A Note on Wormholes in Slightly Modified Gravitational Theories

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Abstract. Wormholes that meet the flare-out condition violate the weak energy condition in classical general relativity. The purpose of this note is to show that even a slight modification of the gravitational theory could, under certain conditions, avoid this violation. The first part discusses some general criteria based on the field equations, while the second part assumes a specific equation of state describing normal matter, together with a particular type of shape function. The analysis is confined to wormholes with zero tidal forces.

PACS numbers: 04.20.Jb, 04.20.Gz, 04.50.-h

Keywords: wormholes, $f(R)$ gravity

1. Introduction

Interest in modified theories of gravity has increased greatly since the discovery that our Universe is undergoing an accelerated expansion. In particular, $f(R)$ modified gravity replaces the Ricci scalar $R$ in the Einstein-Hilbert action

$$S_{EH} = \int \sqrt{-g} \, R \, d^4x$$

by a nonlinear function $f(R)$:

$$S_{f(R)} = \int \sqrt{-g} \, f(R) \, d^4x.$$
Wormhole geometries in $f(R)$ modified gravitational theories are discussed in Ref. [10]. Ref. [3] assumes a noncommutative-geometry background in constructing wormhole geometries in $f(R)$ gravity.

It is well known that wormholes in classical general relativity (GR) require a violation of the null energy condition, usually calling for the use of “exotic matter” [11]. Such matter must be confined to a very narrow band around the throat [2]. Moreover, it is shown in Refs. [4] and [5] that for a wormhole to be compatible with quantum field theory, it is necessary to strike a balance between reducing the size of the exotic region and the degree of fine-tuning of the metric coefficients required to achieve this reduction. So dealing with a very small region suggests that a small modification of the gravitational theory may take the place of exotic matter, analogous to the way that the smearing effect in noncommutative geometry can replace such exotic matter [7]. So in studying the effect of the slightly modified gravity, we concentrate mainly on the vicinity of the throat.

A sufficiently small modification of the gravitational theory is likely to be consistent with observation.

2. WORMHOLE GEOMETRIES IN SLIGHTLY MODIFIED $f(R)$ GRAVITY

To describe a spherically symmetric wormhole spacetime, we take the metric to be [11]

$$ds^2 = -e^{\Phi(r)}dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

Here we recall that $b = b(r)$ is called the shape function and $\Phi = \Phi(r)$ the redshift function. For the shape function we must have $b(r_0) = r_0$, where $r = r_0$ is the radius of the throat of the wormhole. In addition, $b'(r_0) < 1$ and $b(r) < r$ to satisfy the flare-out condition [11]. These restrictions result in the violation of the weak energy condition in classical general relativity, especially in the vicinity of the throat.

Regarding the redshift function, we normally require that $\Phi(r)$ remain finite to prevent an event horizon. In the present study involving $f(R)$ gravity, we need to assume that $\Phi(r) \equiv \text{constant}$, so that $\Phi' \equiv 0$. Otherwise, according to Lobo [10], the analysis becomes intractable.

Our next task is to define what is meant by slightly modified gravity. To this end, we list the gravitational field equations in the form used by Lobo [10]:

$$\rho(r) = F(r) \frac{b'(r)}{r^2},$$

$$p_r(r) = -F(r) \frac{b(r)}{r^3} + F'(r) \frac{rb'(r) - b(r)}{2r^2} - F''(r) \left[ 1 - \frac{b(r)}{r} \right].$$
We will see in the next section that this condition is actually met when $s$ is sufficiently large. Consider a simple example. In the vicinity of the throat, $rb$ now. The flare-out condition implies that $F/r$ is close to zero near the throat, let us disregard the last term for $b/r$ is small positive constant. At $r = r_0$, $F(r_0) = 1$, $F'(r_0) = -a$, and $F''(r_0) = -a^2$. Substituting in Eq. (8), we get

$$ (rb' - b) \frac{2 - ar_0}{2r_0^3} + a^2 \left( 1 - \frac{b}{r} \right) \geq 0. $$

While it is possible in principle to obtain $f(R)$ from $F(r)$, our goal is more modest: how to define slightly modified $f(R)$ gravity. To this end, we observe that the above field equations reduce to the Einstein equations for $\Phi' \equiv 0$ whenever $F \equiv 1$. Consequently, comparing Eqs. (4) and (7), a slight change in $F$ results in a slight change in $R$, which, referring to Eqs. (1) and (2), characterizes $f(R)$ modified gravity. So we may quantify the notion of slightly modified gravity by assuming that $F(r)$ remains close to unity and relatively “flat,” i.e., both $F'(r)$ and $F''(r)$ remain relatively small in absolute value. To discuss wormholes, we need the additional assumption that $F'(r_0)$ is negative. (We will see in the next section that this condition is actually met when $F(r)$ is computed from a known shape function.) $F''(r)$ will be discussed later. Observe that in Eq. (4), $F$ behaves like a dimensionless scale factor.

Suppose the shape function meets the flare-out condition $b'(r_0) < 1$. By continuity, $b(r) < r$ in the immediate vicinity of the throat. Our goal is to show that in this region, we may have $\rho + p_r \geq 0$, as well as $\rho + p_t \geq 0$, thereby satisfying the weak energy condition. From Eqs. (4) and (5)

$$ \rho + p_r = \frac{F b'}{r^2} - F \frac{b}{r^3} + F' \frac{r b' - b}{2r^2} - F'' \left( 1 - \frac{b}{r} \right) = $$

$$ (rb' - b) \left( \frac{F}{r^3} + \frac{F'}{2r^2} \right) - F'' \left( 1 - \frac{b}{r} \right) \geq 0. $$

Since $1 - b/r$ is close to zero near the throat, let us disregard the last term for now. The flare-out condition implies that $rb'(r) - b(r) < 0$; so we must have

$$ \frac{F}{r^3} + \frac{F'}{2r^2} \leq 0. $$

Given that $F(r)$ is very close to unity near the throat and that $F'(r) < 0$ and relatively small in absolute value, $F/r_0^3 + F'/2r_0^2$ can only be negative if $r_0$ is sufficiently large. Consider a simple example. In the vicinity of the throat, suppose that

$$ F = 2 - e^{a(r-r_0)}; \quad F' = -e^{a(r-r_0)} a \quad \text{and} \quad F'' = -e^{a(r-r_0)} a^2, $$

where $a$ is a small positive constant. At $r = r_0$, $F(r_0) = 1$, $F'(r_0) = -a$, and $F''(r_0) = -a^2$. Substituting in Eq. (8), we get

$$ (rb' - b) \frac{2 - ar_0}{2r_0^3} + a^2 \left( 1 - \frac{b}{r} \right) \geq 0. $$
provided that \(2 - a r_0 \leq 0\). We conclude that
\[
 r_0 \geq \frac{2}{a}. \tag{9}
\]
For example, if \(a = 0.001 \text{ m}^{-1}\), then \(r_0 \geq 2 \text{ km}\). The general conclusion is that
\[
 r_0 \geq \frac{2}{-F'(r_0)}, \quad \text{where} \quad F'(r_0) < 0. \tag{10}
\]
Observe that \(F''(r_0)\) must either be negative or negligibly small. (In the above example, it is actually both.) Finally, by Eq. (6), we also have \(\rho + p_t \geq 0\).

The closer \(F'(r_0)\) is to zero, the larger \(r_0\) has to be to meet the condition \(\rho + p_r \geq 0\). Returning to Einstein gravity, if \(F'(r_0) \to 0\), then \(r_0 \to \infty\), and we do not get a wormhole. In this case, then, the existence of a wormhole requires that \(\rho + p_r < 0\), the usual violation of the null energy condition in GR. As we have seen, we also get \(\rho + p_r < 0\) whenever \(F'(r_0) \geq 0\).

### 3. A Known Shape Function

The above analysis assumes a small change in the Ricci scalar, induced by a small change in \(F\). An alternative approach is to assume a certain equation of state and a known shape function and then determine \(F\) in the vicinity of the throat. To make the analysis tractable, \(b(r)\) must be relatively simple, yet typical enough to yield a reasonably general conclusion.

The equation of state to be used for now assumes normal matter:
\[
p_r = \omega \rho, \quad 0 < \omega < 1. \tag{11}
\]
For the shape function we use the form
\[
b(r) = r_0^{1-\alpha} r^\alpha, \quad 0 < \alpha < 1. \tag{12}
\]
Observe that \(b(r_0) = r_0\) and \(b'(r_0) = \alpha < 1\), so that the flare-out condition has been met.

As before, we want \(F(r_0) \approx 1\), but for computational convenience, we assume that \(F(r_0) = 1\). From Eqs. (4) and (5),
\[
\omega F r_0^{1-\alpha} \frac{\alpha r^{-\alpha-1}}{r^2} = -F r_0^{1-\alpha} \frac{\alpha r^\alpha}{r^3} + F'\frac{r_0^{1-\alpha} \alpha r^{-\alpha-1} - r_0^{1-\alpha} r^\alpha}{2r^2} - F'' \left(1 - \frac{r_0^{1-\alpha} r^\alpha}{r}\right). \tag{13}
\]
According to Lobo [10], the existence of \(F''\) makes it virtually impossible in most cases to get an exact solution. While the last term is once again zero at the throat, we still need to be concerned with the vicinity of the throat. To this end, we need to assume that \(\alpha\) is close to unity, so that the last term is close to zero. What remains is easy enough to solve. After simplifying, we obtain the linear differential equation
\[
F'(r) + \frac{2(1 + \alpha \omega)}{1 - \alpha} \frac{1}{r} F(r) = 0,
\]
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where \( r \) is near the throat. The solution is

\[
F(r) = \left( \frac{r}{r_0} \right)^{-2(1+\alpha\omega)/(1-\alpha)},
\]

which satisfies the condition \( F(r_0) = 1 \) and so \( F(r) \approx 1 \) in the vicinity of the throat. Having satisfied the normal-matter equation, Eq. (11), in the vicinity of the throat, as well as the flare-out condition, we conclude that the wormhole is sustained due to the modified gravity.

Even though the last term in Eq. (13) was neglected, we are still dealing with a second-order equation; so we need to consider \( F'(r) \) in the vicinity of the throat:

\[
F'(r_0) = -2(1 + \alpha\omega) \left( \frac{r}{r_0} \right)^{-2(1+\alpha\omega)/(1-\alpha)-1} \left| \frac{1}{r_0} \right|_{r=r_0} = -2(1 + \alpha\omega) \frac{1}{1 - \alpha} \frac{r}{r_0} < 0,
\]

i.e., \( F'(r_0) < 0 \), as in the previous section, but we still want \( |F'(r_0)| \) to be relatively small in order to remain close to Einstein gravity. So once again, \( r_0 \) has to be sufficiently large:

\[
r_0 \geq \frac{2}{-F'(r_0)} \frac{1 + \alpha\omega}{1 - \alpha}.
\]

For example, if \( \alpha = 0.99 \), \( F'(r_0) = -0.3 \), and \( \omega = 0.5 \), we obtain \( r_0 \geq 1 \) km. (It is readily checked that if \( |F'(r_0)| \) is small, then so is \( |F''(r_0)| \).

The parameter \( \omega \) was chosen to describe normal matter. However, solution (14) is valid for any \( \omega \). Thus if \( \omega = -1 \), which is equivalent to assuming Einstein’s cosmological constant \([1]\), inequality (16) reduces to inequality (10). If \( \omega < -1 \), we are dealing with phantom energy, which is known to support wormholes in classical GR \([6, 14, 8, 15]\). In the present situation, we still need to consider \( \alpha \). So if we assume that

\[
-\frac{1}{\omega} < \alpha < 1,
\]

then \((1+\alpha\omega)/(1-\alpha)\) becomes negative and condition (16) is automatically satisfied for all \( r_0 \). We conclude that phantom energy can also support wormholes in our slightly modified gravitational theory.

A remark concerning \( F'' \): Retaining \( F'' \) is possible if \( b(r) \) is sufficiently simple. Suppose \( b(r) = ar, \ a > 0 \). Then \( b(r_0) = ar_0 \approx r_0 \), provided that \( r_0 \) is large compared to \( a \). Then Eq. (8) becomes

\[
F'' + \frac{1}{r^2} a(1 + \omega) \frac{1}{1 - \alpha} F = 0, \quad F(r_0) = 1, \quad 0 < \omega < 1.
\]
The solution is
\[ F(r) = \left( \frac{r}{r_0} \right)^{\frac{1}{2}[1+\sqrt{1-4a(1+\omega)/(1-a)}]} . \] (17)

Given that \( a \) is a small constant, \( F'(r_0) \approx 1/r_0 \) and \( F''(r_0) \approx -1/r_0^2 \). Since \( r_0 \) is assumed to be large, \( F'(r) \) and \( |F''(r)| \) are small in the vicinity of the throat.

4. Conclusion

Wormholes that meet the flare-out condition violate the weak energy condition in classical general relativity. It is shown in the first part of this note that a slight modification of the field equations, and hence of the gravitational theory, could avoid this violation. The modification calls for a change in \( F(r) \), which induces a change in \( R \) in the Einstein-Hilbert action: for every \( r \), \( F(r) \) remains close to unity, while \( F'(r) \) and \( F''(r) \) are relatively small in absolute value. The weak energy condition is met provided that \( r_0 \), the radius of the throat, is sufficiently large and that \( F'(r_0) < 0 \); \( F''(r_0) \) must be either negative or negligibly small.

The second part assumes the equation of state \( p = \omega \rho \), \( 0 < \omega < 1 \), thereby representing normal matter, as well as a shape function of the form \( b(r) = r_0^{1-\alpha} r^\alpha \), \( 0 < \alpha < 1 \); hence \( b'(r_0) = \alpha < 1 \). The equation of state yields the solution \( F(r) = (r/r_0)^{-2(1+\alpha\omega)/(1-\alpha)} \) in the vicinity of the throat, where \( \alpha \) is close to unity and \( r_0 \) is sufficiently large. Since the wormhole consists of ordinary matter, its survival must be attributed to the modified gravity. Moreover, a small modification would likely be consistent with observation.

The assumption that \( \omega \) is positive is not actually necessary: in particular, phantom dark energy can support a wormhole in slightly modified gravitational theory, as well as in GR. Zero tidal forces are assumed throughout.

References


Received: September, 2013