Symmetric Identities of the $q$-Euler Polynomials

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Abstract

In this paper, we study some symmetric identities of $q$-Euler numbers and polynomials. From these properties, we derive several identities of $q$-Euler numbers and polynomials.
1 Introduction

The Euler polynomials are defined by the generating function to be

\[
\frac{2}{e^t + 1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (\text{see } [2-6]).
\]  

(1.1)

with the usual convention about replacing \( E_n(x) \) by \( E_{n,q}(x) \).

When \( x = 0 \), \( E_n = E_n(0) \) are called the Euler numbers. Let \( q \in \mathbb{C} \) with \( |q| < 1 \). For any complex number \( x \), the \( q \)-analogue of \( x \) is defined by \( [x]_q = \frac{1-q^x}{1-q} \). Note that \( \lim_{q \to 1} [x]_q = x \). Recently, T. Kim introduced a \( q \)-extension of Euler polynomials as follows:

\[
F_q(t, x) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n e^{[n+x]_q t} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}, \quad (\text{see } [7,8]).
\]  

(1.2)

When \( x = 0 \), \( E_{n,q} = E_{n,q}(0) \) are called the \( q \)-Euler numbers. From (1.2), we note that

\[
E_{n,q}(x) = (q^x E_q + [x]_q)^n
= \sum_{l=0}^{n} \binom{n}{l} q^x E_{l,q} [x]_q^{n-l}, \quad (\text{see } [7,8]),
\]  

(1.3)

with the usual convention about replacing \( E_q^l \) by \( E_{l,q} \).

In [8], Kim introduced \( q \)-Euler zeta function as follows:

\[
\zeta_{E,q}(s, x) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} F_q(-t, x) \, dt
= [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{[n+x]_q^s},
\]  

(1.4)

where \( x \neq 0, -1, -2, \ldots \), and \( s \in \mathbb{C} \).

From (1.4), we have

\[
\zeta_{E,q}(-m, x) = E_{m,q}(x),
\]  

(1.5)

where \( m \in \mathbb{Z}_{\geq 0} \).
Recently, Y. Simsek gave recurrence symmetric identities for \((h, q)\)-Euler polynomials and the alternating sums of powers of consecutive \((h, q)\)-integers(see [9]) and Y. He gave some interesting symmetric identities of Carlitz’s \(q\)-Bernoulli numbers and polynomials(see [1]). In this paper, we study some new symmetries of the \(q\)-Euler numbers and polynomials, which is the answer to an open question for the symmetric identities of Carlitz’s type \(q\)-Euler numbers and polynomials in [5]. By using our symmetries for the \(q\)-Euler polynomials we can obtain some identities between \(q\)-Euler numbers and polynomials.

2 Symmetric identities of \(q\)-Euler polynomials

In this section, we assume that \(a, b \in \mathbb{N}\) with \(a \equiv 1 \pmod{2}\) and \(b \equiv 1 \pmod{2}\). First, we observe that

\[
\frac{1}{[2]_q^a} \zeta_{E,q^a}(s, bx + \frac{b+1}{a}) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{na}}{[n + bx + \frac{b}{a}]_q^s} = \sum_{n=0}^{\infty} [b+1]_q^s \sum_{i=0}^{b-1} \frac{(-1)^{i+bn} q^{a(i+bn)}}{[ab(x+n)+bj+ai]_q^s}.
\]

(2.1)

Thus, by (2.1), we get

\[
\frac{[b]_q^s}{[2]_q^a} \sum_{j=0}^{a-1} (-1)^j q^{bj} \zeta_{E,q^a}(s, bx + \frac{b+1}{a}) = [b]_q^s \sum_{i=0}^{b-1} \sum_{n=0}^{\infty} (-1)^{i+bn} q^{a(i+bn)} \sum_{j=0}^{a-1} \frac{q^{aj+bn}(s, ax + \frac{a}{b})}{[ab(x+n)+bj+ai]_q^s}.
\]

(2.2)

By the same method as (2.2), we get

\[
\frac{[a]_q^s}{[2]_q^b} \sum_{j=0}^{b-1} (-1)^j q^{aj} \zeta_{E,q^b}(s, ax + \frac{a+1}{b}) = [a]_q^s \sum_{i=0}^{b-1} \sum_{n=0}^{\infty} q^{bi+aj+bn} (-1)^{i+n+j} \sum_{j=0}^{a-1} \frac{q^{aj+bn}(s, ax + \frac{a}{b})}{[ab(x+n)+bj+ai]_q^s}.
\]

(2.3)

Therefore, by (2.2) and (2.3), we obtain the following theorem.

**Theorem 2.1.** For \(a, b \in \mathbb{N}\) with \(a \equiv 1 \pmod{2}\), \(b \equiv 1 \pmod{2}\),

\[
[2]_q^b [b]_q^a \sum_{j=0}^{a-1} (-1)^j q^{bj} \zeta_{E,q^a}(s, bx + \frac{b+1}{a}) = [2]_q^a [a]_q^b \sum_{j=0}^{b-1} (-1)^j q^{aj} \zeta_{E,q^b}(s, ax + \frac{a+1}{b}).
\]
By (1.5) and Theorem 2.1, we obtain the following theorem.

**Theorem 2.2.** For \( n \in \mathbb{Z} \geq 0 \) and \( a, b \in \mathbb{N} \) with \( a \equiv 1 \pmod{2} \), \( b \equiv 1 \pmod{2} \), we have

\[
[2]_q b \binom{2}{a} [a]_q \sum_{j=0}^{a-1} (-1)^j q^{bj} E_{n,q}^a (bx + \frac{bj}{a}) = [2]_q b \binom{2}{b} [b]_q \sum_{j=0}^{b-1} (-1)^j q^{aj} E_{n,q}^b (ax + \frac{aj}{b}).
\]

From (1.3), we note that

\[
E_{n,q}(x + y) = (q^{x+y} E_q + [x + y]_q)^n
= (q^{x+y} E_q + q^y [y]_q + [x]_q)^n
= (q^x (q^y E_q + [y]_q) + [x]_q)^n
= \sum_{i=0}^{n} \binom{n}{i} q^{xi} (q^y E_q + [y]_q)^i [x]_q^{n-i}
= \sum_{i=0}^{n} \binom{n}{i} q^{xi} E_{i,q}(y) [x]_q^{n-i}.
\]

Therefore, by (2.4), we obtain the following proposition.

**Proposition 2.3.** For \( n \geq 0 \), we have

\[
E_{n,q}(x + y) = \sum_{i=0}^{n} \binom{n}{i} q^{xi} E_{i,q}(y) [x]_q^{n-i}
= \sum_{i=0}^{n} \binom{n}{i} q^{(n-i)x} E_{n-i,q}(y) [x]_q^i.
\]

Now, we observe that
we have

For Theorem 2.4.

Therefore, by Theorem 2.2, (2.6) and (2.7), we obtain the following theorem.

\[
\sum_{j=0}^{a-1}(-1)^j q^j b_j E_n, q^a (bx + \frac{bj}{a})
\]

\[
= \sum_{j=0}^{a-1}(-1)^j q^j b_j \sum_{i=0}^{n} \binom{n}{i} q^{ia(\frac{bj}{a})} E_i, q^a (bx) \left[ \frac{bj}{a} \right]_q^{n-i}
\]

\[
= \sum_{j=0}^{a-1}(-1)^j q^j b_j \sum_{i=0}^{n} \binom{n}{i} q^{(n-i)j} E_{n-j, q^a} (bx) \left[ \frac{bj}{a} \right]_q^i
\]

\[
= \sum_{i=0}^{n} \binom{n}{i} \left( \frac{[b]_q}{[a]_q} \right)^i E_{n-i, q^a} (bx) \sum_{j=0}^{a-1}(-1)^j q^{j(n+1-i)} [j]^i
\]

\[
= \sum_{i=0}^{n} \binom{n}{i} \left( \frac{[b]_q}{[a]_q} \right)^i E_{n-i, q^a} (bx) S^*_n, q^a (a),
\]

where \( S^*_n, q^a (a) = \sum_{j=0}^{a-1}(-1)^j q^{(n+1-i)j} [j]^i. \)

From (2.5), we can derive

\[
[2]q^a[a]^n \sum_{j=0}^{a-1}(-1)^j q^j E_n, q^a (bx + \frac{bj}{a}) = [2]q^b \sum_{i=0}^{n} \binom{n}{i} [a]^{n-i} [b]^i E_{n-i, q^a} (bx) S^*_n, q^a (a).
\]

(2.6)

By the same method as (2.6), we get

\[
[2]q^b[b]^n \sum_{j=0}^{b-1}(-1)^j q^j E_n, q^b (ax + \frac{aj}{b}) = [2]q^a \sum_{i=0}^{n} \binom{n}{i} [b]^{n-i} [a]^i E_{n-i, q^b} (ax) S^*_n, q^b (b).
\]

(2.7)

Therefore, by Theorem 2.2, (2.6) and (2.7), we obtain the following theorem.

**Theorem 2.4.** For \( n \in \mathbb{Z}_{\geq 0} \) and \( a, b \in \mathbb{N} \) with \( a \equiv 1 \mod 2 \), \( b \equiv 1 \mod 2 \), we have

\[
[2]q^a \sum_{i=0}^{n} \binom{n}{i} [a]^{n-i} [b]^i E_{n-i, q^b} (bx) S^*_n, q^b (a) = [2]q^a \sum_{i=0}^{n} \binom{n}{i} [b]^{n-i} [a]^i E_{n-i, q^b} (ax) S^*_n, q^b (b),
\]

where \( S^*_n, q^a (a) = \sum_{j=0}^{a-1}(-1)^j q^{(n+1-i)j} [j]^i. \)
It is easy to show that
\[ [x]_q u + q^x [y + m]_q (u + v) = [x + y + m]_q (u + v) - [x]_q v. \] (2.8)

Thus, by (2.8), we get
\[ e^{[x]_q u} \sum_{m=0}^{\infty} q^m (-1)^m e^{[y + m]_q q^x (u + v)} = e^{-[x]_q v} \sum_{m=0}^{\infty} q^m (-1)^m q^{[x + y + m]_q (u + v)}. \] (2.9)

The left hand side of (2.9) multiplied by \([2]_q\) is given by
\[
\begin{align*}
[2]_q e^{[x]_q u} & \sum_{m=0}^{\infty} q^m (-1)^m e^{[y + m]_q q^x (u + v)} \\
& = e^{[x]_q u} \sum_{n=0}^{\infty} q^n x E_{n,q} (y) \frac{(u + v)^n}{n!} \\
& = \left( \sum_{n=0}^{\infty} \frac{[x]_q^n}{n!} \right) \left( \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} q^{(k+n)x} E_{k+n,q} (y) \frac{u^k v^n}{k! n!} \right) \\
& = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \sum_{k=0}^{m} \binom{m}{k} q^{(k+n)x} E_{k+n,q} (y) [x]_q^{n-k} \right) \frac{u^m v^n}{m! n!}.
\end{align*}
\] (2.10)

The right hand side of (2.9) multiplied by \([2]_q\) is given by
\[
\begin{align*}
[2]_q e^{-[x]_q v} & \sum_{m=0}^{\infty} (-1)^m q^m e^{[x + y + m]_q (u + v)} \\
& = e^{-[x]_q v} \sum_{n=0}^{\infty} E_{n,q} (x + y) \frac{(u + v)^n}{n!} \\
& = \left( \sum_{n=0}^{\infty} \frac{(-[x]_q)^n}{n!} \right) \left( \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} E_{m+k,q} (x + y) \frac{u^m v^k}{m! k!} \right) \\
& = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} E_{m+k,q} (x + y) (-[x]_q)^{n-k} \right) \frac{u^m v^n}{m! n!} \\
& = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} E_{m+k,q} (x + y) q^{(n-k)x} [-x]_q^{n-k} \right) \frac{u^m v^n}{m! n!}.
\end{align*}
\] (2.11)

Therefore, by (2.10) and (2.11), we get
\[
\begin{align*}
\sum_{k=0}^{m} \binom{m}{k} q^{(n+k)x} E_{m+k,q} (y) [x]_q^{n-k} &= \sum_{k=0}^{n} \binom{n}{k} q^{(n-k)x} E_{m+k,q} (x + y) [-x]_q^{n-k} \\
& = (2.12)
\end{align*}
\]
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References


Received: November 1, 2013