**Note on q-Deformed Pseudo-Differential Operators**

**and its Supersymetric Extension**

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**Abstract**

In this paper we present some interesting results on q-deformed pseudo differential operators and its algebraic structures. We present also the super symmetric extension of such algebra.

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**1. The algebra of pseudo-differential operators (PDO)**

Noting $\mathcal{A}_s^{(p,q)}$ the algebra of pseudo-differential operators PDO with three quantum numbers $s, p$ and $q$ describing respectively the conformal weight, the lower and higher degrees of type[1,2,3]

$$\mathcal{L} = \sum_{i=p}^{q} u_{s,i} \partial^i.$$  \hspace{1cm} (1)

The symbol $\partial^i$ stands for the differentiation with respect to the coordinates $x$. The ring of analytic functions $\mathcal{A}^{(0,0)} = \mathcal{R}$ of arbitrary conformal weight and vanishing lowest and highest degrees $(0,0)$. The huge Lie algebra of PDO of arbitrary conformal weight and degrees is:

$$\mathcal{A} = \bigoplus_{s \in \mathbb{Z}} \bigoplus_{p \leq q} \mathcal{A}_s^{(p,q)}.$$
The Leibnitz rules:

\[ \partial^k u_s(z) = \sum_{i=0}^{k} \binom{k}{i} u_s^{(i)}(z) \partial^{k-i} \]

(2)

for local differential operators \( k \geq 0 \) and

\[ \partial^{-k} u_s(z) = \sum_{i=0}^{\infty} (-1)^i \binom{k+l-1}{i} u_s^{(i)}(z) \partial^{-k+i} \]

(3)

for the non-local differential operators \( i \geq 0 \) such that

\[ \partial^k \partial^{-k} u_s(z) = u_s(z) \]

(4)

3. q-deformed pseudo-differential operators

3.1. The q-derivation:

Following [2,3,4], the q-derivation is defined as follows

\[ \partial_q(f) = \frac{f(qx) - f(x)}{(q-1)x} \]

(5)

with \( q \neq 1 \). As an operator, \( \partial_q \) acts as follows

\[ \partial_q \circ f = (\partial_q f) + \eta_q(f) \partial_q, \]

(6)

The q-shift operator \( \eta_q \) is defined as

\[ \eta_q(f(x)) = f(qx). \]

(7)

Note that \( \eta \) is a linear function in term of \( f \). The non-local differential operator \( \partial_q^{-1} \) acts as follows

\[ \partial_q^{-1} \circ f = \eta_q^{-1} \partial_q^{-1} + \sum_{k \geq 1} (-1)^k q^{-k(k+1)/2} (\eta_q^{-k-1}(\partial_q f)) \partial_q^{-k-1}. \]

(8)

We simply derive the previous \( q \)-deformed Leibnitz rule for non local operators by using the following relation

\[ (\partial_q^{-1} \circ \partial_q) f = (\partial_q \circ \partial_q^{-1}) f = f. \]

(9)

Note by the way that \( \eta_q \) does not commute with \( \partial_q \),

\[ \partial_q^m(\eta_q^k(f)) = q^m \eta_q^k(\partial_q^m f), \quad k,m \in \mathbb{Z}. \]

(10)

In general we have

\[ \partial_q^m \circ f = \sum_{k \geq 0} \binom{n}{k} \eta_q^{n-k}(\partial_q^k f) \partial_q^{n-k}, \]

(11)

for all \( n \). In the last equation, the q-binomials take the form
Note on \( q \)-deformed pseudo-differential operators

\[
\binom{n}{k}_q = \frac{(n)_q (n-1)_q \ldots (n-k+1)_q}{(1)_q (2)_q \ldots (k)_q}, \quad \binom{n}{0}_q = 1,
\]

and the \( q \)-numbers are given by

\[
(n)_q = \frac{q^n - 1}{q - 1},
\]

### 3.2 The algebra of \( q \)-PDO

This is the algebra of \( q \)-differential operators of arbitrary conformal weight and degrees[4],

\[
\mathcal{A} [\partial_q, u_i] = \bigoplus_{m \in \mathbb{Z}} \bigoplus_{n \leq m} \mathcal{A}^{(m,n)}_s [\partial_q, u_i]
\]

with

\[
\mathcal{A}^{(m,n)}_s [\partial_q, u_i] = \left\{ \sum_{i=0}^{n} u_{s-i} \partial_q^i \mid u_{s-i} \in \mathcal{A}^{(0,0)} \right\}
\]

The Classical Limit:

Once setting \( q \to 1 \), we recover the standard formulas, we have[4]

\[
\partial_q \circ f \to \partial \circ f, \quad \text{with} \quad \eta(f) = f
\]

### 4. Super symmetric extension of \( q \)-PDO

We define the ring \( \Sigma[D_q] \) of differential super symmetric \( q \)-differential operators as polynomials in \( D_q = \partial_q + \theta \partial_q \)

\[
\Sigma[D_q] = \bigoplus_{n \in \mathbb{Z}} \bigoplus_{m \leq n} \Sigma^{(m,n)}_s[D_q] \quad m, n \in \mathbb{Z}
\]

where \( \Sigma^{(m,n)}_s[D_q] \) is the space of super symmetric \( q \)-differential operators type

\[
\Sigma^{(m,n)}_s[D_q] \quad \text{behaves as a} \quad (1 + q - p) \quad \text{dimensional superspace}. \quad \text{Note also that the ring } \mathcal{R} \text{ of all graded superfields can be decomposed as}
\]

\[
\mathcal{R} = \Sigma^{(0,0)}[D_q] \bigoplus \bigoplus_{k \in \mathbb{Z}} \Sigma^{(0,0)}_k[D_q]
\]

where \( \Sigma^{(0,0)}_k \) is the set of superfield \( u_k(\hat{z}) \) indexed by half integer conformal spin \( k/2, k \in \mathbb{Z} \). In the classical limit, we find the results obtained in the references[5,6,7]. More information on this super symmetric algebra will be presented in our next paper.
5. Concluding remarks

This work aims principally to present a some useful formulas of q-deformed pseudo differential operators and its super symmetric extension.

We note that the q-deformed Lie bracket $[\ldots]$ acts as

$$[\ldots]: \mathcal{A}_s^{(m,n)} \times \mathcal{A}_s^{(m,n)} \rightarrow \mathcal{A}_s^{(m,2n)},$$

(20)

and if we imposing the closure, one gets strong constraints on the spin $s$ and the degrees parameters $(m,n)$ namely $s=0$ and $m \leq n \leq 0$.

Further note that the space of PDO admit a Lie algebra’s structure with respect to the bracket for the vector fields of conformal weight 0 and for the scalar differential pseudo operators of higher degree -1.

The conformal factorization: $(f_0 \partial_q + f_1)^o (g_0 \partial_q + g_1) = h_0 \partial_q^2 + h_1 \partial_q + h_2$ with $h_0$, $h_1$ and $h_2$ are the functions in $f_0$, $f_1$, $g_0$ and $g_1$ are called Miura transformation and it is invariant under the q-PDO transformations.

References


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