Reissner-Nordstrom Solution for Non-Rotating Elliptical Charged Celestial Objects

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Abstract

We derived Reissner-Nordstrom solution for non-rotating charged celestial objects with Elliptical in shaped, such as stars and/or elliptical galaxies. This metric is capable to describing the behavior of gravitational field outside any charged ellipsoidal celestial objects like stars, galaxies or Quasi-stellar object.

Keywords: General relativity, Reissner-Nordstrom solution, elliptical charged object, Einstein-Maxwell’s field equation

1 Introduction

The most famous and standard solution of Einstein’s field equation for static and non-rotating objects are spherical Schwarzschild($Q = 0$)and Reissner-Nordstrom metric($Q \neq 0$)[1,2,3]. These two solutions discovered around 1916-1918 well describing the behavior of gravitational field of any massive and large size celestial bodies like planets, stars, galaxies and other astrophysical objects [4]. All of these equations for simplicity are expressed in the spherical coordinate system therefore for non-spherical or elliptical shape objects, we need to solve Einstein’s field equation in elliptical coordinate system. In fact sphere is the special case of spheroid [5]. By solving the spheroid solution of Einstein’s field equation, therefore we actually have found the general solution to describe the behavior of gravitational field outside the object. In this paper
we consider the general form of static elliptical line elements in flat Minkowski spacetime in spherical coordinate \((t, r, \theta, \phi)\) as follow \([6,7]\)

\[
ds^2 = c^2 e^{2(\beta(t,r))} dt^2 - e^{2(\gamma(t,r))} \left( \frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2} \right) dr^2 - \left( r^2 + a^2 \cos^2 \theta \right) d\theta^2 - \left( r^2 + a^2 \right) \sin^2 \theta d\phi^2 \tag{1}
\]

where \(\beta(t, r)\) and \(\gamma(t, r)\) are the unknown function which had to be determined and \(a\) is semi-principal axis along \(x\) and \(y\)-axis.

## 2 Component of Electromagnetic Field, Ricci And Stress-Energy Tensors

The non-zero components of electromagnetic field tensor is given as \([8]\)

\[
F_{\mu\nu} = \begin{pmatrix}
0 & E(r, \theta) & 0 & 0 \\
-E(r, \theta) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\tag{2}
\]

where \(E(r, \theta)\) is electric field as a function of \(r\), and \(\theta\) as below

\[
E = E(r, \theta) = \frac{C}{(r^2 + a^2) \sin^2 \theta + r^2 \cos^2 \theta} \tag{3}
\]

and \(C\) is a constant of boundary condition. Ricci tensor for line element \([Eq1]\) calculated with using Maxima V5.31.2\([9]\). We have determined \(\beta(t, r)\) and \(\gamma(t, r)\) for ellipsoidal shape, where, for \(a = 0\), elliptical Reissner-Nordstrom metric becomes standard spherical Reissner-Nordstrom metric and therefore the Ricci tensor are,

\[
R_{00} = R_{tt} = \dot{\beta} \dot{\gamma} - \dot{\gamma} \dot{\gamma} - \dot{\gamma}^2 - e^{2(\beta-\gamma)} \left( \beta'' + \beta'^2 - \gamma' \beta' + \frac{2\beta'}{r} \right) R_{01} = R_{tr} = \frac{\dot{\gamma}}{r}
\]

\[
R_{11} = R_{rr} = \beta'' + \beta'^2 - \gamma' \beta' + \frac{2\gamma'}{r} - e^{2(\beta-\gamma)} \left( \dot{\beta} \dot{\gamma} - \dot{\gamma} \dot{\gamma} - \dot{\gamma}^2 \right)
\]

\[
R_{22} = R_{\theta\theta} = -e^{-\gamma} \left( \beta' r - \gamma' r + 1 \right) - 1,
\]

\[
R_{33} = R_{\phi\phi} = \left[ -e^{-\gamma} \left( \beta' r - \gamma' r + 1 \right) - 1 \right] \sin^2 \theta \tag{4}
\]

where \(\dot{\beta}, \ddot{\beta}, \beta', \beta'', \dot{\gamma}, \ddot{\gamma}\) and \(\gamma'\) have their usual meaning. We assumed that the charge particle is in static form therefore the components of magnetic field
are zero. We have to note that [Eq3] for spherical shape \((a = 0)\), becomes
\[
E(r) = C/r^2
\]
and thereafter the stress-energy tensor components are,
\[
\begin{align*}
T_{00} &= T_{tt} = \frac{E^2(r)}{2}e^{-2\gamma(t,r)}, \\
T_{11} &= T_{rr} = \frac{E^2(r)}{2}e^{-2\beta(t,r)}, \\
T_{01} &= T_{tr} = 0, \\
T_{22} &= T_{\theta\theta} = \frac{r^2E^2(r)}{2}e^{-2[\beta(t,r) + \gamma(t,r)]}, \\
T_{33} &= T_{\phi\phi} = T_{\theta\theta}\sin^2\theta
\end{align*}
\]  
and for \(R_{01} = 0\), then \(\dot{\gamma} = 0\) and this implies \(\gamma = \gamma(r)\) and simplify we get
\[
\beta(t, r) = -\gamma(r) = \beta(r)
\]

3 The Elliptical Reissner-Nordstrom Metric

With using [Eq4] and [Eq5], for Ricci tensor \(R_{\theta\theta}\) and after simplification we get,
\[
2e^{2\beta(r)}\dot{\beta}r + e^{2\beta(r)} = 1 - \frac{H}{r^2}
\]
which implies \(\frac{\partial e^{2\beta(r)}}{\partial r} = 1 - \frac{H}{r^2}\), where \(H\) is constant. Integrate this equation with respect to \(r\) and we obtain
\[
e^{\gamma(r)} = 1 - \frac{\lambda}{r} + \frac{\delta}{r^2}
\]
here \(\lambda\) and \(\delta\) are constant of integration. Using \(\beta(r) = -\gamma(r)\) we have
\[
e^{\gamma(r)} = e^{-\beta(r)} = 1/e^{\beta(r)} = \frac{1}{1 - \frac{\lambda}{r} + \frac{\delta}{r^2}} = \left(1 - \frac{\lambda}{r} + \frac{\delta}{r^2}\right)^{-1}
\]
and by, substituting the results in [Eq1] then,
\[
ds^2 = c^2\left(1 - \frac{\lambda}{r} + \frac{\delta}{r^2}\right)dt^2 - \left(1 - \frac{\lambda}{r} + \frac{\delta}{r^2}\right)^{-1}\left(\frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}\right)dr^2
\]
\[
- (r^2 + a^2\cos^2\theta)d\theta^2 - (r^2 + a^2)\sin^2\theta d\phi^2
\]
where \(\lambda\) and \(\delta\) are nothing but Schwarzschild and Reissner-Nordstrom radius as a function of mass, \(M\) and charge, \(Q\) of ellipsoidal object. These radii take the value of \(\lambda = \frac{2GM}{c^2}\) and \(\delta = \frac{Q^2G^2}{4\epsilon_0c^4}\) where \(G\) is gravitational constant and \(\epsilon_0\) is permittivity of free space. We can see the generalization of this equation where at \(a = 0\), this line element reduce to the well known standard spherical symmetric solution of Schwarzschild and Reissner-Nordstrom metric.

4 Conclusion

In summary, we have derived the solution of couple Einstein-Maxwell’s field equation for elliptical charged object. This equation is no doubt that cover
the standard spherical solutions as well as the spheroid shape object determined by the parameter, \( a \). It is also important in the study concerning the behaviour of gravitational field for extremely large celestial bodies like stars, planets, galaxies, Qso and as well as black hole.

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References


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