P-branes, Grand Unified Theory and Holography

E.B. Torbrand Dhrif

Allevagen 41, 18639 Vallentuna, Stockholm, Sweden
eric.torbrand@gmail.com

Abstract. In this paper we look at the possibilities for P-brane and Grand Unified Theory holographics. We assume U(N) Chan-Paton color degrees of freedom on physical space-time, this to make the connection to gauge theory. The results are valid when there exists a well defined parabolic problem on some foliated manifolds such as AdS spaces and tubes \( X^D \times [0, \infty) \), \( D \) either the background dimension or the effective space-time dimension and \([0, \infty)\) another time space which we use to obtain the elliptic(hyperbolic after deformation) problem from a parabolic problem. As an astray, we compute a first guess for P-brane correlators, ALE S-matrices and partition functions.

1. SOME DEFINITIONS

We are working with a non-linear sigma model(See Gell-Mann et al [5]) and have in the simplest case the equation for flat target space metric \( G_{mn} = \delta_{mn} \);

\[
\Box^{(p+1)}X^m = \frac{1}{\sqrt{G}} \partial_\mu (G^{\mu\nu} \sqrt{G} \partial_\nu) X^m = 0
\]

for our P-brane. \( \mu, \nu \) are indices who live on our brane. \( m \) is a index on the target space. So \( X^m \) lives on the target space and \( X^\mu \) lives on the P-brane. This follows from the equation of motion for the Lagrangean

\[
S = \int \Sigma \, d^{p+1}X^\mu \sqrt{|G_{\mu\nu}|} \partial_\mu X^m \partial_\nu X^n G(X^m[X^\mu])_{mn} G^{\mu\nu}[X^\mu]
\]
For non-constant $G_{mn}(X^i)$ the implied equation of motion for this lagrangean is non-linear. We generalize further the above definitions.

**Definition 1.1.**

$$S = \int_{\Sigma} d^{p+1}X^\mu \sqrt{|G_{\mu\nu}|} \partial_\mu X^{m,\alpha} \partial_\nu X^{n,\beta} G(X^{m,\alpha}[X^\mu])_{mn} G^{\mu\nu}[X^\mu] K_{\alpha\beta}$$

Furthermore, we require the fields $X^m$ to take coefficients in the Lie algebra $\mathcal{U}(N)$ (Suitable by the Coleman-Mandula Theorem). $\alpha$ and $\beta$ are Lie algebra representation module indices for, say, the Killing metric $K_{\alpha\beta}$ in the adjoint representation, or an ordinary representation module. We also have Lorentz invariance and assume this object to live in a module of a irreducible representation of the Poincare group. The latter is required to make sense of multi-brane systems, just as multiparticle systems in elementary particle physics. Slightly more generally, we have;

$$S_{\text{Bosonic}} = \int_{\Sigma} d^{p+1}X^\mu \sqrt{|G_{\mu\nu}|} (\nabla_\mu X)^{m,\alpha} (\nabla_\nu X)^{n,\beta} G(X^{m,\alpha}[X^\mu])_{mn} G^{\mu\nu}[X^\mu] K_{\alpha\beta}$$

or equivalently

$$S_{\text{Bosonic}} = tr_\mathfrak{g} \int_{\Sigma} d^{p+1}X^\mu \sqrt{|G_{\mu\nu}|} (\nabla_\mu X)^{m} (\nabla_\nu X)^{n} G(X^{m,\alpha}[X^\mu])_{mn} G^{\mu\nu}[X^\mu]$$

with $\nabla_\mu = \partial_\mu + \omega_\mu + A_\mu$. $\omega$ is the Riemann connection and $A$ is the gauge connection. The $\mathfrak{g}$ is the appropriate semi-simple Lie algebra, such as of $U(N)$. Of course this is for the bosonic sector. Some people would like to set $X^m \mapsto E^m$ the vielbein and also $\nabla \mapsto D$ the Dirac operator in the vielbein representation $\Gamma^a = E^a + E^{a*}$, however we do not address this. The fermionic sector follows a standard Dirac action;

$$S_{\text{fermionic}} = \int_{\Sigma} \sqrt{|G_{\mu\nu}|} d^{p+1}X^\mu \bar{\psi} iD\psi$$

where $D$ now is the gauged spinor connection working on a spinor module.

**Definition 1.2.** A AdS space is the space described by the metric

$$ds^2 = \frac{1}{y^2}(-dt^2 - dy^2 + dx_1^2 + \cdots + dx_n^2)$$

where the coordinates are real belonging to the patch $\mathbb{R}^D$ which is foliated by constant $y$ slices into the metric

$$ds^2 = -dt^2 + dx_1^2 + \cdots + dx_n^2$$

also here with coordinates in $\mathbb{R}^{D-1}$. 
2. The usual situation, but for P-branes

We note the quite trivial calculation

\[ Z_{\text{SUGRA},D=10} = \det(D/2)_{X^{10}=\text{AdS}^5 \times X^5} = \det(D/2)_{\text{AdS}^5} \det(D)_{X^5}^2 \]

\[ Z_{\text{General},D=10} = Z(FP_{\pm})_{X^{10}=\text{AdS}^5 \times X^5} = Z_{\text{SUGRA},D=10} = Z(-D/2)_{\text{AdS}^5} Z(ibD)_{X^5}^2 \]

in the appropriate sectors. Here \( F \) is the SUSY Hamiltonian that determines the physical system and \( P_{\pm} \) is a GSO-type projection defined in the appendix. With our conventions \( P_+ \) projects on bosonic states and \( P_- \) projects on fermionic states. Note one field on the left and two fields on the right.

We notice

\[ Z_{\text{SUGRA},D=10} = \det(D/2)_{\text{AdS}^5} \det(D)_{X^5}^2 = Z_{\text{Einstein}}^{\text{AdS}^5} \times |Z_{\text{Gauge}}|_{X^5}^2 \]

We need to show

\[ \det(D/2)_{\text{AdS}^5} = Z_{\text{Einstein}}^{\text{AdS}^5} \]

Look at the action functionals or Lagrangeans

\[ S_{\text{Einstein}} = \int R \sqrt{G} d^Dx, \]
\[ S_{\text{SUGRA}} = \int \theta^* D^2 \theta = \int \theta^* \Box \theta + R \sqrt{G} d^Dx \]

since we normalize by cluster decomposition or by letting the field \( \theta \) (a vielbein) go to a constant at infinity on each leaf, we get

\[ S_{\text{Einstein}} = \int R \sqrt{G} d^Dx, \]
\[ S_{\text{SUGRA}} \mapsto S_{\text{Einstein}} \]

This is somewhat like exposing a suitable boundary conditions, since asymptotically for ALE spaces the correlators are described by

\[ \langle \cdots \rangle \sim \Pi_i e^{-k_i^2 \Delta t} \]

which you normalize by (the standard picture comes from integrating this with respect to \( \Delta t = s \)). This then be used as an approximate picture at flat infinity. However much more generally you can also see this as the Schrodinger interaction picture for the Schrodinger operator or Hamiltonian \( H_a \)
\[ H = D^2 = \Box + \frac{\text{Ricci}}{4\pi G} + F \]

\[ H_{\text{int}} = \frac{\text{Ricci}}{4\times 4\pi G} = \frac{\text{Ricci}}{16\pi G} + \cdots \]

The Hamiltonians are more generally given by

\[ H_{\text{boson}} = H_{\text{even}} = -D^2 \implies H_{\text{even,int}} = \frac{\text{Ricci}}{16\pi G} + \cdots \]

\[ H_{\text{fermion}} = H_{\text{odd}} = iD \]

Also

\[ H_{\text{even,int}} = -L_{\text{even,int}} = \frac{-\text{Ricci}}{16\pi G} + \cdots \]

You could insert mass terms in the above. Since any physical system would be described by the direct sum of the above two operators it follows that the above factorization would go through for any system comprised by fermions and bosons, this, of course, in the so called exact setting, or 1-loop. Indeed even grand unified theory adheres to this.

**Remark 2.1.** We can show the above in another manner. Set \( X^D = X \times Y \) a factorization of the space-time \( X^D \). Then, since bosons and spin-averaged fermions are both described by the relevant terms of \( D^2 \), which decomposes as

\[ D^2_{X \times Y} = D^2_X + D^2_Y \]

on the Cartesian product \( X \times Y \), we have;

\[ Z(FP_\pm) = Tr(q^D \tilde{X}_{\times Y}) = Tr(q^P \tilde{X}) \times Tr(q^P \tilde{Y}) \]

\[ = Z_Y^{\text{Gravity}} Z_X^{\text{Gauge,Spinaveraged}} = Z_Y^{\text{Gravity}} |Z_X^{\text{Gauge}}|^2 \]

We recognize this;

\[ D^2_{Y|\text{Boson}} = \Box + \frac{R}{4} = \Box + \frac{\text{Ricci}}{16\pi G} \]

\[ D^2_{X|\text{Fermion}} = \Box + F \]

so we have the correct Hamiltonians in our partition function pieces after restrictions to the appropriate sectors. Notice that the bosonic(graviton) term has support in the bulk, \( Y \), and the fermionic term on the remaining degrees of freedom, that is, \( X \).
3. The Case for P-branes

A calculation gives, for the \( L^2 \) setting or compact scenario an amplitude for our P-branes

\[
Z[k] = \int DX^\mu e^{-S_{P-brane}} e^{-ik_m X^m} = \Pi_{i<j} e^{2\alpha'G(x_i,x_j)[\Omega]_{k_i k_j} + 2\pi i <\omega_M | \Omega | \omega_M >}
\]

as a first naive ansatz. \( \Omega \) here is the matrix given by

\[
\Omega = \phi_i \wedge *\phi_j, \phi \in H^l(X, \mathbb{C})
\]

for some \( l \). * is a Hodge star here. Notice by Poincare duality we have \( \phi \) is dual to \( *\phi \) (by the usual Hilbert space anti-isomorphism). Obviously, a matrix element for the S-matrix is given by

\[
<X_1 \cdots X_n > = \Delta X_1 \cdots \Delta X_n = \partial_{k_1} \cdots \partial_{k_n} \ln Z[k]_{|k=0}
\]

We now wish to investigate the matrix \( \Omega \), but this is a difficult task, as we do know from, e.g. the theory of 4-manifolds, that there are exotic structures, see the book by Donaldson, Kronheimer[4]. For membranes it is conjectured that the situation is simpler, actually for the compact case we have the same topological functors in, say, the deRham cohomology subset as in compact Riemann surface theory. Monodromy, that is homomorphism classes of the fundamental groups \( \pi_i(\Sigma) \), might be harder.

4. The Metric Beta Function of a Brane

The metric beta function is computable to

\[
\beta_{\mu\nu}^G = e^{\alpha' [Ric_{\mu\nu}] \Delta t} - 1_{\mu\nu}
\]

5. Behaviour of Branes and Holography

The simplest partition function without momentum or current terms, is given by

\[
Z = \Pi_{n \in \mathbb{Z}^+} (1 - q^n)^{D_{eff}}
\]

\[
D_{eff} = D - (p + 1) = 10 - (p + 1) = 9 - p
\]

Note the case \( p = 1 \) to be well known from the torus in String Theory. Since we now realize that this brane lives in effective dimension \( D_{eff} \), we get a lot of dualities by following the reasoning in section 2;
\[ Z_{\text{General}} := Z_{P\text{-brane}} = Z_{X^{D_{\text{eff}}}} = Z_{AdS^N \times X^{D_{\text{eff}}-N}} = Z_{\text{Einstein}} |Z_{Gauge}^{D_{\text{eff}}-N}|^2; \]

\[ 0 \leq N \leq D_{\text{eff}} \]

Please note; one dynamical field on the left, two dynamical fields on the right. Here we have

\[ Z_{Gauge}^{D_{\text{eff}}-N} \]

is gauge theory and lives on some manifold \( X^{D_{\text{eff}}-N} \) of dimension \( D_{\text{eff}} - N \).

Note this;

\[ Z_{\text{even}} = Z_{\text{Einstein}} \]
\[ Z_{\text{odd}} = Z_{\text{Gauge}} \]

This is even true in grand unified theory, since the superdeterminant manipulations go through for any theory and we have that the Einstein action comes up in the even sector as \( H_{\text{int,even}} \) above. We thus have a second set of dualities given by

\[ Z_{\text{General}} := Z_{P\text{-brane}} = Z_{AdS^N} |Z_{Gauge}^{D_{\text{eff}}-N}|^2; \]

\[ 0 \leq N \leq D = 10 \]

where there is one field dynamical on the left, either a boson or a fermion, and two dynamical fields on the right again, both one fermion (that lives on space-time \( X \)) and one boson (that lives on \( AdS^N \));

\[ Z_{\text{even}} = Z_{\text{Einstein}} \]
\[ Z_{\text{odd}} = Z_{\text{Gauge}} \]

For the sake of clearness we make it obvious that we now set

\[ \partial_\mu := \partial_\mu + A_\mu + \omega_\mu = \nabla_\mu \]

on both sides. Then \( G(x_i, x_j)[\Omega] \) in our P-brane amplitudes also depends on the background connections \( \omega \) and \( A \). You could insert a mass term too.

Currents for the above identities are obtained by functional differentiation with respect to the appropriate background connection, either the Riemann and spin connection \( \omega \) or gauge connection or potential \( A \). You could assume that we have a flat infinity for these models, and that the traces involved are well-defined.
Remark 5.1. I think we can generalize to space-times not only such that
\[ X^D = X^M \times X^N \]
but also fibrations \( X^D \to X^M \) with fibre \( X^N \), such as Thom spaces. We must ask; How does twisting of such a bundle affect the results?

Remark 5.2. The case \( Y = X^N = AdS^N \) is of course standard in the literature.

6. Appendix

Write the formula
\[ F = H_{\text{even}} \oplus H_{\text{odd}} \]
Then we define the generalized Schrödinger equation we used above as;
\[ F\Psi = i\partial_t \Psi \]
It’s solutions are given by
\[ \Psi(x, t) = e^{-iFt}\Psi(0, x) = \int d^D x' K(x, x'; t)\Psi(0, x') \]
The propagators are given, after continuation to Euclidean space, by
\[ G(x, x') = \frac{1}{F + i\epsilon} = \int_0^{\infty} ds e^{-Fs} = \int_0^{\infty} ds K(x, x'; s) \]
We have, for non-massive fields,
\[ FP_+ = H_{\text{even}} = H_{\text{bosonic}} = -\nabla^2 + \hat{A}^2 = -\square - \frac{\text{Ricci}}{4 \times 4\pi G} \]
\[ FP_- = H_{\text{odd}} = H_{\text{fermionic}} = i\partial_\mu = i(\partial_\mu + A_\mu)\Gamma^\mu \]
For massive fields (SUSY implies the same mass for the boson and fermion)
\[ FP_+ = H_{\text{even}} = H_{\text{bosonic}} = -\nabla^2 + \hat{A}^2 = -\square - m^2 - \frac{\text{Ricci}}{4 \times 4\pi G} \]
\[ FP_- = H_{\text{odd}} = H_{\text{fermionic}} = i\partial_\mu = i(\partial_\mu + A_\mu)\Gamma^\mu + m \]
Where
\[ P_\pm = \frac{1 \pm (-1)^F}{2} \]
such that
\[ 1 = P_+ + P_- \]
is the GSO projection on either bosonic states, $P_+$, or fermionic states, $P_-$. The partition functions for such systems are given by the standard literature by expressions involving determinants to 1-loop. We also state the following;

Remark 6.1. The scalar part of the bosonic Hamiltonian $D^2$ is

$$D^2 = \Box + \frac{\text{Ricci}}{4} + \cdots$$

so most reasoning goes by fiat to other boson sectors, but not to gauge bosons in the bulk. For gauge bosons, such as gluons, in the bulk you will have to add a background field strength term $F$ in the vielbein representation of the Clifford algebra. This is, however, not the standard Maldacena conjecture.

Remark 6.2. Chiral versions of these Holographic principles are obtained by the usual insertion of

$$\frac{1 \pm \Gamma^{D+1}}{2}$$

the usual chirality operator before or after the relevant operator in the trace or determinant for an 'exact' or 1-loop amplitude.

References


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