (26) analogous to an equation describing the motion of a particle, we make the following replacements:

\[ m' = -\frac{3c^3 a}{4\pi G}, \]  

(29)

\[ V = a^3 c^2 k_{(w)}^2 \frac{a_n}{a^n}. \]  

(30)
After replacing (29) and (30) into (26), we obtain
\[
\left(\nabla S\right)^2 + V - \frac{h^2}{2m'} \nabla^2 R = 0. \tag{31}
\]
This equation is very similar to the modified Hamilton-Jacobi equation for a particle with Hamiltonian \( H = 0 \) and classical potential \( V \), except that the mass of a particle \( m \) is now replaced by \( m' \). In the context of particle, the momentum of a particle is equal to the differentiation of phase function with respect to position. Similarly, the momentum \( P_a \) (17) in the context of cosmological field is given as follows:
\[ P_a = \nabla S. \tag{32} \]
Equation (32) is called the guidance equation. We proceed and make a further step by differentiating both sides of equation (32) with respect to time \( t \):
\[
\frac{d}{dt} P_a = \frac{d}{dt} (\nabla S). \tag{33}
\]
Equation (32) and equation (33) are used to derive the first and second Friedmann equations respectively.

4 Derivation of Friedmann equations \((k = 0)\)

First of all, we compute the physical quantities that are needed to derive the Friedmann equations. The results are given as follows:
\[
\nabla S = -\sqrt{\frac{3k_{(w)}c^4}{2\pi G}} a^{\frac{n}{2}} \tag{34},
\]
\[
\frac{d}{dt} (\nabla S) = -\sqrt{\frac{3k_{(w)}c^4}{2\pi G}} \left[ \frac{2\dot{a}a^{\frac{n}{2}+1} - \frac{n}{2}(n/2)\dot{a}a^{\frac{n}{2}+1}}{a^n} \right], \tag{35}
\]
\[
\frac{d}{dt} P_a = -\frac{3\dot{a}c^3}{4\pi G} - \frac{3a\ddot{a}c^3}{4\pi G}. \tag{36}
\]
Equation (36) is obtained by applying a gauge choice of \( N = 1 \) in equation (17). Substituting (17) and (34) into (32), we obtain an equation as follows [16]:
\[
\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2} \left( \frac{k_{(w)}}{a^n} \right), \tag{37}
\]
where \( \frac{k_{(w)}}{a^n} = \varepsilon \). Equation (37) is the spatially flat first Friedmann equation in
Derivation of Friedmann’s acceleration equation

which the value of \( n \) is dependent on the type of energy density that dominates the universe. To obtain the second Friedmann equation, we substitute (35) and (36) into (33). After doing some arrangements and taking a gauge choice of \( N = 1 \), we obtain the following equation:

\[
\left( \frac{\dot{a}}{a} \right) = \frac{2ck_{[w]}}{a^n} \left( \frac{8\pi G}{3c^3} - \frac{2\pi G n}{3c^3} \right) - \left( \frac{\dot{a}}{a} \right)^2 \tag{38}
\]

From (37) and (14), we now replace the terms \( \left( \frac{\dot{a}}{a} \right)^2 \) and \( n \) by \( \frac{8\pi G k_{[w]}}{3c^2 a^n} \) and \( 3(1+w) \) respectively. The equation (38) thus reduces to the following equation:

\[
\left( \frac{\dot{a}}{a} \right) = \frac{4\pi G}{3c^2} \left( \frac{k_{[w]}}{a^n} \right) - \frac{4\pi G}{c^2} \left( \frac{w k_{[w]}}{a^n} \right) \tag{39}
\]

We also have that \( \frac{k_{[w]}}{a^n} = \varepsilon \) and \( \left( \frac{w k_{[w]}}{a^n} \right) = p \) from (13) and (12). Thus equation (39) can be rewritten as follows:

\[
\left( \frac{\dot{a}}{a} \right) = \frac{4\pi G}{3c^2} \left( \varepsilon + 3p \right) \tag{40}
\]

Equation (40) is indeed the second Friedmann equation (Friedmann’s acceleration equation).

5 Discussion and conclusions

The second Friedmann equation (Friedmann’s acceleration equation) as well as the first Friedmann equation (\( k = 0 \)) are derived by applying the de Broglie-Bohm interpretation to the wave function of the universe. The de Broglie-Bohm interpretation is able to provide the continuously varying values of a variable. Our universe is indeed evolving continuously with time. The Friedmann equations are shown to result from taking the asymptotic form of Bessel functions, or in other words, we need the following condition: \( \frac{2}{6-n} \sqrt{\frac{3k_{[w]} e^4}{2\pi G h^2 a^{\frac{3-n}{2}}}} \gg 1 \) in order to obtain the Friedmann equations. Hence, Friedmann equations are the equations of evolution of the universe based on the specific condition. To obtain the general form of equations of evolution, we have to consider the whole Bessel functions.
and not only the asymptotic form of Bessel functions. In general, we can differentiate both sides of equation (32) with respect to time \( t \) for any number of times. In the situation: \( a \to 0 \), such as in the early time of the universe, the condition will not be satisfied. Hence we are to conclude that in quantum cosmology, the Friedmann equations as well as Einstein’s field equations are only applicable for the large scale universe.

**ACKNOWLEDGEMENT**

One of the authors, Ch’ng would like to thank the Ministry of Higher Education (MOHE), Malaysia for providing MY PhD scholarship.

**References**


Received: March 25, 2013