On a Wave Solution of $f(R)$ Theory of Gravity in Peres Space-Time

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Abstract

Modified theories of Gravity have gained increased attention over the last few years due to combined motivation coming from various branches of physics. Among various alternatives to the Einstein theory of gravity, especially $f(R)$ theories of gravity have received more importance due to number of interesting results in cosmology and astrophysics. Pandey [6] gave an $f(R)$ theory of gravity to obtain conformally invariant gravitational waves in which field equations have the form given by (3). In this paper we have obtained a wave solution of $f(R)$ theory of gravity in a special case of generalized Peres spacetime taken by Pandey [1] for $n = 2$. Some important features of the solution is discussed and is compared with the form of solution obtained by Pandey [1].

Keywords: Conformal invariance, Peres space-time, Gravitational wave

1 Introduction

One of the most remarkable non-Newtonian features of the relativistic theory of gravitation is the possibility of existence of gravitational waves. General relativity predicts that ripples in space-time curvature can propagate with the speed of light through otherwise empty space. These ripples are gravitational waves. They are meaningfully comparable with electromagnetic waves except
for their non-conformal invariance. This is because the gravitational wave equations in the background of Friedman universe can be reduced to \([2, 3]\)

\[
\frac{a''}{a} + \mu \left( n^2 - \frac{a''}{a} \right) = 0,
\]

where \(a(\eta)\) is the scale factor of Friedman universe and \(\eta\) is the wave number such that the physical wave length is given by \((\frac{2\pi a}{n})\). The effective potential \(\frac{a''}{a}\) in (1) where prime denotes derivatives with respect to \(\eta\) and is related to cosmic time \(t\) by \(cdt = a(\eta) d\eta\) distinguishes this equation from the ordinary wave equations in the Minkowski world. The fact that \(\frac{a''}{a}\) is non-zero except for \(a = \text{const}\), and \(a = a_0\eta\) is a manifestation of the so-called conformal non-invariance of gravitational wave equations.

Hence gravitational waves are an inevitable consequence of Einstein theory of gravitation derivable from Hilbert Lagrangian. It is, therefore, necessary to change Hilbert Lagrangian to modify Einstein field equations to obtain conformally invariant gravitational waves equations.

Over the years alternative theories have been postulated. For example, Weyl [4] suggested the invariant \(R^2\) to make the field action scale invariant (and to unify gravitation with electromagnetism). Another attractive alternative suggested is \(R^3/2\) so that the coupling constant in matter Lagrangian is dimensionless.

Because of non-conformal invariance of gravitational wave equations, to nullify the manifestation of gravitation as evident from equation (1) without any special choice of scale factor, Pandey [1] gave an \(f(R)\) theory of gravity considering Lagrangian in the form \([3, 5]\)

\[
L = R + \sum_{n=2}^{N} C_n \left( \frac{l^2 R}{6l^2} \right)^n \quad \text{or, equivalently,} \quad L = R + \sum_{n=2}^{N} a_n R_n, \tag{2}
\]

where \(l\) is the characteristic length and \(C_n\) are the dimensionless arbitrary coefficients corresponding to the values of \(n\). They are introduced to nullify the manifestation of gravitation. The values \(n = 0\) and \(n = 1\) result in Hilbert Lagrangian, that is, Einstein theory. Therefore \(n\) begins from \(n = 2\) onwards.

By applying variational principle to this action, Pandey [6] obtained the following field equations:

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \sum_{n=2}^{N} n a_n R_{\mu\nu}^{n-1} \left[ R_{\mu\nu} - \frac{R g_{\mu\nu}}{2n} - \frac{n(n-1)}{R} (R_{\mu;\nu} - g_{\mu\nu} \Box R) \right. \\
- \left\{ \frac{(n-1)(n-2)}{R^2} \right\} (R_{\alpha;\mu} R_{\nu;\alpha} - g_{\mu\nu} R_{\alpha;\alpha}) \right] = k T_{\mu\nu}. \tag{3}
\]
Here $T_{\mu\nu} = (-g)^{\frac{1}{2}} \left( \frac{\delta L_s}{\delta g^{\mu\nu}} \right)$ stands for the energy momentum tensor responsible for the production of the gravitational potential $g_{\mu\nu}$. It can be seen that $T_{\mu\nu} = 0$ holds for these field equations as it is in case of Einstein general relativity.

2 Peres Spacetime

Peres [7] considered a space-time represented by the metric

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2 - 2f(x, y, z - t)(dz - dt)^2, \quad (4)$$

which can also be put in the form

$$ds^2 = -dx^2 - dy^2 - (1 - E)dz^2 - 2Ezdt + (1 + E)dt^2, \quad (5)$$

by replacing $-2f$ by $E$. Evidently $E$ is a function of $x, y$ and $z - t$. The space-time of Peres belong to second class of Petrov’s classification and the source of $f$ (or $E$), as interpreted by Peres, is a null electromagnetic field. This space-time has been studied by Takeno [8] in detail concerning its phase velocity, coordinate conditions, energy momentum pseudo tensor, etc., and also the solutions of various field equations obtained in this space-time have been called plane wave-like.

In consideration of the above Pandey [1] proposed a generalization of Peres space-time represented by the metric (4), (5), above as

$$ds^2 = -Adx^2 - Bdy^2 - (1 - E)dz^2 - 2Ezdt + (1 + E)dt^2, \quad (6)$$

where $A = A(z, t)$, $B = B(z, t)$, $E = E(x, y, z, t)$. We consider the metric (6) in the following form

$$ds^2 = -dx^2 - dy^2 - (1 - E)dz^2 - 2Ezdt + (1 + E)dt^2, \quad (7)$$

where $A = 1$, $B = 1$, $E = E(x, y, z, t)$.

3 Field Equations

Higher order field equation (3) for $n = 2$ reduces to the following form

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + 2a_2 R \left[ R_{\mu\nu} - \frac{Rg_{\mu\nu}}{4} - \frac{2}{R} (R_{\mu;\nu} - g_{\mu\nu} \Box R) \right] = kT_{\mu\nu}. \quad (8)$$
The field equations (8) with $T_{\mu\nu} = 0$ for the metric (7) are

$$\frac{1}{2}\left[-\partial_y^2 E - \partial_x^2 E - 4a_2 U \left(\frac{1}{4}(1 + E)U + \frac{1}{2}(-\partial_y^2 E - \partial_x^2 E - (1 + E)U) + \left[-((1 + E)\partial_x E + (2 + E)\partial_t E)(-\partial_x U) - 2\partial_t^2 U + \partial_y E\partial_y U + \partial_x E\partial_x U + (\partial_t E + E(\partial_x E + \partial_t E))(\partial_t U) + 2(1 + E)(\partial_y^2 U + \partial_x^2 U + (\partial_x^2 - \partial_t^2)U)\right] / U\right)\right] = 0. \tag{9}$$

$$\frac{1}{2}\left[-\partial_x \partial_z E - \partial_x \partial_t E + 2a_2 \left(-2\partial_x^2 E\partial_x E + 2(\partial_x \partial_z E)(\partial_t \partial_x E) - 6(\partial_x E)(\partial_t \partial_z E) + 2(\partial_t \partial_z E)(\partial_t \partial_z E) + (\partial_t \partial_z E)(\partial_t \partial_z E) + 4\partial_t \partial_z^2 E + (\partial_t \partial_z E)(\partial_t \partial_z E) - 6(\partial_t E)(\partial_z \partial_z E) - 8\partial_z \partial_z \partial_z^2 E - 2(\partial_x E)(\partial_x^3 E) + 4\partial_x^3 \partial_x E\right)\right] = 0. \tag{10}$$

$$\frac{1}{2}(-U)\left(1 + a_2(-U - 8(\partial_y^2 U + 2\partial_x^2 U + (\partial_x^2 - \partial_t^2)U)/U)\right) = 0. \tag{11}$$

$$\frac{1}{2}\left[-\partial_y \partial_z E - \partial_y \partial_t E + 2a_2 \left(-2\partial_y^2 E\partial_y E + 2(\partial_y \partial_z E)(\partial_t \partial_y E) - 6(\partial_y E)(\partial_t \partial_z E) + 2(\partial_t \partial_z E)(\partial_t \partial_y E) + (\partial_t \partial_y E)(\partial_t \partial_y E) + 4\partial_t \partial_y^2 E + (\partial_t \partial_y E)(\partial_t \partial_y E) - 6(\partial_y E)(\partial_z \partial_y E) - 8\partial_y \partial_y \partial_y E - 2(\partial_y E)(\partial_y^3 E) + 4\partial_y^3 \partial_y E\right)\right] = 0. \tag{12}$$

$$4a_2 \left[\partial_x \partial_y \partial_z^2 E + 2\partial_t \partial_x \partial_y \partial_z E + \partial_t^2 \partial_x \partial_y E\right] = 0. \tag{13}$$

$$\frac{1}{2}(-U)\left(1 + a_2(-U - 8(2\partial_y^2 U + \partial_x^2 U + (\partial_x^2 - \partial_t^2)U)/U)\right) = 0. \tag{14}$$

$$\frac{1}{16}\left[\partial_y^2 E + \partial_x^2 E - a_2 U \left(-EU + 2(\partial_y^2 E + \partial_x^2 E + EU) + \left[4\left(-2\partial_t \partial_x U - \partial_t E + E(\partial_x E + \partial_t E))\partial_t U - \partial_y E\partial_y U - \partial_x E\partial_x U - ((-1 + E)(\partial_x E + E(\partial_x E + \partial_t E))\partial_t U - 2E(\partial_y^2 U + \partial_x^2 U + (\partial_x^2 - \partial_t^2)U)\right)\right]/U\right]\right] = 0. \tag{15}$$
\[
\frac{1}{2} \left[ \frac{1}{2} \left[ \begin{array}{c}
\partial_x \partial_z E + \partial_x \partial_t E + 2a_2 \left( 2\partial_z^2 E \partial_x E + 4\partial_z^2 \partial_x E - 2(\partial_x \partial_z E)(\partial_t \partial_z E)
\right)
+ 6(\partial_x E)(\partial_z \partial_x^2 E) - 2(\partial_x \partial_z E)(\partial_t \partial_z E) - (\partial_z^2 E)(\partial_x \partial_z E + \partial_z \partial_t E)
+ 8\partial_t \partial_x \partial_z^2 E - (\partial_x \partial_z E)(\partial_t \partial_z E) - (\partial_t \partial_z E)(\partial_t \partial_z E) + 6(\partial_t E)
\right] + 4\partial_x \partial_z \partial_t^2 E + 2(\partial_x E)(\partial_t^3 E)\right] = 0.
\]

Adding equations (17) and (12), since
\[
\frac{1}{2} \left[ \partial_y \partial_z E + \partial_y \partial_t E + 2a_2 \left( 2\partial_z^2 E \partial_y E + 4\partial_z^2 \partial_y E - 2(\partial_y \partial_z E)(\partial_t \partial_z E)
\right)
+ 6(\partial_y E)(\partial_t \partial_z^2 E) - 2(\partial_y \partial_z E)(\partial_t \partial_z E) - (\partial_z^2 E)(\partial_y \partial_z E + \partial_y \partial_t E)
+ 8\partial_t \partial_y \partial_z^2 E - (\partial_y \partial_z E)(\partial_t \partial_z E) - (\partial_t \partial_y E)(\partial_t \partial_z E) + 6(\partial_t E)
\right] + 4\partial_y \partial_z \partial_t^2 E + 2(\partial_y E)(\partial_t^3 E)\right] = 0.
\]

Adding equations (18) and subtracting twice equation (15) from it we obtain
\[
\left( \partial_z \partial_t^2 E + 4\partial_y \partial_z \partial_t^2 E + 2(\partial_y E)(\partial_t^3 E) \right] = 0.
\]

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\[
\frac{1}{2} \left[ \left( \partial_x^2 E - \partial_z^2 E - 4a_2 U \left( \frac{1}{4}(-1 + E)U + \frac{1}{2}(-\partial_y^2 E - \partial_z^2 E - (-1 + E)U \right)
+ \left[ -2\partial_x^2 U - ((-1 + E)\partial_x E + E\partial_t E)(\partial_t U) + \partial_x E\partial_x U
+ \partial_x E\partial_t U - ((-2 + E)\partial_x E + (-1 + E)\partial_t E)(\partial_t U) + 2
\right.
\right.
\left.
\left.
\left. (-1 + E) \left( \partial_y^2 U + \partial_z^2 U + (\partial_z^2 - \partial_x^2) U \right) \right] / U \right) \right] = 0.
\]

where \( U = (\partial_x^2 + 2\partial_y \partial_x + \partial_t^2) E \), \( \partial_x = \partial \partial_x \), \( \partial_y = \partial \partial_y \), \( \partial_z = \partial \partial_z \), \( \partial_t = \partial \partial_t \), etc.

We now determine \( E \) such that it satisfies all the above field equations. Adding equation (16) and (10), since \( a_2 \neq 0 \) we obtain
\[
\partial_x (\partial_x + \partial_t)^3 E = 0.
\]

Adding equations (17) and (12), since \( a_2 \neq 0 \) we obtain
\[
\partial_y (\partial_x + \partial_t)^3 E = 0.
\]

Equation (19) and (20) yields on integration form of \( E \) as
\[
E = f_1(x, y, z) + z f_2(x, y, z) + z^2 f_3(x, y, z) + f(z, t) \quad (Z = z - t).
\]

Adding equations (9) and (18) and subtracting twice equation (15) from it we obtain
\[
(\partial_z + \partial_t)^2 U = 0.
\]
Equation (22) in view of (21) yield the form of $E$ as

\[ E = f_1(x, y, Z) + z f_2(x, y, Z) + z^2 f_3(x, y, Z) + z^3 f_4(Z). \] (23)

Subtracting equation (14) from equation (11), since $a_2 \neq 0$ we obtain

\[ (\partial_x^2 - \partial_y^2) U = 0. \] (24)

Also equation (13), since $a_2 \neq 0$ can be written as

\[ \partial_x \partial_y U = 0. \] (25)

If we restrict $f_3$ in its dependence on $x$ and $y$ then the form of $E$ given by (23) becomes

\[ E = f_1(x, y, Z) + z f_2(x, y, Z) + z^2 f_3(Z) + z^3 f_4(Z). \] (26)

which clearly satisfies equation (24) and (25).

Hence when the form of $E$ is given by (26), then the field equations (9)—(18) are identically satisfied.

4 Conclusion

Equation (22) is obtained assuming that $U = (\partial_x^2 + 2 \partial_x \partial_t + \partial_t^2) E$ is non zero. It is evident that the value of $E$ given by (26) satisfies this condition.

Pandey [1] obtained the wave solution of the Einstein vacuum field equations in generalized Peres space-time, in which the form of $E$ obtained was

\[ E = f_1(x, y, Z) + f_2(Z) + z f_3(Z). \]

Our solution of $f(R)$ gravity contains extra terms in the form $z f(x, y, Z)$ and $z^2$ and $z^3$ multiple of $f(Z)$.

We saw that as some of the metric coefficients do depend on $x$ and $y$ the solution can be called plane wave-like following Takeno [9]. Also since

\[ A = B = 1 \text{ and } E = E(x, y, Z), \]

we get the plane wave-like solutions obtained by Peres [7] namely (5).

References

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