A Brute Force Calculation
of the General QFT Diagram

E.B. Torbrand Dhirf

Allevagen 41, 18639 Vallentuna, Stockholm, Sweden
eric.torbrand@gmail.com

Abstract. In this paper we compute the general QFT diagram in a class of theories which we define, namely the class of t’Hooft-Veltman renormalizable theories. The calculation uses complex analysis to calculate the diagram in question in a non-iterative way.

1. Introduction

Many people have worked on the area of renormalization and evaluation of Feynman diagrams. Most notably t’Hooft and Veltman were awarded the Nobel prize for their discovery of dimensional regularization but the scheme had precedents in terms of naive cut-off regularization, essentially introducing a ultraviolet cut-off, and Pauli regularization where one introduced a fictious mass. This is so classical work that references are hardly needed. However, some work has also been done lately that might be of interest, see Dillig(2004) for relationships between two standard regularization schemes(dimensional and cut-off) and Ratsimbarinson(2005) for introduction of Hopf algebras into a more systematic way of renormalization. Another issue is also the very computation of the integrals involved, which is precisely what we deal with in this paper.

In this paper we shall evaluate the general Q.F.T. diagram for a large class of theories in a non-iterative way invented by the author during the winter 1998. The main tool for this will be advanced complex analysis, used to evaluate Feynman diagrams non-iteratively for special cases when the diagram can be expressed a sum of higher dimensional Euler integrals. Although we
do not enter into this here, this can also be used to provide different and more consistent renormalization schemes at high loop order. The expression of the Feynman integrals directly as a function of complex dimension allows us to directly implement the economical method of t’Hooft-Veltman dimensional regularization. We begin with a definition

**Definition 1.1.** We define the multi-index $\mathbf{\gamma}$ as

$$\mathbf{\gamma} = (1, 1, 1, \cdots, 1)$$

**Definition 1.2.** A theory is called continuable if

1. It has propagators of type $\frac{p(k_1)}{k_1^2 + m^2}$, $p(k_1)$ a polynomial in $k_1 \in \mathbb{R}^n$ and vertices of type $g(k_1, \cdots, k_d)$ a matrix-valued polynomial in attached momenta $k_\sigma \in \mathbb{R}^n$.
2. All expressions of the theory can be continued to complex space-time dimension.

**Example 1.1.** QED, tensor boson gravity, and phi-fourth are continuable. Any non handed part of the standard model is also continuable. GWS is however not continuable since it involves expressions with $\Gamma_5$, something that cannot be extended in a straightforward way to complex space-time dimension.

We shall now use this definition to prove that in continuable theories there is a renormalized formula for the arbitrary diagram. We first do a little detour to provide the necessary tool;

1.0.1. A Generalized Euler Integral of Higher Complex Dimensions. We will use the observation that the integrals of quantum field theory tend to be higher dimensional analogues of Euler integrals. In particular that means that some families of (complex) deformations of these definite integrals are expressible in terms of gamma functions. We define the function $^1 B_2 : \mathbb{C}^d \to \mathbb{C}$

$$B_2(\zeta, d, z, \Delta) \equiv \int_{\mathbb{S}^d} \frac{r^\zeta}{(r^2 + \Delta^2)^{\frac{d}{2}}} d^d x$$

where $r = ||(x_1, x_2, \cdots, x_d)||$ is the euclidean norm function in $\mathbb{E}^d, d \in \mathbb{C}$, analytically continued to $\mathbb{C}^d$. In the region $\text{Re}[\zeta + d] > 0, \text{Re}[z - \zeta - d] > 0, \text{Re}[d] > 0$ this integral is a holomorphism of several complex variables and can be expressed in terms of gamma functions, and in $\mathbb{C}^d$ it is a meromorphism expressible in ditto functions. If we let $\mu(S^{d-1}) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(d/2)}$ denote the formal (Lebesgue)mass of the $d$-dimensional sphere we have

$^1$Called beta-two.
\[ B_2(\zeta, d, z, \Delta) = \mu(S^{d-1}) B_{\Delta} \left( \frac{\zeta + d}{2}, \frac{z - \zeta - d}{2} \right) = \frac{2\pi^2}{\Gamma(d/2)} \frac{B(\zeta + d, z - \zeta - d)}{2\Delta} \]

with \( B \) the ordinary beta function. In particular this implies by symmetry on the regions we are interested in for

\[ x = (-ix_0, x_1, x_2, \cdots, x_d) \]

\[ \int_{E^d} \frac{x_{\mu}x_{\nu}}{(r^2 + \Delta^2)^{d/2}} = -\frac{g_{\mu\nu}}{d} B_2(2, d, z, \Delta) \]

\[ \int_{E^d} \frac{x_{\mu}x_{\nu}x_{\alpha}x_{\rho}}{(r^2 + \Delta^2)^{d/2}} = +\frac{g_{\mu\nu}g_{\alpha\rho} + g_{\mu\alpha}g_{\nu\rho} + g_{\mu\rho}g_{\nu\alpha}}{d(d+2)} B_2(4, d, z, \Delta) \]

or in full generality for non-vanishing cases of the last type with \( \alpha \) a multiindex, \( \chi_0 \mod 2 \) a characteristic function of the even integers and \( \{\} \) the un-normalized symmetrizer of \(|\alpha|\) symbols,

\[ \int_{E^d} \frac{x_\alpha}{(r^2 + \Delta^2)^{d/2}} = \chi_0 \mod 2(|\alpha|) \frac{(-1)^{|\alpha|}}{2^{\alpha_1-\alpha_\alpha_\alpha}(d-\alpha_1-\alpha_\alpha_1)!} B_2(|\alpha|, d, z, \Delta) \equiv B_2, \alpha(|\alpha|, d, z, \Delta), \ \alpha \in \mathbb{N}^d. \]

making integrals at arbitrary order of perturbation theory easy to evaluate by t’Hooft-Veltman dimensional regularization.

1.0.2. The Main Theorem. We now recollect the pieces of the theorem and give a proof.

**Theorem 1.1.** Assume that a theory is continuable, then the general diagram is expressible through one and the same closed formula.

We postpone the proof in order to give an of example that will illustrate the vital parts of the proof and formula.

We shall evaluate the above diagram using a slightly unusual technique, instead of using iterated integration we shall integrate directly in \( 2d = 8 + 2\epsilon \) dimensional momentum space. We have vertices \(-ie\gamma^\mu\), and propagators \( \frac{i}{k^2 - m} \). Let the integration over internal momenta be implicit and \( Tr \) be the usual
Example 1.2.

\[ \Pi^*_{2\text{LOOP},1}. \]

\[ = \Pi^*_{2\text{LOOP}} \]

\[ \Pi^*_{2\text{LOOP},2}. \]

**Figure 1.** A Feynman diagram of simple 2-loop radiative corrections in QED. We evaluate the latter one, \( \Pi^*_{2\text{LOOP},2} \), as our first illustrative example.

trace over the Dirac algebra, then with \( m \) electron mass and \( e < 0 \) the fundamental charge we have have from physics

\[
\Pi^*_{2\text{LOOP},2} = Tr[-ie\gamma^\mu \frac{p-k-m}{(p-k)^2-m^2} - ie\gamma^\nu \frac{k-h-m}{(p-k)^2-m^2} + ie\gamma^\mu \frac{k+h+m}{(p-k)^2-m^2} - i\gamma^\nu \frac{k+m}{(p-k)^2-m^2}] \\
= -ie^4 Tr[\gamma^\mu \frac{p-k+m}{(p-k)^2-m^2} \gamma^\nu \frac{p-k-h+m}{(p-k)^2-m^2} \gamma^\mu \frac{k+h+m}{(k+q)^2-m^2} \gamma^\nu \frac{k+m}{(k+q)^2-m^2}] \\
= \text{Feynman parameters} \ \{x, y, z, w\} \\
= \int \frac{dk^d}{(2\pi)^d} \frac{dq^d}{(2\pi)^d} \int_{[0,1]^4} d\xi dy dz dw \\
\supress Tr[\gamma^\mu (p-k+m) \gamma^\nu (p-k-h+m) \gamma^\mu (k+h+m) \gamma^\nu (k+m)]
\]

We now combine the sum of denominators-it is seen to be a quadric of momenta, thus elementary algebra gives that this quadric can be written as \(^2 \|\Lambda^{-1}(k \oplus q) + T\|^2 - \Delta^2 \) with \( \Lambda, T \) a Feynman parameter dependent matrix and \( \| \cdot \| \) the norm induced by the metric \( h = g_1 \oplus g_2 \), \( g_i \) usual Minkowski space metrics. Hence a simple change of variables gives, setting \( t = \Lambda^{-1}(k \oplus q) \|_{k \rightarrow -ik_4, q \rightarrow -iq_4} \) together with setting the numerator to \( \sum A^\alpha t_\alpha \) \(^3 \) that we can finally see the use of the previous section;

\(^2\)Orthogonal diagonalization of the quadric and a little algebra suffices to show this for example.

\(^3\)The matrices \( A^\alpha \) depend on dimension in the general case, although they do not do so in this specific case. Expressions in momenta before transformation must be expressed in the new variables \( t \), so their induces a covariant transformation of the matrices \( A^\alpha \) by \( \Lambda \), something that is supressed later.
\[
\Pi_{2\text{LOOP}}^2 = (-i)^2 \text{det}(\Lambda) \sum A^\alpha t\alpha \ln(\Delta^2 + \Delta^2)
= (-i)^2 \text{det}(\Lambda) \sum A^\alpha B_{2,\alpha}(z, d, z, \Lambda)_{|z=|z|, d=2d=8+2\epsilon, z=10}
= (-i)^2 \frac{-ie^{i\delta}}{(2\pi)^2} \Gamma(\Delta) \sum \left[ A^\alpha \chi_0 \mod 2(|\alpha|) \frac{(-1)^{|\alpha|}}{2^{|\alpha|^2}} \frac{g^{|\alpha|^2}}{(4\pi)^2} \frac{(d-2)!}{(d+1/\theta - 2)!} B_{2}(\zeta, d, z, \Lambda)_{|z=|z|, d=2d=8+2\epsilon, z=10}
\right]
\]

In the above \( A \) must be transformed covariantly with \( \Lambda \), this is something that we assume from now on. The calculation of the \( A^\alpha \) in the example above can be made and consists of some tedious algebra \(^4\). The main point is that we now have an answer- that was reached without intermediate use of countert-terms. Also, the procedure outlined has great generality, for we can evaluate the Laurent series expansion of \( B_2 \) once and for all for all cases. Thus 'tHooft-Veltman regularization, the above tricks and the formula for the above Euler integrals will give us our answers.

Let us give the general formula for the Laurent expansion. It might not be understandable why we choose to give such a formula, but we will need the exhaustiveness to achieve generality. In even background dimension \( d_0 \in \mathbb{N} \) the following case covers the non-trivial cases I have seen so far \(^5\).

\[
\frac{1}{(2\pi)^2} B_2(\zeta, d, z, \Delta)
= \frac{1}{4\pi \Delta} \frac{\Gamma(\zeta + d_0 + \frac{1}{2})}{\Gamma(\frac{d_0 + 1}{2})} \left[ (-1)^{\frac{z-d_0-\zeta}{2}} \frac{\gamma - \sum 1 \leq n \leq \frac{z-d_0-\zeta}{2}, n \in \mathbb{Z}}{\frac{1}{n}} + O(\epsilon) \right],
\]

or in still greater generality

\(^4\)The main impetus for inventing my function was to have some device for writing down the regularized result for any Feynman integral at any order of perturbation theory. Of course there still remains the evaluation of the integrals over Feynman parameters, which is best made numerically. The above procedure still holds at high loop, and there it can be advisable to also handle the calculation of \( \Lambda \) and the Dirac algebra on computer too.

\(^5\)However there are cases \( z - d - \zeta \leq 0 \) where regularization is not needed. Direct evaluation in terms of gamma functions is then possible according to the previous section, so those cases are trivial in some sense.
\[
\begin{align*}
\frac{1}{(2\pi)^d} B_{2,\alpha}(\zeta, d, z, \Delta) &= \\
&= \frac{1}{(4\pi)^{d/2} \Delta^{d/2}} \frac{\Gamma(\frac{d+\zeta}{2})}{\Gamma(\frac{d}{2})} \chi_{0 \text{ mod } 2} (|\alpha|)(-1)_{\frac{|\alpha|}{2}} g_{(\alpha_1 \cdots \alpha_n)} (d_0-2)!! \times \\
&\quad \left[ \frac{-2}{\epsilon} - \left[ \ln(\frac{A^2}{\pi}) \right] \right]_{\frac{|\alpha|}{2}}^{\frac{|\alpha|}{2}} + \frac{1}{2} \left( \gamma - \sum_{1 \leq n \leq |\alpha|} \zeta ight) - 2 \frac{\partial_d (d-2)!!}{(d-2)!!} [d=d_0] + O(\epsilon) ,
\end{align*}
\]

\[
\frac{1}{D^\beta} = \frac{1}{B(\beta)} \int_{I=[\beta]} (z^{\beta-\gamma}) \delta(\sum \zeta_i - 1) \frac{1}{(\sum \zeta_i D^\beta)^\beta}
\]

\[\beta\] a multiindex. \(D\) the denominators. The measure used is \(d\zeta^{-\gamma}\). It all now all falls down to using the above formula, Wick rotate, then make a change of variables and express the end result as a superposition of relevant Laurent series. Let us call this general diagram \(\mathcal{M}\), \(p\) the degree of the polynomial in the numerator and set \(dl = d_0 l + \epsilon\), \(\epsilon \in \mathbb{C}\), \(|\epsilon| > 0\) small enough.

**Example 1.3.** In QED \(p\) is the number of internal fermions and at four-loop in dimension 6 we have \(dl = 6 \cdot 4 = 24\).
Then we have
\[
\mathcal{M} = (-i)^{\delta} \det(A) \sum_{0 \leq |\alpha| \leq p} \left[ A^\alpha \mid d = d_0, + \partial_{d} A^{\alpha} \mid d = d_0 \epsilon + O(\epsilon^2) \right] \frac{1}{(2\pi)^d} B_{2|\alpha, d, z, \Delta} \\
= (-i)^{\delta} \det(A) \sum_{0 \leq |\alpha| \leq p} \left[ A^\alpha \mid d = d_0, + \partial_{d} A^{\alpha} \mid d = d_0 \epsilon + O(\epsilon^2) \right] \\
\left[ \chi_{C_0}(z - d_0 - \zeta) \frac{1}{(4\pi)^d} \Gamma(\frac{d}{2}) \frac{\Delta^\zeta - d_0 \epsilon}{2} \right] \left[ - \frac{\det z}{\epsilon} \right] \chi_{0 \mod 2} (|\alpha|) \frac{1}{2^{\frac{|\alpha|}{2}}} \left[ g_{(\alpha \ldots g_{\alpha}) (d_0 - 2)!} \right] \\
\frac{\det z}{\epsilon} \left[ - \frac{\det z}{\epsilon} \right] \chi_{0 \mod 2} (|\alpha|) \frac{1}{2^{\frac{|\alpha|}{2}}} \left[ g_{(\alpha \ldots g_{\alpha}) (d_0 - 2)!} \right] \\
\left[ \chi_{C_0}(z - d_0 - \zeta) \frac{1}{(4\pi)^d} \Gamma(\frac{d}{2}) \frac{\Delta^\zeta - d_0 \epsilon}{2} \right] \left[ - \frac{\det z}{\epsilon} \right] \chi_{0 \mod 2} (|\alpha|) \frac{1}{2^{\frac{|\alpha|}{2}}} \left[ g_{(\alpha \ldots g_{\alpha}) (d_0 - 2)!} \right]
\]
\[
\frac{\det z}{\epsilon} \left[ - \frac{\det z}{\epsilon} \right] \chi_{0 \mod 2} (|\alpha|) \frac{1}{2^{\frac{|\alpha|}{2}}} \left[ g_{(\alpha \ldots g_{\alpha}) (d_0 - 2)!} \right]
\]
\[
\frac{\det z}{\epsilon} \left[ - \frac{\det z}{\epsilon} \right] \chi_{0 \mod 2} (|\alpha|) \frac{1}{2^{\frac{|\alpha|}{2}}} \left[ g_{(\alpha \ldots g_{\alpha}) (d_0 - 2)!} \right]
\]
\[
\frac{\det z}{\epsilon} \left[ - \frac{\det z}{\epsilon} \right] \chi_{0 \mod 2} (|\alpha|) \frac{1}{2^{\frac{|\alpha|}{2}}} \left[ g_{(\alpha \ldots g_{\alpha}) (d_0 - 2)!} \right]
\]
\[
\frac{\det z}{\epsilon} \left[ - \frac{\det z}{\epsilon} \right] \chi_{0 \mod 2} (|\alpha|) \frac{1}{2^{\frac{|\alpha|}{2}}} \left[ g_{(\alpha \ldots g_{\alpha}) (d_0 - 2)!} \right]
\]

Where \( C_+ \) are the complex numbers of positive real part and \( C_0 \) the complement in \( C \). Thus desupressing the integral over Feynman parameters we get for \( d_0 \geq 2 \)

\[
\mathcal{M} = \frac{1}{B(|\beta|)} \int d \gamma_{|\beta|} \delta(\sum \zeta - 1) \zeta^\beta - \gamma (-i)^{\delta} \det(A) \sum_{0 \leq |\alpha| \leq p} \left[ \left[ \chi_{C_0}(z - d_0 - \zeta) \frac{1}{(4\pi)^d} \Gamma(\frac{d}{2}) \frac{\Delta^\zeta - d_0 \epsilon}{2} \right] \left[ - \frac{\det z}{\epsilon} \right] \chi_{0 \mod 2} (|\alpha|) \frac{1}{2^{\frac{|\alpha|}{2}}} \left[ g_{(\alpha \ldots g_{\alpha}) (d_0 - 2)!} \right] \right. \\
\frac{\det z}{\epsilon} \left[ - \frac{\det z}{\epsilon} \right] \chi_{0 \mod 2} (|\alpha|) \frac{1}{2^{\frac{|\alpha|}{2}}} \left[ g_{(\alpha \ldots g_{\alpha}) (d_0 - 2)!} \right]
\]

the regularized formula for the general field theory diagram in the continual class of theories. Subtracting at a suitable subtraction point, or employing minimal subtraction, i.e. projecting away the purely meromorphic germ at the origin in complex \( \epsilon \)-space, we have our renormalized diagram. Some remarks are to be done: This evaluation of the general diagram for this class of theories may at first hand seem sensitive to the order of integration over loop dimensions, however this problem is taken care of by regarding the whole integral as a meromorphism over several variables. Slicing up the integration region as a hypercube, letting the side of the cube go to infinity and integrating iteratively over cubes pertaining to the various loop momenta we note that in euclidean time we have a holomorphism in the form of a meromorphic function.
transversally by the general Paley-Wiener theorem from several complex variables. Furthermore the integral is invariant under order of integration by Funabini-Tonelli's theorem from integration theory for each successive cube in the limit taken.

2. A View Towards the Future and Conclusions

In this article we calculated the general diagram of perturbative QFT for the class of dimensionally renormalizable theories with vertices that are at most matrix-valued polynomials in momentum space. A tentative result was obtained but we believe that further checks of the formula must be made, e.g. by numerical means. We remark that this formula may or may not imply a reordering of the perturbation theory series. The task of checking this formula at high loop and what kind of field theory redefinitions it implies is daunting, but the check may well be partly overcome by instead directly trying to see if the answers are the physical ones.

References


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