Quaternionic Vector Coherent States for Spin-Orbit Interactions

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Abstract

This work addresses the study of two-dimensional electron gas in a perpendicular magnetic field in the presence of both Rashba and Dresselhaus spin-orbit (SO) interactions, with effective Zeeman coupling. Exact analytical expressions are found for the eigenvalue problem giving the Landau levels and the associated eigenstates, thanks to an appropriate method of parametrization used to quantize the physical system. Such a system exhibits interesting properties of quaternions for which vector coherent states are built. Similarity with the Weyl-Heisenberg group and some relevant properties of these vectors are investigated and discussed.

Keywords: Spin-orbit interactions; quaternionic vector coherent states

1 Introduction

Spin orbit Hamiltonian systems continue to attract increasing interest in recent years due to their prevalence in various physical applications as in semiconductors and spintronics. For details on relevant applications, see [15] and

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references therein. To cite here a few, let us mention that in the paradigmatic Datta-Das spin transistor, the spin of the electron passing through the device is controlled by the Rashba spin-orbit (SO) interaction [12], which in turn can be varied by the application of gate voltages. The Rashba interaction stems from the structural inversion asymmetry (SIA) introduced by a heterojunction or by surface or external fields. In semiconductors with narrower energy gap (InGaAs, AlGaAs), this effect is expected to be stronger. It has been shown experimentally that the Rashba spin-orbit interaction can be modified up to 50% by external gate voltages [9], [10]. In addition to the Rashba coupling, there is also a material-intrinsic Dresselhaus spin-orbit interaction. This originates from the bulk inversion asymmetry (BIA) of the crystal, and can be relatively large in semiconductors like InSb/InAlSb [5]. Both Rashba and Dresselhaus interactions contribute to the spin-dependent splitting of the band structure of the host material, leading to dramatic spin-dependent phenomena for electrons or holes in semiconductors.

Globally, the determination of the eigenvalues and eigenstates of such systems is crucial for the understanding and the computation of a number of their important physical properties [13]. All these aspects motivate the investigations of the properties of the SO Hamiltonian systems. Recently, standard and nonlinear vector coherent states (VCS) have been constructed for relevant generalized models in quantum optics for nonlinear spin-Hall effect [4] and for the Jaynes-Cummings model in the rotating wave approximation [3]. See also references quoted in these works. Despite all these investigations, it is worthy of attention to note that only a few exactly analytically solvable eigenvalue problems associated to SO interactions exist in the literature. Besides, in the most interesting case of the description of the more general case, in which both Rashba and Dresselhaus interactions are present with arbitrary strength, in the effective Zeeman coupling, exact solutions to the eigenvalue problem are still lacking to our best knowledge of the existing works. The present paper aims, among others, at fulfilling such a gap. We give explicit exact analytical solutions to the addressed eigenvalue problem and build quaternionic VCS (QVCS) for the spinor Landau levels. Similarity properties in connection with the Weyl-Heisenberg group is examined.

The paper is organized as follows. Section 2 deals with the model description in two gauge symmetries corresponding to the two orientations of the magnetic field. Then, the eigenvalue problem is addressed and solved. In section 3, VCS and QVCS are formulated and some physical properties of the latter states are investigated. Then, follows the conclusion.

2 Physical Hamiltonians

2.1 Hamiltonian with Zeeman coupling

The Hamiltonian of a particle system with mass M and charge e with Zeeman coupling g , in a magnetic field can be defined by

$$\mathcal{H} = \frac{1}{2M} \left(\mathbf{p} - \frac{e}{c} \mathbf{A}(\mathbf{r}, \mathbf{t}) \right)^2 - g \frac{\mu}{2} \boldsymbol{\sigma} \cdot \mathbf{B} \quad (1)$$

where in the term $g\mu/2 \boldsymbol{\sigma} \cdot \mathbf{B}$, the symbol $\mu = e\hbar/2M$ denotes the Bohr magneton, g the factor of coupling related to the particle and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \equiv (\sigma_1, \sigma_2, \sigma_3)$ the vector of Pauli spin matrices.

Let us first consider the gauge symmetry $\mathbf{A} = (-\frac{B}{2}y, \frac{B}{2}x)$. Then, the generalized Hamiltonian (1) becomes for an electron with charge $-e$,

$$\mathcal{H}_{10}(\mathbf{p}, \mathbf{r}) \equiv \mathcal{H}_{10} = \frac{1}{2M} \left[\left(p_x - \frac{eB}{2c} y \right)^2 + \left(p_y + \frac{eB}{2c} x \right)^2 \right] - \frac{g\mu B}{2} \sigma_z. \quad (2)$$

Following the general scheme of canonical quantization, we can introduce the coordinate and momentum operators $R_i, P_i, i = 1, 2$ corresponding to the classical coordinates x, y and their conjugate momenta p_x, p_y , respectively, so that the canonical commutation relations

$$[R_i, P_j] = i\hbar\delta_{ij}, \quad \text{for } R_i = X, Y \quad \text{and} \quad P_i = P_x, P_y, \quad i = 1, 2 \quad (3)$$

hold. Using the change of variables

$$Z = X + iY, \quad P_z = \frac{1}{2}(P_x - iP_y), \quad (4)$$

and the creation and annihilation operators $b^\dagger, b$ defined as

$$b^\dagger = -2iP_{\bar{z}} + \frac{eB}{2c}Z, \quad b = 2iP_z + \frac{eB}{2c}\bar{Z}, \quad (5)$$

and such that

$$[b, b^\dagger] = 2M\hbar\omega_c \quad (6)$$

where $\omega_c = \frac{eB}{Mc}$ (the cyclotron frequency), the Hamiltonian (2) can be re-expressed in the manner

$$H_{10} = \hbar\omega_c \left(b^\dagger b + \frac{\mathbf{1}}{2} \right) \sigma_0 + \hbar\omega_c \xi \sigma_z, \quad \xi = -g\mu Mc/2\hbar e, \quad (7)$$

with $\sigma_0 = \mathbb{I}_2$, the identity matrix, where the annihilation and creation operators b and $b^\dagger$ have been rewritten as follows:

$$b = \sqrt{2M\hbar\omega_c}b', \quad b^\dagger = \sqrt{2M\hbar\omega_c}b'^\dagger, \quad (8)$$

with

$$[b', b'^\dagger] = \mathbf{1}. \quad (9)$$

The eigenvalues related to H_{1_0} are given by

$$E_n^\pm = \hbar\omega_c \left(n + \frac{1}{2} \right) \mp \hbar\omega_c \xi. \quad (10)$$

The corresponding eigenstates, denoted by $\begin{pmatrix} |\Phi_n\rangle \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ |\Phi_n\rangle \end{pmatrix}$, span the Hilbert space $\hat{\mathfrak{H}} := \mathbb{C}^2 \otimes \mathfrak{H}$, with $\mathfrak{H} := \text{span}\{|\Phi_n\rangle\}_{n=0}^\infty\}$, $|\Phi_n\rangle$ corresponding to the number operator $N = b'^\dagger b'$ orthonormalized eigenstates $|n\rangle = (1/\sqrt{n!})(b'^\dagger)^n|0\rangle$, and

$$\hat{\mathfrak{H}} = \left\{ |\chi^+ \otimes \Phi_n\rangle = \begin{pmatrix} |\Phi_n\rangle \\ 0 \end{pmatrix}, |\chi^- \otimes \Phi_n\rangle = \begin{pmatrix} 0 \\ |\Phi_n\rangle \end{pmatrix}, n = 0, 1, 2, \dots \right\}, \quad (11)$$

where χ^+ and χ^- , given by $\chi^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\chi^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, form an orthonormal basis of $\mathbb{C}^2$.

In opposite, in the gauge symmetry $\mathbf{A} = (\frac{B}{2}y, -\frac{B}{2}x)$, the Hamiltonian is given by

$$\mathcal{H}_{2_0}(\mathbf{p}, \mathbf{r}) \equiv \mathcal{H}_{2_0} = \frac{1}{2M} \left[\left(p_x + \frac{eB}{2c}y \right)^2 + \left(p_y - \frac{eB}{2c}x \right)^2 \right] + \frac{g\mu B}{2}\sigma_z, \quad (12)$$

with the same change of variables as in (4). The annihilation and creation operators given, respectively, by

$$d = -2iP_{\bar{z}} - \frac{eB}{2c}Z, \quad d^\dagger = 2iP_z - \frac{eB}{2c}\bar{Z} \quad (13)$$

satisfy the following commutation relations

$$[d, d^\dagger] = 2M\hbar\omega_c. \quad (14)$$

The corresponding canonically quantized Hamiltonian can be written as

$$H_{2_0} = \hbar\omega_c \left(d'^{\dagger} d' + \frac{\mathbf{1}}{2} \right) \sigma_0 + \hbar\omega_c \xi' \sigma_z, \quad (15)$$

where $\xi' = -\xi$, through the annihilation and creation operators d and $d^{\dagger}$ built as follows:

$$d = \sqrt{2M\hbar\omega_c} d', \quad d^{\dagger} = \sqrt{2M\hbar\omega_c} d'^{\dagger} \quad (16)$$

with

$$[d', d'^{\dagger}] = \mathbf{1}. \quad (17)$$

The eigenvalues and eigenstates related to the Hamiltonian H_{2_0} are obtained as previously.

2.2 Eigenvalues and eigenstates of general spin-orbit interactions Hamiltonians

Through a reparametrization using the Fock operators or by making use of projections and defining the Pauli matrices $\sigma_{\pm} = (\sigma_x \pm i\sigma_y)/2$, the generalization of Rashba H_R and Dresselhaus H_D SO interactions can be introduced as:

$$\begin{aligned} H_R &= i\hbar\omega_c \gamma \left(\lambda(N)(b'^{\dagger})^k \sigma_- - \overline{\lambda(N)} b'^k \sigma_+ \right), \\ H_D &= \hbar\omega_c \beta \left(\lambda(N) b'^k \sigma_- + \overline{\lambda(N)} (b'^{\dagger})^k \sigma_+ \right) \end{aligned} \quad (18)$$

where γ and β are dimensionless parameters defined by $\gamma = \gamma_0 \sqrt{\frac{2M}{\hbar^3 \omega_c}}$ and $\beta = \beta_0 \sqrt{\frac{2M}{\hbar^3 \omega_c}}$ with γ_0, β_0 being the Rashba and Dresselhaus coupling constants, respectively. For two values of the parameter ε , there follow also two cases:

(i) $\varepsilon = +$,

$$B_{k,+}^+ = \lambda(N) b'^{-k} = \lambda(N) b'^k, \quad B_{k,+}^- = \overline{\lambda(N)} b'^{+k} = \overline{\lambda(N)} (b'^{\dagger})^k; \quad (19)$$

(ii) $\varepsilon = -$,

$$B_{k,-}^+ = \lambda(N) b'^{+k} = \lambda(N) (b'^{\dagger})^k, \quad B_{k,-}^- = -\overline{\lambda(N)} b'^{-k} = -\overline{\lambda(N)} b'^k, \quad (20)$$

where $\lambda(N)$ is a nonvanishing complex valued function of the number operator N , $\overline{\lambda(N)}$ denotes its complex conjugate and k is a strictly positive integer, $k = 1, 2, \dots$. The expressions in (i) and (ii) can be abridged as

$$B_{k,\varepsilon}^+ = \lambda(N)b'^{-\varepsilon k}, \quad B_{k,\varepsilon}^- = \varepsilon \overline{\lambda(N)}b'^{\varepsilon k}, \quad (21)$$

and denoting the generalized dimensionless SO interaction by

$$V_{k,\varepsilon} = B_{k,\varepsilon}^+ \sigma_- + B_{k,\varepsilon}^- \sigma_+, \quad (22)$$

the relations (18) take the following compact forms

$$H_R = i\hbar\omega_c\gamma V_{k,-}, \quad H_D = \hbar\omega_c\beta V_{k,+}. \quad (23)$$

Recall that in the study of a spin-1/2 particle of charge $-e$ subjected to a spin-orbit interaction in a perpendicular magnetic field $\mathbf{B} = -B\hat{z} = \nabla \times \mathbf{A}$ and to an electric field $\mathbf{E} = E\hat{y}$ along the y axis [13], using the introduced lowering operator a , the both Rashba and Dresselhaus couplings are given by $V_R = i\sqrt{2}\eta_R(a\sigma_- - a^\dagger\sigma_+)$ and $V_D = \sqrt{2}\eta_D(a^\dagger\sigma_- + a\sigma_+)$ which lead to the generalized dimensionless spin-orbit interaction established in [4] as: $U_{k,\epsilon} = C_{k,\epsilon}^+ \sigma_+ + C_{k,\epsilon}^- \sigma_-$, where $C_{k,\epsilon}^+ = a^{-\epsilon k} \lambda^\epsilon(N)$, $C_{k,\epsilon}^- = \lambda^{-\epsilon}(N) a^{\epsilon k}$, with $\epsilon = \pm 1$.

From (22) and (23), the generalized Hamiltonians describing a 2D electron gas mixing with full Rashba and Dresselhaus SO interactions, and effective Zeeman coupling in a perpendicular magnetic field background are given in both relevant gauge symmetries by

$$H_{1,2SO} = \frac{1}{2M} \left[\left(p_x \mp \frac{eB}{2c} y \right)^2 + \left(p_y \pm \frac{eB}{2c} x \right)^2 \right] \mp \frac{g\mu B}{2} \sigma_z + V_{SO}, \quad (24)$$

where V_{SO} stands for the generalized Rashba or Dresselhaus spin-orbit terms given in (23).

Remark 2.1 *The relevant exactly solvable cases known in the literature [16] correspond to $\beta = 0$, $\gamma = 0$ and $\gamma = \pm\beta$ and concern with the gauge in which the constant magnetic field points along the positive z -direction. The eigenstates and eigenvalues of the associated resulting Hamiltonians are given by the expressions:*

- For $\beta = 0$ and nonvanishing Rashba SO interaction term, which couples the up-spin level $(\Phi_{n-1}, 0)$ to the down-spin state $(0, \Phi_n)$, $n \geq 1$

$$\begin{aligned} \psi_n^+ &= \begin{pmatrix} \cos \theta_n \Phi_{n-1} \\ \sin \theta_n \Phi_n \end{pmatrix}, & \psi_n^- &= \begin{pmatrix} -\sin \theta_n \Phi_{n-1} \\ \cos \theta_n \Phi_n \end{pmatrix} \\ E_n^\pm &= \hbar\omega_c n \mp \frac{\delta}{2} \sqrt{1 + 4\gamma^2 n \hbar^2 \omega_c^2 / \delta^2}, \end{aligned} \quad (25)$$

in which $\delta = \hbar\omega_c(1 + 2\xi)$. The mixing angle θ_n is given by $\tan 2\theta_n = 2\sqrt{n}\gamma\hbar\omega_c/\delta$ and varies from 0 (when $\gamma = 0$) to $\pi/4$ (for weak magnetic field or strong SO interaction). Each Landau level $\Phi_n \uparrow\downarrow$ splits into two levels ψ_n^- and ψ_{n+1}^+ with the splitting gap $\Delta = E_n^- - E_{n+1}^+$;

- For $\gamma = 0$ and nonvanishing Dresselhaus SO interaction term which couples $(\Phi_n, 0)$ only to the $(0, \Phi_{n-1})$, $n \geq 1$

$$\begin{aligned} \psi_n^+ &= \begin{pmatrix} \cos \theta_n \Phi_n \\ \sin \theta_n \Phi_{n-1} \end{pmatrix}, & \psi_n^- &= \begin{pmatrix} -\sin \theta_n \Phi_n \\ \cos \theta_n \Phi_{n-1} \end{pmatrix} \\ E_n^\pm &= \hbar\omega_c n \mp \frac{\delta}{2} \sqrt{1 + 4\beta^2 n \hbar^2 \omega_c^2 / \delta^2}, \end{aligned} \quad (26)$$

with $\delta = \hbar\omega_c(-1 + 2\xi)$, $\tan 2\theta_n = 2\sqrt{n}\beta\hbar\omega_c/\delta$.

It is also worth noticing that for the most usual widespread situation corresponding to $\lambda(N) = -i\mathbf{I}$, $k = 1$, the cases of $\gamma = \pm\beta$ and $|\beta| < |\gamma|$ have been treated with or without the Zeeman coupling term, using the perturbation theory. For more details, see [15]. In the latter case, the Rashba and Dresselhaus SO interactions take the simpler forms in the gauge $\mathbf{A} = (-\frac{B}{2}y, \frac{B}{2}x)$, where H_R and H_D are obtained from (19) and (20) by

$$\begin{aligned} H_R &= i\hbar\omega_c\gamma(-ib'^\dagger\mathbf{I}\sigma_- - ib'\mathbf{I}\sigma_+) = \hbar\omega_c\gamma(b'^\dagger\sigma_- + b'\sigma_+) \\ &= \hbar\omega_c\gamma \begin{pmatrix} 0 & b' \\ b'^\dagger & 0 \end{pmatrix}, \end{aligned} \quad (27)$$

$$\begin{aligned} H_D &= \hbar\omega_c\beta(ib'^\dagger\mathbf{I}\sigma_+ - ib'\mathbf{I}\sigma_-) = \hbar\omega_c\beta(ib'^\dagger\sigma_+ - ib'\sigma_-) \\ &= \hbar\omega_c\beta \begin{pmatrix} 0 & ib'^\dagger \\ -ib' & 0 \end{pmatrix}, \end{aligned} \quad (28)$$

respectively.

In the sequel, we turn back to the more general situation when the interaction terms (27) and (28) are coupled with the generalized Hamiltonians H_{1_0} and H_{2_0} investigated in the previous section, describing a 2D electron in magnetic field with a Zeeman coupling term.

Analogous expressions are readily obtained for $H_{2_{SO}} = H_{2_0} + V_{SO}$ in the second gauge, when the constant magnetic field points along the negative z -direction, by replacing δ by δ' defined with ξ' .

The spinor states are formulated as follows:

- For $\beta = 0$ and $\gamma \neq 0$

$$\psi_n^+ = \cos \theta_n \begin{pmatrix} \Phi_{n-1} \\ 0 \end{pmatrix} \oplus \sin \theta_n \begin{pmatrix} 0 \\ \Phi_n \end{pmatrix}$$

$$\begin{aligned}
&= \cos \theta_n (\chi^+ \otimes \Phi_{n-1}) \oplus \sin \theta_n (\chi^- \otimes \Phi_n), \\
\psi_n^- &= -\sin \theta_n \begin{pmatrix} \Phi_{n-1} \\ 0 \end{pmatrix} \oplus \cos \theta_n \begin{pmatrix} 0 \\ \Phi_n \end{pmatrix} \\
&= -\sin \theta_n (\chi^+ \otimes \Phi_{n-1}) \oplus \cos \theta_n (\chi^- \otimes \Phi_n); \tag{29}
\end{aligned}$$

- For $\beta \neq 0$ and $\gamma = 0$

$$\begin{aligned}
\psi_n^+ &= \cos \theta_n \begin{pmatrix} \Phi_n \\ 0 \end{pmatrix} \oplus \sin \theta_n \begin{pmatrix} 0 \\ \Phi_{n-1} \end{pmatrix} \\
&= \cos \theta_n (\chi^+ \otimes \Phi_n) \oplus \sin \theta_n (\chi^- \otimes \Phi_{n-1}), \\
\psi_n^- &= -\sin \theta_n \begin{pmatrix} \Phi_n \\ 0 \end{pmatrix} \oplus \cos \theta_n \begin{pmatrix} 0 \\ \Phi_{n-1} \end{pmatrix} \\
&= -\sin \theta_n (\chi^+ \otimes \Phi_n) \oplus \cos \theta_n (\chi^- \otimes \Phi_{n-1}). \tag{30}
\end{aligned}$$

In the weak Rashba SO coupling limit case (γ negligible), we can expand $\mathcal{E}_n^\pm = E_n^\pm - E_0^\pm$ using the approximation $(1 + \varepsilon)^{1/2} \simeq 1 + \varepsilon/2$, $\varepsilon \ll 1$ and, by taking $\hbar = 1$, get from (25)

$$\mathcal{E}_n^\pm = \omega_\pm(\gamma)n, \quad \omega_\pm(\gamma) = \omega_c \left(1 \mp \frac{2\gamma^2}{2 - gc} \right). \tag{31}$$

Define

$$\rho_\pm(n) := \mathcal{E}_1^\pm \mathcal{E}_2^\pm \cdots \mathcal{E}_n^\pm \tag{32}$$

with the increasing order $\mathcal{E}_1^\pm < \mathcal{E}_2^\pm < \cdots < \mathcal{E}_n^\pm$, such that

$$\rho_\pm(n) = \prod_{q=1}^n \omega_\pm(\gamma)q = n!(\omega_\pm(\gamma))^n. \tag{33}$$

2.3 Spectral decomposition

Introduce the passage operators from the Hilbert space $\mathcal{S}$ with eigenbasis $\{|\psi_n^\pm\rangle\}$ to $\{|\chi^\pm \otimes \Phi_n\rangle\}$ and vice versa, $\chi^\pm$ being the natural basis of $\mathbb{C}^2$, given by

$$\mathcal{U}|\chi^\pm \otimes \Phi_n\rangle = |\psi_n^\pm\rangle \quad \mathcal{U}^\dagger|\psi_n^\pm\rangle = |\chi^\pm \otimes \Phi_n\rangle \tag{34}$$

where $\mathcal{U}, \mathcal{U}^\dagger$ expand as

$$\mathcal{U} = \sum_{n=0,\pm}^{\infty} |\psi_n^\pm\rangle \langle \chi^\pm \otimes \Phi_n| \quad \mathcal{U}^\dagger = \sum_{n=0,\pm}^{\infty} |\chi^\pm \otimes \Phi_n\rangle \langle \psi_n^\pm| \quad (35)$$

respectively. $\mathcal{U}, \mathcal{U}^\dagger$ are obtained as mutually adjoint through the following identities:

$$\mathcal{U}^\dagger \mathcal{U} = \sum_{n=0,\pm}^{\infty} |\chi^\pm \otimes \Phi_n\rangle \langle \chi^\pm \otimes \Phi_n| = \mathbb{I}_2 \otimes I_{\mathfrak{H}}, \quad \mathcal{U} \mathcal{U}^\dagger = \sum_{n=0,\pm}^{\infty} |\psi_n^\pm\rangle \langle \psi_n^\pm| = \mathbb{I}_{\mathcal{S}}, \quad (36)$$

where $\mathbb{I}_2 \otimes I_{\mathfrak{H}}$ and $\mathbb{I}_{\mathcal{S}}$ are the identity operators on $\mathbb{C}^2 \otimes \mathfrak{H}$ and $\mathcal{S}$, respectively. Then, the Hamiltonian $H_{1_{SO}}$ can be rewritten in a diagonal form as below:

$$\mathbb{H}_{1_{SO}}^{red} = \mathcal{U}^\dagger H_{1_{SO}} \mathcal{U} = \sum_{n=0,\pm}^{\infty} |\chi^\pm \otimes \Phi_n\rangle \mathcal{E}_n^\pm \langle \chi^\pm \otimes \Phi_n|. \quad (37)$$

3 Canonical QVCS and VCS

In this section, we first provide the canonical QVCS related to the Hamiltonian $\mathbb{H}_{1_{SO}}^{red}$ and then discuss the VCS construction with their connection with the QVCS. Then, statistical properties of the constructed QVCS are investigated through studying some expectations and the uncertainty relation. Besides, the signal-to-quantum-noise ratio in the QVCS and their time evolution are also discussed.

3.1 Canonical QVCS

The QVCS are built using the complex representation of quaternions by 2×2 matrices [2]. Using the basis matrices,

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad (38)$$

where σ_1, σ_2 and σ_3 are the usual Pauli matrices, a general quaternion is written as

$$q = x_0 \sigma_0 + i \underline{x} \cdot \underline{\sigma} \quad (39)$$

with $x_0 \in \mathbb{R}$, $\underline{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $\underline{\sigma} = (\sigma_1, -\sigma_2, \sigma_3)$. Thus,

$$\mathbf{q} = \begin{pmatrix} x_0 + ix_3 & -x_2 + ix_1 \\ x_2 + ix_1 & x_0 - ix_3 \end{pmatrix}. \quad (40)$$

Introducing the polar coordinates as:

$x_0 = r \cos \eta$, $x_1 = r \sin \eta \sin \phi \cos \psi$, $x_2 = r \sin \eta \sin \phi \sin \psi$,
 $x_3 = r \sin \eta \cos \phi$, where $r \in [0, \infty)$, $\phi \in [0, \pi]$ and $\eta, \psi \in [0, 2\pi]$, it follows that

$$\mathbf{q} = A(r)e^{i\eta\sigma(\hat{n})} \quad (41)$$

where

$$A(r) = r\sigma_0, \quad \sigma(\hat{n}) = \begin{pmatrix} \cos \phi & e^{i\psi} \sin \phi \\ e^{-i\psi} \sin \phi & -\cos \phi \end{pmatrix} \quad \text{and} \quad \sigma(\hat{n})^2 = \sigma_0 = \mathbb{I}_2. \quad (42)$$

The field of quaternions is denoted by $\mathbb{H}$, this algebra being generated by $\{1, \hat{i}, \hat{j}, \hat{k}\}$, where $\hat{i}, \hat{j}, \hat{k}$ are imaginary units. The real quaternions denoted by $Q_{\mathbb{R}}$ are given by

$$Q_{\mathbb{R}} = \{q = x_0 + x_1\hat{i} + x_2\hat{j} + x_3\hat{k} \mid \hat{i}^2 = \hat{j}^2 = \hat{k}^2 = -1, \hat{i}\hat{j}\hat{k} = -1, x_0, x_1, x_2, x_3 \in \mathbb{R}\}, \quad (43)$$

where q can be written as $q = q(x_0, \underline{x}) = x_0 + \underline{i} \cdot \underline{x}$, with $\underline{i} = (\hat{i}, \hat{j}, \hat{k})$ and $\underline{x} = (x_1, x_2, x_3)$ given as in (39). The complex quaternions denoted by $Q_{\mathbb{C}}$ are given by $q = x_0 + ix_1\hat{i} + ix_2\hat{j} + ix_3\hat{k} = x_0 + i\underline{i} \cdot \underline{x}$.

Remark that in the definition of the Hamiltonian given in (7), the harmonic oscillator part of H_{10} , given by $\hbar\omega_c (b^\dagger b' + \frac{1}{2}) \sigma_0$, can be rewritten in the supersymmetric quantum mechanics as

$$H^{SUSY} = \begin{pmatrix} H^b & 0 \\ 0 & H^f \end{pmatrix} = \hbar\omega_c \begin{pmatrix} b^\dagger b' & 0 \\ 0 & b' b^\dagger \end{pmatrix}, \quad (44)$$

where the supersymmetric partner Hamiltonians, H^b, H^f are given by $H^b = H_{10SC} - \frac{\hbar\omega_c}{2}\mathbf{1}$ and $H^f = H_{10SC} + \frac{\hbar\omega_c}{2}\mathbf{1}$, with $H_{10SC} = \hbar\omega_c (b^\dagger b' + \frac{1}{2})$, respectively. This Hamiltonian can also be rewritten as $H^{SUSY} = \{Q^\dagger, Q\}$, with $Q = \sqrt{\hbar\omega_c} \begin{pmatrix} 0 & 0 \\ b' & 0 \end{pmatrix} = \sqrt{\hbar\omega_c} b' \sigma_-$ and $Q^\dagger = \sqrt{\hbar\omega_c} \begin{pmatrix} 0 & b^\dagger \\ 0 & 0 \end{pmatrix} = \sqrt{\hbar\omega_c} b^\dagger \sigma_+$ the supercharges. The annihilation operator defined on $\mathbb{C}^2 \otimes \mathfrak{H}$ by

$$\mathcal{A}' = \begin{pmatrix} b' & 0 \\ 0 & b' \end{pmatrix} \quad (45)$$

satisfies the commutation relation $[H^{SUSY}, \mathcal{A}'] = -\hbar\omega_c \mathcal{A}'$. The connection with the supersymmetric harmonic oscillator in this study appears by obtaining the QVCS denoted by $|\mathbf{q}; \pm\rangle$, known in the literature as the *quaternionic*

canonical coherent states [2], given on $\mathbb{C}^2 \otimes \mathfrak{H}$ as follows

$$|\mathbf{q}; \pm\rangle = \frac{e^{-\frac{r^2}{2}}}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{\mathbf{q}^n}{\sqrt{n!}} |\chi^{\pm} \otimes \Phi_n\rangle, \quad (46)$$

where $\mathbf{q}$ is given in (41), as the eigenstates of the operator $\mathcal{A}'$ in (45).

The QVCS related to the Hamiltonian $\mathbb{H}_{1SO}^{red}$ can be defined as

$$|\mathbf{q}; \pm\rangle_1 = (\mathcal{N}_{(\pm)}(|\mathbf{q}|))^{-1/2} \sum_{n=0}^{\infty} \frac{\mathbf{q}^n}{\sqrt{\rho_{(\pm)}(n)}} |\chi^{\pm} \otimes \Phi_n\rangle. \quad (47)$$

The variables $\mathbf{q}$ are as in (41). The 2×2 matrix-valued functions $A(r), \sigma(\hat{n})$ satisfy the following properties [2] (assumed to hold almost everywhere with respect to appropriate measures):

$$\sigma(\hat{n}) = \sigma(\hat{n})^\dagger \quad [A(r), A(r)^\dagger] = 0 \quad [A(r), \sigma(\hat{n})] = 0. \quad (48)$$

The normalization constants $\mathcal{N}_{(\pm)}(|\mathbf{q}|)$ ensured by the relations

$$\sum_{\pm} {}_1\langle \mathbf{q}; \pm | \mathbf{q}; \pm \rangle_1 = 1, \quad (49)$$

are given, with respect to (48) for $|\mathbf{q}; \pm\rangle_1$ by

$$\mathcal{N}_{(\pm)}(|\mathbf{q}|) = \mathcal{N}_{(\pm)}(r) = 2 \sum_{n=0}^{\infty} \frac{r^{2n}}{n! (\omega_{(\pm)}(\gamma))^n} = 2e^{\frac{r^2}{\omega_{(\pm)}(\gamma)}}. \quad (50)$$

The $\mathbb{H}_{1SO}^{red}$ QVCS (47) satisfy on the Hilbert space $\mathbb{C}^2 \otimes \mathfrak{H}$ the following resolution of the identity

$$\sum_{\pm} \int_{\mathcal{D}} |\mathbf{q}; \pm\rangle_1 W_{(\pm)}(r) {}_1\langle \mathbf{q}; \pm | d\nu = \mathbb{I}_2 \otimes I_{\mathfrak{H}}, \quad (51)$$

$W_{(\pm)}(r)$ is the density function and $\mathcal{D} = \mathbb{R}^+ \times (0, 2\pi] \times S^2$, where S^2 is the surface of the unit two-sphere on which the points with coordinates (ϕ, ψ) are defined. The measure $d\nu$ is such that $d\nu(r, \eta, \phi, \psi) = r dr d\eta d\Omega(\phi, \psi)$, with $d\Omega(\phi, \psi) = \frac{1}{4\pi} \sin \phi d\phi d\psi$.

The relation (51) translates in the Stieljes moment problems

$$\int_0^{+\infty} r^{2n+1} \varrho_{(\pm)}(r) dr = \rho_{(\pm)}(n), \quad (52)$$

which are solved by

$$\varrho_{(\pm)}(r) = \frac{2}{\omega_{(\pm)}(\gamma)} e^{-\frac{r^2}{\omega_{(\pm)}(\gamma)}} \quad (53)$$

where the density functions are given by

$$W_{(\pm)}(r) = \frac{\mathcal{N}_{(\pm)}(r)\varrho_{(\pm)}(r)}{2\pi}. \quad (54)$$

Treating the vectors $|\mathbf{q}; +\rangle_1$ and $|\mathbf{q}; -\rangle_1$ as elements of a basis, we shall define a general QVCS for $\mathbb{H}_{1SO}^{red}$ (resp. $\mathbb{H}_{2SO}^{red}$) as a linear combination,

$$|\mathbf{q}, \chi\rangle = c_+ |\mathbf{q}; +\rangle_1 + c_- |\mathbf{q}; -\rangle_1 \quad (55)$$

where $c_{\pm} \in \mathbb{C}$, $|c_+|^2 + |c_-|^2 = 1$ and $\chi = \sum_{\pm} c_{\pm} \chi^{\pm}$.

It is worth noticing that the constructed QVCS are related to the Weyl-Heisenberg group G_{W-H} as observed in [2]. Indeed, from the relations

$$e^{\mathbf{q} \otimes b'^{\dagger} - \mathbf{q}^{\dagger} \otimes b'} = e^{-(1/2)[\mathbf{q} \otimes b'^{\dagger}, -\mathbf{q}^{\dagger} \otimes b']} e^{\mathbf{q} \otimes b'^{\dagger}} e^{-\mathbf{q}^{\dagger} \otimes b'}, \quad [\mathbf{q}^{\dagger} \otimes b', \mathbf{q} \otimes b'^{\dagger}] = r^2 \mathbb{I}_2 \otimes I_3 \quad (56)$$

since

$$e^{-\mathbf{q} \otimes b'} |\chi^{\pm} \otimes \Phi_0\rangle = |\chi^{\pm} \otimes \Phi_0\rangle, \quad (57)$$

we can write

$$\frac{1}{\sqrt{2}} e^{[\frac{\mathbf{q}}{(\omega_{(\pm)}(\cdot))^{1/2}} \otimes b'^{\dagger} - \frac{\mathbf{q}^{\dagger}}{(\omega_{(\pm)}(\cdot))^{1/2}} \otimes b']} |\chi^{\pm} \otimes \Phi_0\rangle = \frac{e^{-\frac{r^2}{2\omega_{(\pm)}(\cdot)}}}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{\mathbf{q}^n}{\sqrt{n! (\omega_{(\pm)}(\cdot))^n}} \times |\chi^{\pm} \otimes \Phi_n\rangle, \quad (58)$$

where $\omega_{(\pm)}(\cdot)$ stands for $\omega_{(\pm)}(\beta)$ and $\omega_{(\pm)}(\gamma)$, respectively. Therefore, in the case of the Hamiltonian H_{1SO} , the QVCS can be rewritten by use of the operators $\tilde{U}_{1,\gamma}^{(\pm)}(0, \mathbf{q})$, $\tilde{U}_{1,\beta}^{(\pm)}(0, \mathbf{q})$ related to Rashba and Dresselhaus SO interactions as

$$|\mathbf{q}; \pm\rangle_1 = \frac{1}{\sqrt{2}} \tilde{U}_{1,\gamma}^{(\pm)}(0, \mathbf{q}) |\chi^{\pm} \otimes \Phi_0\rangle, \quad |\mathbf{q}; \pm\rangle_1^{\sim} = \frac{1}{\sqrt{2}} \tilde{U}_{1,\beta}^{(\pm)}(0, \mathbf{q}) |\chi^{\pm} \otimes \Phi_0\rangle, \quad (59)$$

in analogy with the case of the canonical coherent states [1], where the operators b' and $b'^{\dagger}$ are those given in (8).

The unitary operator $\tilde{U}(0, \mathbf{q})$ has the form [2]

$$\tilde{U}(0, \mathbf{q}) = e^{\mathbf{q} \otimes b'^{\dagger} - \mathbf{q}^{\dagger} \otimes b'} = u(\eta, \phi) \begin{pmatrix} U(0, q, p) & 0 \\ 0 & U(0, q, -p) \end{pmatrix} u(\eta, \phi)^{\dagger}, \quad (60)$$

and for fixed $(\eta, \phi), \eta \in [0, 2\pi), \phi \in [0, \pi]$ and p, q such that $z = \frac{q-ip}{\sqrt{2}}$, the operators $\tilde{U}_{1,\gamma}^{(\pm)}(0, \mathbf{q}), \tilde{U}_{2,\gamma}^{(\pm)}(0, \mathbf{q})$ and $\tilde{U}_{1,\beta}^{(\pm)}(0, \mathbf{q}), \tilde{U}_{2,\beta}^{(\pm)}(0, \mathbf{q})$ are defined in the cases of the Hamiltonians H_{1SO}, H_{2SO} for Rashba and Dresselhaus SO interactions, respectively, by

$$\begin{aligned}\tilde{U}_{1,\gamma}^{(\pm)}(0, \mathbf{q}) &= e^{\left[\frac{\mathbf{q}}{(\omega_{(\pm)}(\gamma))^{1/2}} \otimes b'^{\dagger} - \frac{\mathbf{q}^{\dagger}}{(\omega_{(\pm)}(\gamma))^{1/2}} \otimes b'\right]}, \\ \tilde{U}_{2,\gamma}^{(\pm)}(0, \mathbf{q}) &= e^{\left[\frac{\mathbf{q}}{(\omega_{(\pm)}(\gamma))^{1/2}} \otimes d'^{\dagger} - \frac{\mathbf{q}^{\dagger}}{(\omega_{(\pm)}(\gamma))^{1/2}} \otimes d'\right]},\end{aligned}\quad (61)$$

$$\begin{aligned}\tilde{U}_{1,\beta}^{(\pm)}(0, \mathbf{q}) &= e^{\left[\frac{\mathbf{q}}{(\omega_{(\pm)}(\beta))^{1/2}} \otimes b'^{\dagger} - \frac{\mathbf{q}^{\dagger}}{(\omega_{(\pm)}(\beta))^{1/2}} \otimes b'\right]}, \\ \tilde{U}_{2,\beta}^{(\pm)}(0, \mathbf{q}) &= e^{\left[\frac{\mathbf{q}}{(\omega_{(\pm)}(\beta))^{1/2}} \otimes d'^{\dagger} - \frac{\mathbf{q}^{\dagger}}{(\omega_{(\pm)}(\beta))^{1/2}} \otimes d'\right]},\end{aligned}\quad (62)$$

where the operators d' and $d'^{\dagger}$ are those introduced in (16). They realize unitary (reducible) representations of the Weyl-Heisenberg group G_{W-H} [2] on the Hilbert space $\mathbb{C}^2 \otimes \mathfrak{H}$.

3.2 Vector coherent states related to $\mathbb{H}_{1SO}^{red}$

This subsection deals with the VCS and QVCS construction for the Hamiltonian $\mathbb{H}_{1SO}^{red}$. Each set of vectors satisfies a resolution of the identity on the Hilbert space $\mathbb{C}^2 \otimes \mathfrak{H}$.

The VCS are given on $\mathbb{C}^2 \otimes \mathfrak{H}$ as

$$|Z; \pm\rangle_1 = (\mathcal{N}(Z))^{-1/2} \sum_{n=0}^{\infty} (R(n))^{-1/2} Z^n |\chi^{\pm} \otimes \Phi_n\rangle \quad (63)$$

where $Z = \text{diag}(z_1, z_2), z_1 = r_1 e^{i\varphi_1}, z_2 = r_2 e^{i\varphi_2}, r_1, r_2 \in [0, \infty); \varphi_1, \varphi_2 \in [0, 2\pi)$ and $R(n) = \text{diag}(\rho_+(n), \rho_-(n))$.

In this case, the normalization factor is given by

$$\mathcal{N}(Z) = e^{r_1^2/\omega_+(\gamma)} + e^{r_2^2/\omega_-(\gamma)}. \quad (64)$$

The states $|Z; \pm\rangle_1$ fulfill on $\mathbb{C}^2 \otimes \mathfrak{H}$ the following resolution of the identity

$$\sum_{\pm} \int_{\mathcal{D}_1 \times \mathcal{D}_2} |Z; \pm\rangle_1 W(r_1, r_2)_1 \langle Z; \pm| d\mu = \mathbb{I}_2 \otimes I_{\mathfrak{H}} \quad (65)$$

where the measure $d\mu$ is given on $\mathcal{D}_1 \times \mathcal{D}_2 = \{[0, \infty) \times [0, 2\pi)\}^2$ by $d\mu(r_1, r_2, \varphi_1, \varphi_2) = \frac{1}{4\pi^2} r_1 dr_1 r_2 dr_2 d\varphi_1 d\varphi_2$. From (53) and (54), the density function is now provided by

$$W(r_1, r_2) = \mathcal{N}(Z) \frac{4e^{-r_1^2/\omega_+(\gamma)} e^{-r_2^2/\omega_-(\gamma)}}{\omega_+(\gamma)\omega_-(\gamma)}. \quad (66)$$

Let $\mathcal{Z} = \text{diag}(z, \bar{z})$ where $z = \tilde{r}e^{i\vartheta}$, $\tilde{r} \in [0, \infty)$, $\vartheta \in [0, 2\pi)$. Consider $\mathfrak{Z} = U\mathcal{Z}U^\dagger$, $U \in SU(2)$ with $U = u_{\phi_1}u_\varphi u_{\phi_2}$; $\phi_1, \phi_2 \in [0, 2\pi)$, $\varphi \in [0, \pi]$, and

$$u_{\phi_j} = \begin{pmatrix} e^{i\frac{\phi_j}{2}} & 0 \\ 0 & e^{-i\frac{\phi_j}{2}} \end{pmatrix}, j = 1, 2, \quad u_\varphi = \begin{pmatrix} \cos \frac{\varphi}{2} & i \sin \frac{\varphi}{2} \\ i \sin \frac{\varphi}{2} & \cos \frac{\varphi}{2} \end{pmatrix}. \quad (67)$$

Introduce the quaternion $\mathfrak{Q} = B(\tilde{r})e^{i\vartheta\tilde{\sigma}(\hat{k})}$ with $B(\tilde{r}) = \tilde{r}\mathbb{I}_2$ and

$$\tilde{\sigma}(\hat{k}) = \begin{pmatrix} \cos \varphi & e^{i\varrho} \sin \varphi \\ e^{-i\varrho} \sin \varphi & -\cos \varphi \end{pmatrix} \quad (68)$$

where $\varphi \in [0, \pi]$, $\vartheta, \varrho \in [0, 2\pi]$. For $\phi_1 = \phi_2 = \varrho$ we get $\mathfrak{Z} = \tilde{r}(\mathbb{I}_2 \cos \vartheta + i\tilde{\sigma}(\hat{k}) \sin \vartheta) = \mathfrak{Q}$. Then, the QVCS are given by

$$|\mathfrak{Q}; \pm\rangle_1 = (\mathcal{N}(\tilde{r}))^{-1/2} \sum_{n=0}^{\infty} (R(n))^{-1/2} \mathfrak{Q}^n |\chi^\pm \otimes \Phi_n\rangle \quad (69)$$

with the normalization factor provided by

$$\mathcal{N}(\tilde{r}) = e^{\tilde{r}^2/\omega_+(\gamma)} + e^{\tilde{r}^2/\omega_-(\gamma)}. \quad (70)$$

They satisfy on $\mathbb{C}^2 \otimes \mathfrak{H}$ the following resolution of the identity:

$$\sum_{\pm} \int_{\mathcal{D}} |\mathfrak{Q}; \pm\rangle_1 W(\tilde{r}) {}_1\langle \mathfrak{Q}; \pm| d\mu = \mathbb{I}_2 \otimes I_{\mathfrak{H}} \quad (71)$$

where $\mathcal{D} = \mathbb{R}^+ \times (0, 2\pi] \times S^2$ and $d\mu = \frac{1}{4\pi} \tilde{r} d\tilde{r} \sin \varphi d\varphi d\varrho d\vartheta$ with

$$W(\tilde{r}) = \begin{pmatrix} \frac{\mathcal{N}(\tilde{r})}{\pi} \frac{(2\omega_c)^{n+1}}{(\omega_-(\gamma))^n} \frac{e^{-\tilde{r}^2 \left[\frac{2\omega_c}{\omega_+(\gamma)\omega_-(\gamma)} \right]}}{\omega_+(\gamma)\omega_-(\gamma)} & 0 \\ 0 & \frac{\mathcal{N}(\tilde{r})}{\pi} \frac{(2\omega_c)^{n+1}}{(\omega_+(\gamma))^n} \frac{e^{-\tilde{r}^2 \left[\frac{2\omega_c}{\omega_+(\gamma)\omega_-(\gamma)} \right]}}{\omega_+(\gamma)\omega_-(\gamma)} \end{pmatrix}. \quad (72)$$

Proof.

$$\begin{aligned}
 & \sum_{\pm} \int_{\mathcal{D}} |\mathfrak{Q}; \pm\rangle_1 W(\tilde{r})_1 \langle \mathfrak{Q}; \pm| d\mu \\
 &= \int_{\mathcal{D}} W(\tilde{r}) \times \\
 & \sum_{\pm} \left| (\mathcal{N}(\tilde{r}))^{-1/2} \sum_{n=0}^{\infty} \frac{\mathfrak{Q}^n \chi^{\pm} \otimes \Phi_n}{\sqrt{R(n)}} \right\rangle \left\langle (\mathcal{N}(\tilde{r}))^{-1/2} \sum_{m=0}^{\infty} \frac{\mathfrak{Q}^m \chi^{\pm} \otimes \Phi_m}{\sqrt{R(m)}} \right| d\mu \\
 &= \sum_{\pm} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \int_{\mathcal{D}} \frac{W(\tilde{r})}{\mathcal{N}(\tilde{r}) \sqrt{R(m)R(n)}} \mathfrak{Q}^n |\chi^{\pm}\rangle \langle \chi^{\pm}| \mathfrak{Q}^{\dagger m} \otimes |\Phi_n\rangle \langle \Phi_m| d\mu \\
 &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \int_{\mathcal{D}} \frac{W(\tilde{r})}{\mathcal{N}(\tilde{r}) \sqrt{R(m)R(n)}} B(\tilde{r})^n e^{in\vartheta\tilde{\sigma}(\hat{k})} \left(\sum_{\pm} |\chi^{\pm}\rangle \langle \chi^{\pm}| \right) \\
 & \times B(\tilde{r})^{m\dagger} e^{-im\vartheta\tilde{\sigma}(\hat{k})\dagger} \otimes |\Phi_n\rangle \langle \Phi_m| d\mu.
 \end{aligned}$$

Using the facts

$$\begin{aligned}
 & \sum_{\pm} |\chi^{\pm}\rangle \langle \chi^{\pm}| = \mathbb{I}_2, \\
 & \tilde{\sigma}(\hat{k})^{\dagger} = \tilde{\sigma}(\hat{k}) \quad \text{and} \\
 & \int_0^{2\pi} \int_0^{2\pi} \int_0^{\pi} e^{i(n-m)\vartheta\tilde{\sigma}(\hat{k})} \sin \varphi d\varphi d\varrho d\vartheta = \begin{cases} 0 & \text{if } m \neq n, \\ 8\pi^2 \mathbb{I}_2 & \text{if } m = n, \end{cases} \quad (73)
 \end{aligned}$$

we obtain

$$\begin{aligned}
 & \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \int_{\mathcal{D}} \frac{W(\tilde{r})}{\mathcal{N}(\tilde{r}) \sqrt{R(m)R(n)}} B(\tilde{r})^n e^{in\vartheta\tilde{\sigma}(\hat{k})} \left(\sum_{\pm} |\chi^{\pm}\rangle \langle \chi^{\pm}| \right) \\
 & \times B(\tilde{r})^{m\dagger} e^{-im\vartheta\tilde{\sigma}(\hat{k})\dagger} \otimes |\Phi_n\rangle \langle \Phi_m| d\mu \\
 &= \sum_{n=0}^{\infty} \int_0^{+\infty} \frac{2\pi W(\tilde{r})}{\mathcal{N}(\tilde{r})} |B(\tilde{r})|^{2n} \text{diag} \left(\frac{1}{\rho_+(n)}, \frac{1}{\rho_-(n)} \right) \otimes |\Phi_n\rangle \langle \Phi_n| \tilde{r} d\tilde{r}. \quad (74)
 \end{aligned}$$

Taking

$$W(\tilde{r}) = \begin{pmatrix} \frac{\mathcal{N}(\tilde{r}) (2\omega_c)^{n+1}}{\pi (\omega_-(\gamma))^n} e^{-\tilde{r}^2 \left[\frac{2\omega_c}{\omega_+(\gamma)\omega_-(\gamma)} \right]} & 0 \\ 0 & \frac{\mathcal{N}(\tilde{r}) (2\omega_c)^{n+1}}{\pi (\omega_+(\gamma))^n} e^{-\tilde{r}^2 \left[\frac{2\omega_c}{\omega_+(\gamma)\omega_-(\gamma)} \right]} \end{pmatrix} \quad (75)$$

we get from (74) that

$$\sum_{\pm} \int_{\mathcal{D}} |\mathfrak{Q}; \pm\rangle_1 W(\tilde{r})_1 \langle \mathfrak{Q}; \pm| d\mu$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \text{diag} \left[\frac{(2\omega_c)^{n+1}}{(\omega_-(\gamma))^n} \int_0^{+\infty} \frac{2\tilde{r}^{2n}}{\rho_+(n)} \frac{e^{-\tilde{r}^2 \left[\frac{2\omega_c}{\omega_+(\gamma)\omega_-(\gamma)} \right]}}{\omega_+(\gamma)\omega_-(\gamma)} \tilde{r} d\tilde{r}, \right. \\
&\quad \left. \frac{(2\omega_c)^{n+1}}{(\omega_+(\gamma))^n} \int_0^{+\infty} \frac{2\tilde{r}^{2n}}{\rho_-(n)} \frac{e^{-\tilde{r}^2 \left[\frac{2\omega_c}{\omega_+(\gamma)\omega_-(\gamma)} \right]}}{\omega_+(\gamma)\omega_-(\gamma)} \tilde{r} d\tilde{r} \right] \times \mathbb{I}_2 \otimes |\Phi_n\rangle\langle\Phi_n| \\
&= \mathbb{I}_2 \otimes \sum_{n=0}^{\infty} |\Phi_n\rangle\langle\Phi_n| = \mathbb{I}_2 \otimes I_{\mathfrak{H}}
\end{aligned}$$

where the following moment problems

$$\frac{(2\omega_c)^{n+1}}{(\omega_{\mp}(\gamma))^n} \int_0^{+\infty} \frac{2\tilde{r}^{2n}}{\rho_{\pm}(n)} \frac{e^{-\tilde{r}^2 \left[\frac{2\omega_c}{\omega_+(\gamma)\omega_-(\gamma)} \right]}}{\omega_+(\gamma)\omega_-(\gamma)} \tilde{r} d\tilde{r} = 1 \quad (76)$$

are satisfied.

□

3.3 Physical properties

In this section, we study the physical properties of the QVCS (47) and (69). The generalized annihilation, creation and number operators are introduced and allow the definitions of the quadrature operators. Their actions on the QVCS (47) and (69) are then derived. Next, the expectations of these operators are given in the second subsection. The two last subsections discuss the signal-to-quantum-noise ratio in the QVCS and their time evolution.

3.3.1 Generalized annihilation, creation and number operators

Consider, on the Hilbert space $\mathfrak{H}$, the actions of the generalized annihilation, creation and number operators on the basis vectors $\{|\Phi_n\rangle\}_{n=0}^{\infty}$ as follows:

$$a|\Phi_n\rangle = \sqrt{x_n}|\Phi_{n-1}\rangle, \quad a^\dagger|\Phi_n\rangle = \sqrt{x_{n+1}}|\Phi_{n+1}\rangle, \quad N'|\Phi_n\rangle = x_n|\Phi_n\rangle, \quad (77)$$

where $x_n = n\omega_{(\pm)}(\cdot)$ such that $x_n! = n!(\omega_{(\pm)}(\cdot))^n$, with $x_0! = 1$ by definition.

The corresponding operators for the QVCS (47) are given on $\mathbb{C}^2 \otimes \mathfrak{H}$ by

$$A = \mathbb{I}_2 \otimes a, \quad A^\dagger = \mathbb{I}_2 \otimes a^\dagger, \quad N = \mathbb{I}_2 \otimes N' \quad (78)$$

with the following realization:

$$A|\mathbf{q}; \pm\rangle_1 = \mathbf{q}|\mathbf{q}; \pm\rangle_1, \quad (79)$$

$$A^\dagger|\mathbf{q}; \pm\rangle_1 = (\mathcal{N}_{(\pm)}(|\mathbf{q}|))^{-1/2} \sum_{n=0}^{\infty} \mathbf{q}^n \sqrt{\frac{x_{n+1}}{x_n!}} |\chi^\pm \otimes \Phi_{n+1}\rangle, \quad (80)$$

$$N|\mathbf{q}; \pm\rangle_1 = (\mathcal{N}_{(\pm)}(|\mathbf{q}|))^{-1/2} \sum_{n=0}^{\infty} \frac{\mathbf{q}^n x_n}{\sqrt{x_n!}} |\chi^\pm \otimes \Phi_n\rangle. \quad (81)$$

Furthermore, we have

$$\begin{aligned} [A, A^\dagger]|\mathbf{q}; \pm\rangle_1 &= (\mathcal{N}_{(\pm)}(|\mathbf{q}|))^{-1/2} \sum_{n=0}^{\infty} \mathbf{q}^n \frac{(x_{n+1} - x_n)}{\sqrt{x_n!}} |\chi^\pm \otimes \Phi_n\rangle, \\ [Q, P]|\mathbf{q}; \pm\rangle_1 &= i (\mathcal{N}_{(\pm)}(|\mathbf{q}|))^{-1/2} \sum_{n=0}^{\infty} \mathbf{q}^n \frac{(x_{n+1} - x_n)}{\sqrt{x_n!}} |\chi^\pm \otimes \Phi_n\rangle \\ &= i[A, A^\dagger]|\mathbf{q}; \pm\rangle_1 \end{aligned} \quad (82)$$

where

$$Q = \frac{A + A^\dagger}{\sqrt{2}}, \quad P = \frac{A - A^\dagger}{i\sqrt{2}}. \quad (83)$$

Consider the following annihilation, creation and number operators $\mathcal{A}, \mathcal{A}^\dagger, \mathcal{N}$ given on $\mathbb{C}^2 \otimes \mathfrak{H}$ by

$$\mathcal{A}|\chi^\pm \otimes \Phi_n\rangle = \sqrt{n\omega_\pm(\gamma)} |\chi^\pm \otimes \Phi_{n-1}\rangle,$$

$$\mathcal{A}^\dagger|\chi^\pm \otimes \Phi_n\rangle = \sqrt{(n+1)\omega_\pm(\gamma)} |\chi^\pm \otimes \Phi_{n+1}\rangle, \quad (84)$$

$$\mathcal{N}|\chi^\pm \otimes \Phi_n\rangle = n\omega_\pm(\gamma) |\chi^\pm \otimes \Phi_n\rangle, \quad \mathcal{N} = \mathcal{A}^\dagger \mathcal{A} \quad (85)$$

and introduce the quadrature operators $\mathcal{Q}$ and $\mathcal{P}$ given on $\mathbb{C}^2 \otimes \mathfrak{H}$ as

$$\mathcal{Q} = \frac{1}{\sqrt{2}}(\mathcal{A} + \mathcal{A}^\dagger), \quad \mathcal{P} = \frac{1}{i\sqrt{2}}(\mathcal{A} - \mathcal{A}^\dagger). \quad (86)$$

For the QVCS (69), we obtain on $\mathbb{C}^2 \otimes \mathfrak{H}$ the following relations:

$$\mathcal{A}|\mathfrak{Q}; \pm\rangle_1 = (\mathcal{N}(\tilde{r}))^{-1/2} \sum_{n=0}^{\infty} \frac{\mathfrak{Q}^{n+1}}{\sqrt{n!(\omega_\pm(\gamma))^n}} |\chi^\pm \otimes \Phi_n\rangle, \quad (87)$$

$$\mathcal{A}^\dagger|\mathfrak{Q}; \pm\rangle_1 = (\mathcal{N}(\tilde{r}))^{-1/2} \sum_{n=0}^{\infty} \mathfrak{Q}^n \sqrt{\frac{(n+1)\omega_\pm(\gamma)}{n!(\omega_\pm(\gamma))^n}} |\chi^\pm \otimes \Phi_{n+1}\rangle, \quad (88)$$

$$\mathcal{N}|\mathfrak{Q}; \pm\rangle_1 = (\mathcal{N}(\tilde{r}))^{-1/2} \sum_{n=0}^{\infty} \mathfrak{Q}^n \frac{n\omega_\pm(\gamma)}{\sqrt{n!(\omega_\pm(\gamma))^n}} |\chi^\pm \otimes \Phi_n\rangle. \quad (89)$$

3.3.2 Expectations and uncertainty relation

The expectations of the operators $A, A^\dagger, N, P$ and Q for the QVCS (47) are derived in the quantum state $|\mathbf{q}; \pm\rangle_1$ as follows:

$$\langle A \rangle_{|\mathbf{q}; \pm\rangle_1} = {}_1\langle \mathbf{q}; \pm | A | \mathbf{q}; \pm \rangle_1 = \frac{1}{2} \langle \chi^\pm | \mathbf{q} \chi^\pm \rangle = \frac{r}{2} (\cos(\eta) \pm i \sin(\eta) \cos(\phi)), \quad (90)$$

$$\langle A^\dagger \rangle_{|\mathbf{q}; \pm\rangle_1} = {}_1\langle \mathbf{q}; \pm | A^\dagger | \mathbf{q}; \pm \rangle_1 = \frac{1}{2} \langle \chi^\pm | \mathbf{q} | \chi^\pm \rangle = \frac{r}{2} (\cos(\eta) \mp i \sin(\eta) \cos(\phi)), \quad (91)$$

$$\begin{aligned} \langle Q \rangle_{|\mathbf{q}; \pm\rangle_1} &= \frac{r}{\sqrt{2}} \cos(\eta), \quad \langle P \rangle_{|\mathbf{q}; \pm\rangle_1} = \pm \frac{r}{\sqrt{2}} \sin(\eta) \cos(\phi), \\ \langle N \rangle_{|\mathbf{q}; \pm\rangle_1} &= \frac{1}{2} |\mathbf{q}|^2 = \frac{r^2}{2}, \end{aligned} \quad (92)$$

$$\langle Q^2 \rangle_{|\mathbf{q}; \pm\rangle_1} = \frac{1}{4} (4r^2 \cos^2(\eta) + 1), \quad \langle P^2 \rangle_{|\mathbf{q}; \pm\rangle_1} = \frac{1}{4} (4r^2 \sin^2(\eta) + 1). \quad (93)$$

There results the following uncertainty relation for the QVCS (47):

$$(\Delta Q)_{|\mathbf{q}; \pm\rangle_1}^2 (\Delta P)_{|\mathbf{q}; \pm\rangle_1}^2 \geq \frac{1}{4} \left| \left(\mathcal{N}_{(\pm)}(|\mathbf{q}|) \right)^{-1} \sum_{n=0}^{\infty} \frac{(x_{n+1} - x_n)}{x_n!} \langle \mathbf{q}^n \chi^\pm | \mathbf{q}^n \chi^\pm \rangle \right|^2 \quad (94)$$

which is reduced, in the particular situation where $x_n = n$, $\mathcal{N}_{(\pm)}(|\mathbf{q}|) = 2e^{r^2}$, to a simpler expression:

$$(\Delta Q)_{|\mathbf{q}; \pm\rangle_1}^2 (\Delta P)_{|\mathbf{q}; \pm\rangle_1}^2 \geq \frac{1}{4} \left(\frac{e^{-r^2}}{2} \sum_{n=0}^{\infty} \frac{r^{2n}}{n!} \right)^2 = \frac{1}{16}. \quad (95)$$

Set

$$\mathcal{F}_+(\tilde{r}) = \frac{e^{\tilde{r}^2/\omega_+(\gamma)}}{e^{\tilde{r}^2/\omega_+(\gamma)} + e^{\tilde{r}^2/\omega_-(\gamma)}}, \quad \mathcal{F}_-(\tilde{r}) = \frac{e^{\tilde{r}^2/\omega_-(\gamma)}}{e^{\tilde{r}^2/\omega_+(\gamma)} + e^{\tilde{r}^2/\omega_-(\gamma)}}. \quad (96)$$

In the case of the QVCS (69), we get in the quantum states $|\mathbf{Q}; \pm\rangle_1$ the following relations

$$\langle A \rangle_{|\mathbf{Q}; \pm\rangle_1} = \tilde{r} \mathcal{F}_\pm(\tilde{r}) (\cos(\vartheta) \pm i \sin(\vartheta) \cos(\varphi)),$$

$$\langle \mathcal{A}^\dagger \rangle_{|\Omega; \pm\rangle_1} = \tilde{r} \mathcal{F}_\pm(\tilde{r}) (\cos(\vartheta) \mp i \sin(\vartheta) \cos(\varphi)), \quad (97)$$

$$\langle \mathcal{Q} \rangle_{|\Omega; \pm\rangle_1} = \tilde{r} \sqrt{2} \mathcal{F}_\pm(\tilde{r}) \cos(\vartheta), \quad \langle \mathcal{P} \rangle_{|\Omega; \pm\rangle_1} = \pm \tilde{r} \sqrt{2} \mathcal{F}_\pm(\tilde{r}) \sin(\vartheta) \cos(\varphi), \quad (98)$$

$$\langle \mathcal{A} \mathcal{A}^\dagger \rangle_{|\Omega; \pm\rangle_1} = (\tilde{r}^2 + \omega_\pm(\gamma)) \mathcal{F}_\pm(\tilde{r}), \quad \langle \mathcal{N} \rangle_{|\Omega; \pm\rangle_1} = \tilde{r}^2 \mathcal{F}_\pm(\tilde{r}). \quad (99)$$

The expectations of the operators $\mathcal{Q}^2$ and $\mathcal{P}^2$ are obtained as

$$\begin{aligned} \langle \mathcal{Q}^2 \rangle_{|\Omega; \pm\rangle_1} &= \frac{1}{2} [4\tilde{r}^2 \cos^2(\vartheta) + \omega_\pm(\gamma)] \mathcal{F}_\pm(\tilde{r}), \\ \langle \mathcal{P}^2 \rangle_{|\Omega; \pm\rangle_1} &= \frac{1}{2} [4\tilde{r}^2 \sin^2(\vartheta) + \omega_\pm(\gamma)] \mathcal{F}_\pm(\tilde{r}). \end{aligned} \quad (100)$$

There follow the dispersions given by

$$\begin{aligned} (\Delta \mathcal{Q})_{|\Omega; \pm\rangle_1}^2 &= 2\tilde{r}^2 \mathcal{F}_+(\tilde{r}) \mathcal{F}_-(\tilde{r}) \cos^2(\vartheta) + \omega_\pm(\gamma) \frac{\mathcal{F}_\pm(\tilde{r})}{2}, \\ (\Delta \mathcal{P})_{|\Omega; \pm\rangle_1}^2 &= 2\tilde{r}^2 \mathcal{F}_\pm(\tilde{r}) \sin^2(\vartheta) [1 - \mathcal{F}_\pm(\tilde{r}) \cos^2(\varphi)] \\ &\quad + \omega_\pm(\gamma) \frac{\mathcal{F}_\pm(\tilde{r})}{2}. \end{aligned} \quad (101)$$

If we take $\omega_+(\gamma) = \omega_-(\gamma) = \omega(\gamma)$, then $\mathcal{F}_+(\tilde{r}) = \frac{1}{2} = \mathcal{F}_-(\tilde{r})$ and

$$\begin{aligned} (\Delta \mathcal{Q})_{|\Omega; \pm\rangle_1}^2 &= \frac{1}{4} [\omega(\gamma) + 2\tilde{r}^2 \cos^2(\vartheta)], \\ (\Delta \mathcal{P})_{|\Omega; \pm\rangle_1}^2 &= \frac{1}{4} \left[\omega(\gamma) + 4\tilde{r}^2 \sin^2(\vartheta) \left(1 - \frac{1}{2} \cos^2(\varphi) \right) \right]. \end{aligned} \quad (102)$$

In the case where $\tilde{r}^2 \cos^2(\vartheta) \geq \frac{1}{2}$ and $\tilde{r}^2 \sin^2(\vartheta) \geq \frac{1}{2}$, the above relations can be simplified to yield:

$$(\Delta \mathcal{Q})_{|\Omega; \pm\rangle_1}^2 \geq \frac{1}{4} [\omega(\gamma) + 1] \geq \frac{1}{4}, \quad (\Delta \mathcal{P})_{|\Omega; \pm\rangle_1}^2 \geq \frac{1}{4} [\omega(\gamma) + 1] \geq \frac{1}{4} \quad (103)$$

leading to $(\Delta \mathcal{Q})_{|\Omega; \pm\rangle_1}^2 (\Delta \mathcal{P})_{|\Omega; \pm\rangle_1}^2 \geq \frac{1}{16}$.

3.3.3 Signal-to-quantum-noise ratio in the QVCS

Signal-to-quantum-noise ratio (SNR) is relevant when studying, for example, low-field magnetoresistance in carbon-doped p -type GaAs/AlGaAs heterostructures [7], and the magnetoconductance in InSb/InAlSb quantum wells in which Rashba coupling occurs [6]. For a normalized state $|\phi\rangle$, in terms of the self-adjoint quadrature operator Q , the SNR is defined as [11, 14]

$$\sigma_{|\phi\rangle} := \frac{\langle Q \rangle_{|\phi\rangle}^2}{(\Delta Q)_{|\phi\rangle}^2}. \quad (104)$$

A low SNR indicates that the noise dominates the measurements in the state $|\phi\rangle$. In opposite, a high SNR implies that the measurements are relatively clean in the state $|\phi\rangle$. Evaluate now the SNR for the Hamiltonian $\mathbb{H}_{1SO}^{red}$.

- In the QVCS $|\Omega; \pm\rangle_1$, by definition

$$\sigma_{|\Omega; \pm\rangle_1} := \frac{\langle Q \rangle_{|\Omega; \pm\rangle_1}^2}{(\Delta Q)_{|\Omega; \pm\rangle_1}^2} \quad (105)$$

and from the relations (98) and (101) we obtain the relation:

$$\sigma_{|\Omega; \pm\rangle_1} = \frac{2\tilde{r}^2(\mathcal{F}_{\pm}(\tilde{r}))^2 \cos^2(\vartheta)}{2\tilde{r}^2 \mathcal{F}_+(\tilde{r}) \mathcal{F}_-(\tilde{r}) \cos^2(\vartheta) + \omega_{\pm}(\gamma) \frac{\mathcal{F}_{\pm}(\tilde{r})}{2}}. \quad (106)$$

If $\omega_+(\gamma) = \omega_-(\gamma) = \omega(\gamma)$, i.e. $\mathcal{F}_+(\tilde{r}) = \frac{1}{2} = \mathcal{F}_-(\tilde{r})$, the expression (106) becomes

$$\sigma_{|\Omega; \pm\rangle_1} = \frac{2\tilde{r}^2 \cos^2(\vartheta)}{2\tilde{r}^2 \cos^2(\vartheta) + \omega(\gamma)}. \quad (107)$$

- In the QVCS $|\mathbf{q}; \pm\rangle_1$, we get, by using (92) and (93),

$$\begin{aligned} \sigma_{|\mathbf{q}; \pm\rangle_1} &:= \frac{\langle Q \rangle_{|\mathbf{q}; \pm\rangle_1}^2}{(\Delta Q)_{|\mathbf{q}; \pm\rangle_1}^2} \\ &= \frac{2r^2 \cos^2(\eta)}{2r^2 \cos^2(\eta) + 1}. \end{aligned} \quad (108)$$

Let us take $\tilde{r} = r$ and $\vartheta = \eta$. From (107) and (108), if $\omega(\gamma) = 1$, then $\sigma_{|\mathfrak{Q}; \pm\rangle_1} = \sigma_{|\mathfrak{q}; \pm\rangle_1}$. Furthermore, if $\omega(\gamma) > 1$, then $\sigma_{|\mathfrak{Q}; \pm\rangle_1} < \sigma_{|\mathfrak{q}; \pm\rangle_1}$ implying that the noise dominates the measurements in the QVCS $|\mathfrak{Q}; \pm\rangle_1$ than in the QVCS $|\mathfrak{q}; \pm\rangle_1$. Besides, if $\omega(\gamma) < 1$, then $\sigma_{|\mathfrak{Q}; \pm\rangle_1} > \sigma_{|\mathfrak{q}; \pm\rangle_1}$ indicating that the measurements are relatively clean in the QVCS $|\mathfrak{Q}; \pm\rangle_1$ than in the QVCS $|\mathfrak{q}; \pm\rangle_1$.

3.3.4 Time evolution

The QVCS given in (47) and (69) can be equipped with a parameter τ expressing their time evolution as follows:

$$\begin{aligned} |\mathfrak{q}; \pm, \tau\rangle_1 &:= \mathbb{U}(\tau)|\mathfrak{q}; \pm\rangle_1 \\ &= (\mathcal{N}_{(\pm)}(|\mathfrak{q}|))^{-1/2} \sum_{n=0}^{\infty} \frac{\mathfrak{q}^n}{\sqrt{\rho_{(\pm)}(n)}} e^{-i\tau\mathcal{E}_n^\pm} |\chi^\pm \otimes \Phi_n\rangle, \end{aligned} \quad (109)$$

and

$$\begin{aligned} |\mathfrak{Q}; \pm, \tau\rangle_1 &:= \mathbb{U}(\tau)|\mathfrak{Q}; \pm\rangle_1 \\ &= (\mathcal{N}(\tilde{r}))^{-1/2} \sum_{n=0}^{\infty} (R(n))^{-1/2} \mathfrak{Q}^n e^{-i\tau\mathcal{E}_n^\pm} |\chi^\pm \otimes \Phi_n\rangle \end{aligned} \quad (110)$$

relatively to the time evolution operator $\mathbb{U}(t) := e^{-it\mathbb{H}_{so}^{red}}$ that fulfills the property:

$$\begin{aligned} \mathbb{U}(t)|\mathfrak{q}; \pm, \tau\rangle_1 &:= e^{-it\mathbb{H}_{so}^{red}} |\mathfrak{q}; \pm, \tau\rangle_1 = |\mathfrak{q}; \pm, \tau + t\rangle_1, \\ \mathbb{U}(t)|\mathfrak{Q}; \pm, \tau\rangle_1 &:= e^{-it\mathbb{H}_{so}^{red}} |\mathfrak{Q}; \pm, \tau\rangle_1 = |\mathfrak{Q}; \pm, \tau + t\rangle_1. \end{aligned} \quad (111)$$

Conclusion

In section 2 of this work, we have first investigated the coherent properties of a Hamiltonian describing an electron gas in a constant magnetic field with Zeeman coupling interaction. Then a more general Hamiltonian with both Rashba and Dresselhaus spin-orbit interactions has been examined in details, including a generalization of these interactions. In section 3, we have provided the canonical QVCS related to the studied Hamiltonian, discussed the VCS construction and their connection with the QVCS. Interesting physical features of the constructed vectors, generated by the introduction of generalized

annihilation and creation operators together with the definition on the Hilbert space $\mathbb{C}^2 \otimes \mathfrak{H}$ of quadrature operators have been deduced; expectations of these operators and uncertainty relations have been computed. Furthermore, signal-to-quantum-noise ratio in the constructed QVCS as well as their time-evolution have been discussed. The method of construction developed in this work can be extended to analogous study in the situation of magnetoresistivity and inelastic light-scattering experiments [8], where the interplay between Rashba, Dresselhaus, and Zeeman interactions in a quantum well subjected to an external magnetic field has been studied.

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