Two-Component Form of the New Dirac Equation

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Abstract

The two-component form of the new Dirac equation is obtained for a zero mass particle using a unitary transformation.

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1. Introduction

The new relativistic wave equation proposed by Dirac in 1971 [2] is not symmetric in term of positive and negative values of energy. This equation describes a spinless particle with positive energy, internal structure and non-zero rest mass. The equation has the following form:

\[
\left\{ \frac{\partial}{\partial x_0} + \alpha_r \frac{\partial}{\partial x_r} + m\beta \right\} q \psi = 0
\]  

(1)

where \( \alpha_r \) (\( r = 1, 2, 3 \)) are real \( 4 \times 4 \) matrices and \( \beta \) is an antisymmetric matrix given by

\[
\beta = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
\end{pmatrix}
\]  

(2)

Note that \( \beta^2 = -I \), where I is the unity matrix. The quantity \( q \) is a colon vector. The symbol \( q \) will denote a line vector \( (q_1, q_2, q_3, q_4) \) where \( q_1, q_3 = p_1 \) and \( q_2, q_4 = p_2 \) are the dynamic variables of two harmonic oscillators describing the internal structure of the particle. The quantities \( q_a \) (\( a = 1, 2, 3, 4 \)) satisfy following commuting law

\[
[q_a, q_b] = q_a q_b - q_b q_a = i\beta_{ab}
\]  

(3)

The wave function \( \psi \) is one-component and depends on \( x_0, x_r \) and two commuting quantities \( q_a \) (for example \( q_1 \) and \( q_2 \)). The matrices \( \alpha_r \) (\( r = 1, 2, 3 \)) and \( \beta \) satisfy the Clifford-Dirac algebra relations:
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\[ \alpha_\alpha + \alpha_\beta \alpha_\gamma = 2\delta_{r,s} \]
\[ \alpha_\beta + \beta_\alpha \alpha_\gamma = 0 \quad (r,s = 1, 2, 3) \]  
\[ (4) \]

Introducing the notations \( \partial^\mu \equiv \frac{\partial}{\partial x^\mu} \) and \( \alpha_\alpha = I \) the unity matrix, the equation (1) takes the form:

\[ (\alpha_\mu \partial^\mu + m\beta)q_\psi = 0 \quad (\mu = 0, 1, 2, 3) \]  
\[ (5) \]

2. Two-component equation

One of the possible choices of the real symmetric \( \alpha \) matrix is the following:

\[ \alpha_1 = \begin{pmatrix} 0 & -\sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} \]  
\[ (6) \]

where \( \sigma_0 = I \) and \( \sigma_r \) \( (r = 1, 2, 3) \) are the Pauli matrices.

Next we consider the equation (5) in the case of zero mass. We get the equation

\[ \alpha_\mu \partial^\mu q_\psi = 0 \]  
\[ (7) \]

With the help of the unitary transformation \( \alpha_\mu' = U^{-1}_\mu \alpha_\mu U \), where

\[ U = \frac{1}{2} \begin{pmatrix} 1 & i & 1 & i \\ 1 & -i & -1 & i \\ 1 & i & -1 & -i \\ -1 & i & -1 & i \end{pmatrix}, \]

we transform the matrices \( \alpha_r \) and \( \beta \) in the matrices \( \alpha'_r \) and \( \beta' \) respectively. We get:

\[ \alpha'_1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \quad \beta' = \begin{pmatrix} 0 & -\sigma_0 \\ \sigma_0 & 0 \end{pmatrix} \]

Similarly, the column \( q \) is transformed into a column \( q' = Uq \), equal to:

\[ q' = \frac{1}{2} \begin{pmatrix} q_1 + iq_2 + q_3 + iq_4 \\ q_1 - iq_2 - q_3 + iq_4 \\ q_1 + iq_2 - q_3 - iq_4 \\ -q_1 + iq_2 - q_3 + iq_4 \end{pmatrix} \]

Quantities \( q'_a \) \( (a = 1, 2, 3, 4) \) satisfy the relation
where the matrice $\Delta$ is equal to:

$$\Delta = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}$$

After the transformation, the equation (7) takes the form

$$\alpha' \tilde{\alpha}^\mu q^\mu \psi = 0$$

(8)

Multiplying (8) on the left by the matrix $\beta'$, and introducing the notation

$$\gamma_\mu = \beta' \alpha'$$

we get the following equation:

$$\gamma_\mu \tilde{\alpha}^\mu q^\mu \psi = 0$$

(9)

where

$$\gamma_0 = \begin{pmatrix} 0 & -\sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \quad \gamma_k = \begin{pmatrix} -\sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \quad (k = 1, 2, 3)$$

(10)

As might be expected, equation (7) is invariant under similarity transformation of the $\alpha$ matrices.

Next, we introduce the operators $P_\pm$ (see [1]) given by $P_\pm = \frac{I \pm \gamma_5}{\sqrt{2}}$, where $\lambda_5 = i \gamma_0 \gamma_1 \gamma_2 \gamma_3$.

The matrice $\gamma_5$ anticommute with all matrices $\gamma_\mu$. Acting operators $P_\pm$ on the left of $q^\mu \psi$, we obtain:

$$P_+ q^\mu \psi = \begin{pmatrix} Q_1 \\ Q_2 \\ \bar{Q}_1 \\ \bar{Q}_2 \end{pmatrix} \psi, \quad P_- q^\mu \psi = \begin{pmatrix} \bar{Q}_1 \\ \bar{Q}_2 \\ Q_1 \\ Q_2 \end{pmatrix} \psi$$

(11)

Here, for the elements of four-column, we used the following notations:

$$Q_1 = \frac{1}{\sqrt{2}}(q'_1 + q'_3); \quad \bar{Q}_1 = \frac{1}{\sqrt{2}}(q'_1 - q'_3);$$

$$Q_2 = \frac{1}{\sqrt{2}}(q'_2 + q'_4); \quad \bar{Q}_2 = \frac{1}{\sqrt{2}}(q'_2 - q'_4).$$

Now if we multiply (9) by $P_\pm$ on the left and using (11), we obtain two independent equations for two-component quantities $Q \psi$ and $\bar{Q} \psi$

$$\left(\tilde{\alpha}^0 + \sigma, \tilde{\alpha}' \right) Q \psi = 0$$

(12a)
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\[ \left( \not{\partial} - \sigma \not{\partial} \right) \varphi = 0 \]  

(12b)

where \( Q = \begin{pmatrix} a \\ b \end{pmatrix}, \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \)

The quantities \( Q_a \) and \( Q_b \) (\( a = 1, 2 \)) verify relations:

\[ [Q_a, Q_b] = i \eta_{ab}, \]

\[ [Q_a, \varphi_b] = i \eta_{ab}, \quad (a, b = 1, 2), \]

\[ [Q_a, \varphi_b] = 0 \]

where \( \eta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) and \( \eta^2 = -1 \).

It is obvious, for the continuation, to consider only the equation (12a). It can be written as:

\[ \Pi_a \varphi = 0, \]  

(13)

where \( \Pi_a = (\sigma \not{\partial}^\mu) Q_b, \quad (a, b = 1, 2) \).

The equation (13) is a system of two coupled differential equation for the one-component function \( \psi(x, Q_1) \). The compatibility condition of these equations is:

\[ [\Pi_a, \Pi_b] \psi = 0, \quad (a, b = 1, 2) \]  

(14)

It leads to the Klein-Gordon equation:

\[ \partial_{\mu} \partial^\mu \psi = 0, \quad (\mu = 0, 1, 2, 3) \]  

(15)

Taking the wave function in the form of a plane wave

\[ \psi = \varphi(q_1, q_2) \exp\left\{ -ip^\mu x_\mu \right\} \]

and using the following representation:

\[ q_1 = -i \partial/\partial q_1, \quad q_2 = -i \partial/\partial q_2, \]

we seek a solution of the equation (13), which can already be written as

\[ (q_1 + i \partial/\partial q_1 + iq_2 + \partial/\partial q_2) \varphi = 0, \]

(16)

provided

\[ p_0 = p_1 \neq 0, \quad p_2 = p_3 = 0 \]  

(17)

The solution of the equation (16) is:

\[ \varphi = \exp\left\{ \frac{i}{2} [Q_1^2 + Q_2^2] \right\}. \]  

(18)

Then the general solution of the equation (13) with the conditions (17) will be:
\[ \psi = \exp \left\{ \frac{i}{2} \left[ Q_i^2 + Q_i^2 \right] \right\} \times \exp \left\{ -ip^\mu x_\mu \right\} \]  \hspace{1cm} (19)

References


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